18.155 LECTURE 21
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ABSTRACT. Substitute lecture for Richard Melrose while he's away.

Read Friedlander [1, §2.5] for more about convolutions, [1, §6.1] for a derivation of the forward fundamental solution, and [2] for alternative ways to solve the wave equation.

- Forward Fundamental Solution to the Wave Equation
- One-dimensional problem
- Uniqueness
- Review of Homogeneous Distributions
- General Existence
- Representation Formula and the Cauchy Problem
- Finite Speed of Propagation and the Strong Huygens Principle

0. Forward Fundmental Solution to the Wave Equation

Consider on $\mathbf{R}^{n+1}$ the wave operator

$$\Box = \partial_t^2 - \Delta_x$$

(observe that other authors, such as Melrose, define $\Box$ and Δ with the opposite sign convention, i.e.

$$\Box = D_t^2 - \Delta_x).$$

A *foward fundamental solution* is the unique distribution $E_+ \in \mathcal{S}'(\mathbf{R}^{n+1})$ with $\operatorname{supp} E_+ \subseteq \{t \geq |x|\} =: C$ satisfying

$$\Box E_+ = \delta(t=0)\delta(x=0).$$

1. One-dimensional problem

We will first do the case $n = 1$ to warm up. Set $u = \frac{t-x}{2}$ and $v = \frac{t+x}{2}$. Then

$$\Box E_+ = \partial_u \partial_v E_+ = \frac{1}{2}\delta(u=0)\delta(v=0).$$

We make the ansatz u, v $E_+(u,v) = F(u)G(v)$. Then we should have $F = a_1 H + c_1,\ G = a_2 H + c_2$, where $a_1, a_2, c_1, c_2 \in \mathbf{C}$, $a_1 a_2 = \frac{1}{2}$ and H is the Heaviside function

$$H(s) = \begin{cases} 1, & s \geq 0 \\ 0, & s < 0. \end{cases}$$

Changing coordinates back, we see that

$$\Box E_+ = \frac{1}{2}(H(t-x) + c_1)(H(t+x) + c_2).$$

Since we want E_+ to be supported in $t \geq |x|$, we set $c_1 = c_2 = 0$, and so

$$\Box E_+ = \frac{1}{2}H(t-x)H(t+x).$$

This is of course the usual statement that solutions to the wave equation are the superposition of two travelling waves: one travelling to the right, and one travelling to the left.

2. Uniqueness

We now treat the general case of uniqueness.

If F is any other foward fundamental solution, then

$$F = F * \delta = F * \Box E_+ = \Box F * E = \delta * E = F.$$

In order to define the covolution, we used the following lemma

Lemma 2.1. *Suppose $u, v \in C^{-\infty}(\Omega)$, and the map*

$$\operatorname{supp} u \times \operatorname{supp} v \ni (x, y) \mapsto x + y$$

is proper *in the sense that the preimage of any compact set is compact. Then $u * v \in C^{-\infty}$ is well-defined and satisfies the obvious properties.*

Remark 2.2. Being proper is equivalent to the condition that for all c, $|x + y| \leq c$ implies there is some d so that $|x|, |y| \leq d$.

Proof. If $\varphi \in C_c^\infty(\Omega)$, then we choose a compactly supported cutoff $\chi(x)\eta(y)$ such that $\chi(x)\eta(y) = 1$ if $x + y \in \operatorname{supp} \varphi$ Then set

$$u * v = (\chi u) * (\eta v),$$

where the which is well-defined since both distributions are compactly supported. In other words,

$$\langle u * v, \varphi\rangle = \langle u(x), \langle v(y), \chi(x)\eta(y)\varphi(x + y)\rangle\rangle.$$

It is easy to check that this definition does not depend on χ, η and that it satisfies all the desired properties. $\square$

3. Review of Homogeneous Distributions

Recall homogeneous distributions. We say that $u \in \mathcal{S}'(\mathbf{R}^m)$ is homogeneous of degree d on $\mathbf{R}^m$ if

$$\langle u, f_\lambda\rangle = \lambda^{m+d}\langle u, f\rangle.$$

This is motivated by changing variables if u is a homogeneous function. Observe that the δ function is homogeneous of degree $-m$, and taking a deriative of a homogeneous distribution decreases the degree of homogeneity by 1.

