18.155 LECTURE 7
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Abstract. Notes before and then after lecture.

Read: Notes, Chapter 4 §1,2. As usual I will do things slightly differently.

Before lecture

- Last time I did most of the work to show that the spaces of distributions on open sets, $\mathcal{C}^{-\infty}(\Omega)$, $\Omega \subset \mathbb{R}^n$ form a sheaf. For us the main point is that the support is well-defined. Here is a more utilitarian definition (in the notes) and the proof of equivalence to the one I gave last time:

Proposition 1. *A point $p \in \Omega$, $\Omega \subset \mathbb{R}^n$ open is* not *in the support of $u \in \mathcal{C}^{-\infty}(\Omega)$ (the dual space to $\mathcal{C}^\infty_{\mathrm{c}}(\Omega)$) if and only if there exists $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ such that $\phi(p) \neq 0$ and $\phi u = 0$.*

Here the multiplication by test functions is defined as usual by formal transpose. Namely the product of functions gives

$$\mathcal{C}^\infty(\Omega) \times \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \ni (\phi, \psi) \longrightarrow \phi\psi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \tag{1}$$

is continuous so if $\phi \in \mathcal{C}^\infty(\Omega)$ and $u \in \mathcal{C}^{-\infty}(\Omega)$ we can define the product by

$$\phi u(\psi) = u(\phi\psi) \ \forall\ \psi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \Longrightarrow \phi u \in \mathcal{C}^{-\infty}(\Omega). \tag{2}$$

Proof. If $\phi(p) \neq 0$ then for some ϵ, $|\phi|$ is bounded away from 0 on $\overline{B(p,\epsilon)}$, the closed ball around p and hence $\frac{1}{\phi} \in \mathcal{C}^\infty(B(p,\epsilon))$. Choose a cut-off $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(B(p,\epsilon))$ with $\chi = 1$ in $B(p,\epsilon/2)$. Then $\frac{\chi}{\phi} \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ (extended as zero outside $B(p,\epsilon)$ and hence

$$\chi u = \frac{\chi}{\phi}\phi u = 0. \tag{3}$$

From this it follows that if $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(B(p,\epsilon/2)) \subset \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ then $\mu\chi = \mu$ and $\chi u(\mu) = u(\mu) = 0$. Thus in fact $u = 0$ in $B(p,\epsilon/2)$ (as defined last time) and hence p lies in an open set on which u vanishes, so $p \notin \operatorname{supp}(u)$.

The converse is more obvious, since if $p \notin \operatorname{supp}(u)$ then $u = 0$ on $B(p,\epsilon)$ for some $\epsilon > 0$ and $u(\phi) = 0$ for all $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(B(p,\epsilon))$ and we can certainly arrange that $\phi(p) \neq 0$. □

- This also suggests that we can define the *singular support* as 'the set where a distribution is not equal to a smooth function' by

$$\text{if } u \in \mathcal{C}^{-\infty}(\Omega) \text{ then } p \notin \operatorname{singsupp}(u) \Longleftrightarrow \exists\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \phi(p) \neq 0,\ \phi u \in \mathcal{C}^\infty(\Omega). \tag{4}$$

Exercise 1. Show that the quotient spaces $\mathcal{C}^{-\infty}(\Omega)/\mathcal{C}^{\infty}(\Omega)$ (don't try to give these a toplogy ..) form a sheaf and the singular support of a distribution is the support of its image in this sheaf. Check that $\Omega \setminus \operatorname{singsupp}(u)$ is the largest open subset of Ω to which u restricts to be equal to a smooth function.

You can also think of singsupp as the support of an element of $\mathcal{C}^{-\infty}(\Omega)$ *relative* to the subsheaf $\mathcal{C}^{\infty}(\Omega)$.

- Convolution $\mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)*\mathcal{C}^{\infty}(\mathbb{R}^n) \subset \mathcal{C}^{\infty}(\mathbb{R}^n)$ and then $\mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)*\mathcal{C}^{-\infty}(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$.
- Support and singular support of convolutions,

$$\operatorname{supp}(u*v) \subset \operatorname{supp}(u)+\operatorname{supp}(v),\ \operatorname{singsupp}(u*v) \subset \operatorname{singsupp}(u)+\operatorname{singsupp}(v) \tag{5}$$

where one of u and v has compact support.

