PROBLEM SET 2, 18.155
DUE SEPTEMBER 23, 2016

Questions 3-5 of this problem set constitute a proof of Schwartz' structure theorem. This uses the material in Lecture 3 on operations and Sobolev spaces and the results of Lecture 4 on the Sobolev embedding theorem. A (slightly different) proof is in the notes for the course. If you want to do it early, just notice that if $k \in \mathbb{N}$ and $s > n/2 + k$ then

$$H^s(\mathbb{R}^n) \subset \mathcal{C}^k_\infty(\mathbb{R}^n) \tag{1}$$

[the space of functions with continuous derivatives up to order k bounded on $\mathbb{R}^n$] is a continuous inclusion.

The norms we use on Schwartz space here are

$$\|u\|_k = \sup_{x\in\mathbb{R}^n, |\alpha|\le k} |(1+|x|^2)^{k/2}|D^\alpha_x u(x)|, \ k \in \mathbb{N}_0 = \{0, 1, \dots, \}.$$

The norm on the Sobolev space $H^s(\mathbb{R}^n)$ is

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}|^2 d\xi\right)^{\frac{1}{2}}. \tag{2}$$

(1) (L3) The Dirac delta 'function' $\delta \in \mathcal{S}'(\mathbb{R}^n)$ defined by

$$\delta(\phi) = \phi(0) \ \forall \ \phi \in \mathcal{S}(\mathbb{R}^n)$$

is amongst the most important distributions (it is a measure).

A) Find explicit formulae for the derivatives $\partial^\alpha \delta$ evaluated on test functions

B) Compute the Fourier transform of $\partial^\alpha \delta$.

C) Show that

$$\partial^\alpha \delta \in H^{-|\alpha|-n/2-\epsilon}(\mathbb{R}^n)$$

for $\epsilon > 0$ but not for $\epsilon = 0$.

(2) (L3) Go through the convolution discussion for L^1. That is, show that if $u \in L^1(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ then the convolution integral

$$u * \psi(x) = \int u(y)\psi(x-y)$$

is a well-defined infinitely differentiable function with all derivatives in L^1. Show (you might want to look up and are allowed to use continuity-in-the-mean of L^1 functions) that if $\phi_k(x) = k^{-n}\phi(kx)$ is an approximate identity (as in class) then

$$u * \phi_k \to u \text{ in } L^1(\mathbb{R}^n).$$

Use this to show (trivially, i.e. in one line) that $L^1(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ (meaning an injection) by the usual definition $u \longrightarrow U_u$, $U_u(\phi) = \int u(x)\phi(x)$.

(3) (L4) Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$, show that for some constants $k \in \mathbb{N}$ and $C > 0$

$$|((1+|x|^2)^{-k}u)(\phi)| \le C\|\phi\|_{\mathcal{C}^k},$$
$$\|\phi\|_{\mathcal{C}^k} = \sup_{x\in\mathbb{R}^n, |\alpha|\le k} |\partial^\alpha\phi(x)| \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

(4) (L4) Recall the Sobolev embedding theorem and show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then there exist k and N such that

$$|((1+|x|^2)^{-k}u)(\phi)| \le C_N\|\phi\|_{H^{2N}} \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

(5) (L4) Conclude that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then there exist $f \in L^2(\mathbb{R}^n)$, and $k,\ N \in \mathbb{N}$ such that

$$u = (1+|x|^2)^k(1+\Delta)^N f,\ \Delta = -\sum_{i=1}^n \partial_i^2.$$

(6) (Optional) The formula above has a certain lack of symmetry between 'real space' (multiplication by functions) and 'dual space' (differentiation). First show that there is a similar expression with the order reversed. Suppose you knew (as is indeed the case) that the harmonic oscillator $\Delta+|x|^2$ is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$ and hence on $\mathcal{S}'(\mathbb{R}^n)$ which has the property

(3)
$$(\Delta+|x|^2)^{-1}[(1+|x|^2)^{k/2}H^{-k}(\mathbb{R}^n)] \subset (1+|x|^2)^{(k-1)/2}H^{1-k}(\mathbb{R}^n)\ \forall\ k \in \mathbb{N}.$$

Show that for each $u \in \mathcal{S}'(\mathbb{R}^n)$ there exists $N \in \mathbb{N}$ and $f \in L^2(\mathbb{R}^n)$ such that

(4)
$$u = (\Delta+|x|^2)^N f.$$

(7) (Optional) From the results above, observe that $\mathcal{S}'(\mathbb{R}^n)$ is a union of Hilbert subspaces. Give it the inductive limit topology in which a set is open if it intersects each of these subspaces in an open set. Show that a linear functional on $\mathcal{S}'(\mathbb{R}^n)$ which is

continuous in this topology is given by pairing with an element of $\mathcal{S}(\mathbb{R}^n)$.