There are special homogeneous distributions on $\mathbf{R}$ of degee z supported in $\{x \geq 0\}$. These are defined for $\operatorname{Re}(z)$ large enough by

$$x_+^z = \begin{cases} x_+^z, & x \geq 0, \\ 0, & x < 0. \end{cases}.$$

We will be using modified distributions

$$\chi_+^z(x) = \frac{x_+^z}{\Gamma(z+1)},$$

where $\Gamma(z)$ the Gamma function defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, dt$$

for $\operatorname{Re}(z)$ large and anaytically continued to a meromorphic function using the identity

$$\Gamma(z+1) = z\Gamma(z).$$

The advantage of using the Γ function is that

$$\frac{d}{dx}\chi_+^z = \chi_+^{z-1},$$

which allows us to define χ_+^z via analytic continuation (really, we have just built the regularization into the definition).

In particular, we find that

$$\chi_+^{-k-1} = \frac{d}{dx}^{k+1} \chi_+^0 = \frac{d}{dx} H(x) = \delta^{(k)}(x).$$

In fact, repeated integration by parts shows that if m is large enough,

$$\begin{aligned}\langle \chi_+^z(x), \varphi \rangle = \frac{1}{\Gamma(z+1)} &\left(\int_0^1 x^z \left(\varphi(t) - \sum_{k=0}^{m-1} \varphi^{(k)}(0) \frac{x^k}{k!} \right) \, dx \right) \\ + &\left(\sum_{k=0}^{m-1} \frac{\varphi^{(k)}(0)}{\Gamma(k+z+2)} + \int_1^\infty x^z \varphi(z) \, dz \right).\end{aligned}$$

We also have the following identity for $\mathrm{Re}(z)$ large, and hence for all z by analytic continuation:

$$x\chi_+^z(x) = (z+1)\chi_+^{z+1}(x).$$

4. General Existence

We will try the naive thing and hope it works. Namely, we will look for E_+ to be homogeneous of order $-n+1$ and supported in C. The obvious thing to try is

$$E_+ = \chi_+^{\frac{-n+1}{2}}(t^2 - |x|^2), \tag{4.1}$$

forgetting for the moment that this is not well-defined. Applying $\Box$ and using the chain rule gives that

$$\begin{aligned}\partial_t E_+ &= 2t\chi_+^{(-n-1)/2}(t^2 - |x|^2) \\ \partial_{x_i} E_+ &= 2x_i\chi_+^{(-n-1)/2}(t^2 - |x|^2) \\ \partial_t^2 E_+ &= 4t^2\chi_+^{(-n-3)/2}(t^2 - |x|^2) + 2\chi_+^{(-n-1)/2}(t^2 - |x|^2) \\ \partial_{x_i}^2 E_+ &= 4x_i^2\chi_+^{(-n-3)/2}(t^2 - |x|^2) - 2\chi_+^{(-n-1)/2}(t^2 - |x|^2).\end{aligned}$$

So,

$$\begin{aligned}\Box E_+ &= 2(n+1)\chi_+^{(-n-1)/2}(t^2 - |z|^2) + 4(t^2 - |x|^2)\chi_+^{(-n-3)/2}(t^2 - |x|^2) \\ &= 2(n+1)\chi_+^{(-n-1)/2}(t^2 - |z|^2) + 4\frac{-n-1}{2}\chi_+^{(-n-1)/2}(t^2 - |x|^2) = 0.\end{aligned}$$

This isn't quite what we want. Observe though that the differential of $t^2 - |x|^2$ fails to be surjective at 0, and so E_+ has a singularity there, and we can't differentiate. This suggests that should we be able to define (4.1), the above formal computation should be valid outside of 0, and so $\Box E_+$ is homogeneous of degree $-n-1$ and supported at 0, so must be a multiple of $\delta(t)\delta(x)$, which is what we want.