- Ellipticity of the 'characteristic polynomial' of a partial differential operator $P(D) = \sum\limits_{|\alpha|\le m} c_\alpha D^\alpha$, $P(\xi) = \sum\limits_{|\alpha|\le m} c_\alpha \xi^\alpha$. For a polynomial of degree m the following conditions are equivalent – and called 'ellipticity'
  (1) There is a constant $c > 0$ such that $|P(\xi)| \ge c|\xi|^m$ in $|\xi| > 1/c$ ($\xi \in \mathbb{R}^n$)
  (2) If $P_m(\xi) = \sum\limits_{|\alpha|=m} c_\alpha \xi^\alpha$ is the 'principal part' of P then $P_m(\xi) \ne 0$ on $|\xi| = 1$ in $\mathbb{R}^n$.
  (3) There is a function $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ such that

$$F = \frac{1-\chi}{P(\xi)} \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ satisfies } |D^\beta F(\xi)| \le C_\beta(1+|\xi|)^{-m-|\beta|}\ \forall\ \beta. \tag{6}$$

- The singular support of $Q = P(D)F$ is $\{0\}$.
- Might get this far.
- Q is a *parametrix* for P.
- If $u \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ then $u = Q * P(D)u + u'$, $u' \in \mathcal{C}^\infty(\mathbb{R}^n)$.
- Elliptic regularity: if $u \in \mathcal{C}^\infty(\Omega)$ for some $\Omega \subset \mathbb{R}^n$ open, $P(D)$ is elliptic and $\chi P(D)u \in H^p(\mathbb{R}^n)$ for some $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ and some p then $\phi u \in H^{p+m}(\mathbb{R}^n)$ for any $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$ such that $\chi \ne 0$ on $\operatorname{supp}(\phi)$.

After lecture

I made rather heavy weather of convolution in lecture. Here is a brief summary.

(1) We already know that $\mathcal{S}(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ defined as the convolution integral, where the variables can be switched:

$$\phi * \psi(x) = \int \phi(x-y)\psi(y) = \int \phi(y)\psi(x-y)dy. \tag{7}$$

This integral converges absolutely and Fubini's theorem we can see that

$$\mathcal{F}(\phi * \psi) = \hat{\phi}\hat{\psi}$$

so proving the convolution is in $\mathcal{S}(\mathbb{R}^n)$. We can see this more directly, without using the Fourier transform, by using the triangle equality in the form

$$|x| \le |x-y|+|y| \Longrightarrow (1+|x|) \le (1+|x-y|)(1+|y|) \Longrightarrow (1+|x-y|)^{-1}(1+|y|)^{-1} \le (1+|x|)^{-1}$$

since this shows that the integrand in (7) satisfies

$$|\phi(x-y)\psi(y)| \le C_{2N}(1+|x-y|)^{-2N}(1+|y|)^{-2N} \le C_N(1+|x|)^{-N}(1+|y|)^{-N} \tag{8}$$

so not only is the integrand rapidly decreasing but the result is rapidly decreasing in x. From convergence of the difference quotients (or the formula in terms of Fourier transform) it also follows that

$$\partial_{x_j}(\phi * \psi) = (\partial_{x_j}\phi) * \psi. \tag{9}$$

(2) We can also estimate the support of the convolution. Suppose that say $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ has compact support then the integral will vanish if $\phi(x-y)\psi(y)) \equiv 0$. The product of two smooth functions vanish if they have disjoint supports and

$$\begin{gathered}\operatorname{supp}(\phi(x-\cdot)) = \{y \in \mathbb{R}^n; \exists z \in \operatorname{supp}(\phi), x = z + y\} \Longrightarrow \\ \operatorname{supp}(\phi(x-\cdot)\psi(\cdot)) \subset \{y \in \operatorname{supp}(\psi); \exists\ z \in \operatorname{supp}(\phi),\ x = z + y\}\end{gathered} \tag{10}$$

This set is usually written $\operatorname{supp}(\phi) + \operatorname{supp}(\psi)$, the set of sums of pairs of elements of these sets. So

$$\operatorname{supp}(\phi * \psi) \subset \operatorname{supp}(\phi) + \operatorname{supp}(\psi) \text{ if } \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{11}$$

Here the real statement is that if x is outside this set then the product vanishes for all x in a neighbourhood, i.e. using the fact that the sum is closed. If neither of the elements has compact support the argument does not work because this uniformity fails, and in particular the sum of two closed sets need not be closed.

