PROBLEM SET 1, 18.155
DUE FRIDAY 16 SEPTEMBER, 2016

These problems can be done after Lecture 1. Only solutions to the first 5 problems need to be submitted.

Let me remind you of the arrangements for homework. This should be emailed to me (not our grader) as a pdf file. It is not essential that you work with TeX although it is preferable. If you write out the solutions and scan them to pdf please check that they are readable. Solutions are due on Fridays, but I will not check until Saturday morning.

(1) (L1) Prove (probably by induction) the multi-variable form of Leibniz formula for the derivatives of the product of two (sufficiently differentiable) functions:-

$$\partial^\alpha(fg) = \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \partial^\beta f \cdot \partial^{\alpha-\beta} g.$$

[Obviously you need to work out what the combinatorial coefficients are, or define them by induction at least].

(2) (L1) Consider the norms, for each $N \in \mathbb{N}$, on $\mathcal{S}(\mathbb{R}^n)$

$$\|f\|_N = \sum_{|\beta|+|\alpha|\le N} \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha f(x)|.$$

Show that

$$\|f\|'_N = \sum_{|\beta|+|\alpha|\le N} \sup_{x\in\mathbb{R}^n} |\partial^\alpha (x^\beta f(x))|$$

are equivalent norms, $\|f\|_N \le C_N \|f\|'_N$ and $\|f\|'_N \le C'_N \|f\|_N$.

(3) (L1) Consider $F \in \mathcal{C}^\infty(\mathbb{R}^n)$ which is an infinitely differentiable function of polynomial growth, in the sense that for each α there exists $N(\alpha) \in \mathbb{N}$ and $C(\alpha) > 0$ such that

$$|\partial^\alpha F(x)| \le C(\alpha)(1+|x|)^{N(\alpha)}.$$

Show that multiplication by F gives a continuous linear map $\times F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$.

(4) (L1) Show that if $s \in \mathbb{R}$ then $F_s(x) = (1+|x|^2)^{s/2}$ is a smooth function of polynomial growth in the sense discussed above and that multiplication by F_s is an isomorphism on $\mathcal{S}(\mathbb{R}^n)$.

(5) (L1)

Consider one-point, or stereographic, compactification of $\mathbb{R}^n$. This is the map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$ obtained by sending x first to the point $z = (1, x)$ in the hyperplane $z_0 = 1$ where $(z_0, \dots, z_n)$ are the coordinates in $\mathbb{R}^{n+1}$ and then mapping it to the point $Z \in \mathbb{R}^{n+1}$ with $|Z| = 1$ which is also on the line from the 'South Pole' $(-1, 0)$ to $(1, x)$.

Derive a formula for T and use it to find a formula for the inversion map $I : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ which satisfies $I(x) = x'$ if $T(x) = (z_0, z)$ and $T(x') = (-z_0, z)$. That is, it correspond to reflection across the equator in the unit sphere.

Show that if $f \in \mathcal{S}(\mathbb{R}^n)$ then $I^*f(x) = f(x')$, defined for $x \neq 0$, extends by continuity with all its derivatives across the origin where they all vanish.

Conversely show that if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a function with all derivatives continuous, then $f \in \mathcal{S}(\mathbb{R}^n)$ if I^*f has this property, that all derivatives extend continuously across the origin and vanish there – i.e. they all have limit zero at the origin.

***

(6) (Optional) Prove that $\mathcal{S}(\mathbb{R}^n)$ is a *Montel space* which means that it has an analogue of the Heine-Borel property. Namely, (you have to show that) if $D \subset \mathcal{S}(\mathbb{R}^n)$ is closed and 'bounded' in the sense that for each N there exists C_N such that $\|\phi\|_N \leq C_N$ for all $\phi \in D$, then D is compact.

(7) (Optional)

A) 'Recall' that if $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ is measurable and

$$(1+|x|)^{-N} u \in L^1(\mathbb{R}^n) \tag{1}$$

for some N then $I(u)(\phi) = \int u\phi$, for $\phi \in \mathcal{S}(\mathbb{R}^n)$ defines an element $I(u) \in \mathcal{S}'(\mathbb{R}^n)$.

B) Now, refute the idea that these are the 'most general' functions which define distributions – this is a dangerously vague statement anyway and I'm sure you would not say such a thing. NAMELY observe that

$$u(x) = \exp(i \exp(x))$$

defines an element of $\mathcal{S}'(\mathbb{R})$ and hence conclude that, in a sense you should make clear, so does

$$\exp(x)\exp(i\exp(x)) \tag{2}$$

but that this does NOT satisfy (1) above.