To get around this, we take a slightly different approach. Define for $\mathrm{Re}(z)$ large enough

$$G(z)(t,x) = A(z)1_C(t,x)(t^2 - |x|^2)^z,$$

where $A(z)$ is some meromorphic function given by

$$A(z)^{-1} = \pi^{n/2-1/2}2^{2z+n}\Gamma(z + (n+1)/2)\Gamma(z+1).$$

The Γ normalization will be important for regularization. Then $G(z)$ is certainly a tempered distribution. Observe that $G(z)$ is homogeneous of degree $2z$.

The π factors are chosen so that

$$\int G(z)e^{-t} \, dtdx = 1.$$

To see this, one simply integrates in polar coordinates and uses the duplication formula

$$\Gamma(2z) = \Gamma(z)\Gamma(z+1/2)2^{2z-1}\pi^{-1/2},$$

the formula for the Beta function

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

and the surface are of S^{n-1}

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Indeed, one finds that

$$\int (t^2-|x|^2)^z e^t \, dtdx = \omega_{n-1} \int_0^\infty \int_0^t e^{-t}(t^2-r^2)^z r^{n-1} \, drdt = \frac{1}{2}\omega_{n-1} \left(\int_0^\infty e^{-t} t^{2z+n} \, dt \right) \left(\int_0^1 (1-s)^z s^{n/2-1} \, ds \right).$$

A computation similar to the formal one above shows that

$$\Box G(z) = G(z-1).$$

We use this to analytically continue $G(z)$ as a tempered distribution. Since $G(z)$ is homogeneous of degree $2z$ for $\mathrm{Re}(z)$ large, $G(z)$ extends to be homogeneous of order $2z$ for all $z \in \mathbf{C}$.

We next determine another way to write $G(z)$ which will be more convinient for computations, but which will only work as a distribution on $C_c^\infty(\mathbf{R}^{n+1} \setminus \{0\})$. The idea is to integrate over level sets of $t^2 - |x|^2$. For $\mathrm{Re}(z)$ sufficiently large

$$\begin{aligned}
\langle G(z), \varphi \rangle &= A(z) \int_0^\infty \int_0^t \int_{S^{n-1}} (t^2 - r^2)^z \varphi(t, r, \theta) r^{n-1} \, d\theta dr dt \\
&= \frac{A(z)}{2} \int_0^\infty \int_0^\infty \int_{S^{n-1}} s^z \varphi((s + r^2)^{1/2}, r, \theta))(r^2 + s)^{1/2} \, d\theta dr dt \\
&= \frac{A(z)}{2} \int_0^\infty s^z \widetilde{\varphi(s)} \, ds.
\end{aligned}$$

So, formally at least,

$$\langle G(z), \varphi \rangle = A'(z) \langle \chi_+^z, \widetilde{\varphi} \rangle,$$

where

$$A'(z)^{-1} = \pi^{1/2n-1/2} \Gamma(z + (n+1)/2) 2^{2z+n}.$$

Unfortunately $\widetilde{\varphi}$ is not necessarily smooth. However it is smooth if $\varphi \in C_c^\infty(\mathbf{R}^{n+1} \setminus \{0\})$. Thus by analytic continuation,

$$\langle G(z), \varphi \rangle = A'(z) \langle \chi_+^z, \widetilde{\varphi} \rangle.$$

We remark that this makes sense for all z since Γ has no zeroes; ultimately we will only use this for values of $\Gamma(z + (n+1)/2)$ which we already know.

