18.155 LECTURE 23
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ABSTRACT. Elliptic operators on manifolds

BEFORE LECTURE

- Diffeomorphism invariance of Sobolev spaces
- Partitions of unity on (compact) manifolds
- Sobolev spaces on manifolds
- Linear differential operators
- Symbols and ellipticity
- Elliptic operators are Fredholm
- Laplace Beltrami operator
- deRham and Hodge Theorems

AFTER LECTURE

Three big theorems dealing with differential operators to understand!

(1)

Theorem 1 (deRham). *On any compact $\mathcal{C}^\infty$ manifold the deRham cohomology groups (i.e. linear spaces)*

$$H^k_{\mathrm{dR}}(M) = \left\{u \in \mathcal{C}^\infty(M;\Lambda^k); du = 0\right\}/d\mathcal{C}^\infty(M;\Lambda^{k-1}) \xrightarrow{\simeq} \left\{u \in \mathcal{C}^{-\infty}(M;\Lambda^k); du = 0\right\}/d\mathcal{C}^{-\infty}(M;\Lambda^{k-1}) \tag{1}$$

are naturally isomorphic with the singular (or simplicial or Čech) cohomology groups with complex coefficents.

(2)

Theorem 2 (Hodge). *On any compact $\mathcal{C}^\infty$ manifold equipped with a Riemann metric the space of harmonic forms, the null space of the Laplace-Beltrami operator*

$$H^k_{Ho}(M) = \left\{u \in \mathcal{C}^\infty(M;\Lambda^k); \varDelta u = 0\right\} \xrightarrow{\simeq} H^k_{\mathrm{dR}}(M). \tag{2}$$

(3)

Theorem 3 (Atiyah-Singer)**.** *On any compact $\mathcal{C}^\infty$ manifold there is a commutative diagram*

(3)
$$
\begin{tikzcd}
\{\textit{Elliptic (pseudo-)differential operators on } M\} \arrow[rr, "\textit{symbol data}"] \arrow[dr, "\mathrm{ind}"'] & & K^0(T^*M) \arrow[dl, "\pi_!"] \\
& \mathbb{Z} = K^0(pt) &
\end{tikzcd}
$$

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