Exercise 2. Check that if $C_1 = \{(x,y) \in \mathbb{R}^2; x, y \geq 0,\ xy \geq 1\}$ and $C_2 = \{(x,y) \in \mathbb{R}^2; x \leq 0,\ y \geq 0, xy \leq -1\}$ then $C_1 + C_2$ is not closed.

(3) The existence of $\phi * \psi$ does not depend on the Fourier transform at all, nor do the identities (9) and (11), in the sense that the same arguments show that

$$\phi * \psi \in \mathcal{C}^\infty(\mathbb{R}^n), \quad \text{if } \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ \psi \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{12}$$

since the integral still converges in the same way and the result is smooth. Moreover we can write this in weak form thinking of x as a parameter

$$\phi * \psi(x) = \psi(T_x\check{\phi}),\ \check{\phi}(y) = \phi(-y),\ T_x\mu(y) = \mu(y-x). \tag{13}$$

Then $T_x\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ depends continuously on x as an element of $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ so we can use this as a definition

$$\text{if } \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ u \in \mathcal{C}^{-\infty}(\mathbb{R}^n),\ \phi * u(x) = u(T_x\check{\phi}). \tag{14}$$

Using an argument with the difference quotient for ϕ the identity (9) still holds as does the bound (11). Thus we see that

$$\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) * \mathcal{C}^{-\infty}(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n),\ \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) * \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n). \tag{15}$$

(4) We can write (13) in an 'even weaker' form for the resulting function thought of as a distribution. Namely integrating against a compactly supported test function (and using Fubini again)

$$\phi * u(\mu) = u(\int \phi(x-\cdot)\mu(x)dx) = u(\check{\phi} * \mu). \tag{16}$$

Now, if we use (15) we to we see that this continues to make sense even if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$, $\phi \in \mathcal{C}^{-\infty}_{\mathrm{c}}(\mathbb{R}^n)$ and $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ so with only a modicum of continuity required to be checked we arrive at the important obvervations that

(17) $$\mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) * \mathcal{C}^{-\infty}(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n),\ \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) * \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n) \subset \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$$

where the identity (9) and bound (11) still hold. To prove these we can use a density argument.

Definition 1. A sequence $u_n \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\Omega)$ is said to converge weakly to $u \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\Omega)$ if $u_n(\phi) \longrightarrow u(\phi)$ for all $\phi \in \mathcal{C}^{\infty}(\Omega)$.

Changing the space of test functions there are similar definitions of weak convergence in $\mathcal{S}'(\mathbb{R}^n)$ (so $\phi \in \mathcal{S}(\mathbb{R}^n)$) and $\mathcal{C}^{-\infty}(\Omega)$ (where $\phi \in \mathcal{C}^{\infty}(\Omega)$).

The important point here is that the weak limit, if it exists, is unique and that if u_n converges weakly to u (in any of these three senses) then $\partial_j u_n$ converges weakly to $\partial_j u$. This latter statement just being that $-u_n(\partial_j\phi) \longrightarrow -u(\partial_j\phi)$.

(5)

Exercise 3. Show that the space $\mathcal{C}_{\mathrm{c}}^{\infty}(\Omega)$ is weakly dense in $\mathcal{C}_{\mathrm{c}}^{-\infty}(\Omega)$ and $\mathcal{C}^{-\infty}(\Omega)$ for any open set Ω.

Exercise 4. If $u_n \in \mathcal{C}_{\mathrm{c}}^{\infty}(\mathbb{R}^n)$ converges weakly to $u \in \mathcal{C}_{\mathrm{c}}^{-\infty}(\mathbb{R}^n)$ with $\mathrm{supp}(u_n)$ uniformly bounded then for any $v \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ $u_n * v$ converges weakly to $u * v$.

This takes care of uniqueness issues.

At the end I defined ellipticity of a polynomial and talked about elliptic regularity. This will be done next week.

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