Now, observe that $A'((-n-1)/2) = 0$ since Γ has a pole at -1. Thus, $\operatorname{supp} G((-n-1)/2) \subseteq \{0\}$ as a tempered. It follows that $G((-n-1)/2) = A\delta(t=0)\delta(x=0)$ by homogeneity. By analytic continuation,

$$\Box G((-n+1)/2) = G((-n-1)/2) = A\delta(t=0)\delta(x=0),$$

so $E_+ = A^{-1} G((-n+1)/2)$ is the forward fundamental solution. We may compuate A by testing against e^{-t}, which is valid since we may multiply by a cutoff of C:

$$\langle G(z), e^{-t} \rangle = 1$$

for $\mathrm{Re}(z)$ large, so by analytic continuation.

$$A = \langle G((-n+1)/2), e^{-t} \rangle = 1.$$

5. Representation Formula and the Cauchy Problem

The Cauchy problem asks, given functions u_0, u_1 on $\mathbf{R}^n$ and $f \in C^\infty([0,\infty) \times \mathbf{R}^n)$ with reasonable support in $t \geq 0$ for a distribution, for $u \in C^\infty([0,\infty) \times \mathbf{R}^n)$ such that $\Box u = f$ and $(u, \partial_t u)|_{t=0} = (u_0, u_1)$. We will also show that such a solution is unique.

First, uniqueness. This will follow from the so-called "representation" or "jumps" formula. Extend u, f to be smooth for $t < 0$ (by using Borel's lemma, for instance). We start with the formal computation

$$\begin{aligned}
u &= (uH(t)) * (\delta(t)\delta(x)) = (H(t)u) * \Box E_+ \\
&= \Box(H(t)u) * E_+ = (H(t)f) * E_+ + 2(\delta(t)\partial_t u) * E_+ + \delta'(t) u * E_+ \\
&= f * E_+ + (\delta(t)\partial_t u) * E_+ + \partial_t(\delta(t)u) * E_+
\end{aligned}$$

$$= f * E_+ + (\delta(t)\partial_t u) * E_+ + (\delta(t)u) * \partial_t E_+.$$

All computations are valid by considering the supports. Observe that $\delta(t)\partial_t u = \delta(t)u_1$ and $\delta(t)u = \delta(t)u_0$, where u_0, u_1 are arbitrary smooth extensions into $t \neq 0$. We therefore obtain the representation formula

$$u = f * E_+ + (\delta(t)u_1) * E_+ + (\delta(t)u_0) * \partial_t E_+, \tag{5.1}$$

from which uniqueness follows.

It is possible to prove exstience from this as well, but this would require interpreting $E_+(t) \in \mathcal{S}'(\mathbf{R}^n)$ for each $t \in \mathbf{R}$, which we will not do. Instead we use Borel's lemma. First we will solve the Cauchy Problem in formal power series. Write our propsective solution

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x)t^k,$$

and

$$f(x,t) = \sum_{k=0}^{\infty} f_k(x)t^k$$

as formal power series. We wish to solve for u_k. We are given u_0, u_1 and f_k, and also that

$$(\partial_t^2 - \Delta)u = f,$$

i.e. for $k \geq 0$

$$(k+2)(k+1)u_{k+2}(x) = \Delta u_k + f_k.$$

This lets us solve for smooth functions $u_{k+2}(x)$ inductively. Now, using Borel's lemma, we can find $v \in C^\infty(\mathbf{R}^{n+1})$, compactly supported in time, such that $\partial_t^k v(0,x) = k!u_k(x)$. Set $g = \Box v - f$. Then g vanishes to all orders at $t = 0$, since v solves the formal Cauchy problem. Extend g by 0 to $t \leq 0$ (this is smooth). Now define

$$u := v - g * E_+.$$

Then

$$\Box u = \Box v - \delta(t)\delta(x) * \Box g = f.$$

To show that u solves the Cauchy problem, we just need to show that $(\partial_t^k g) * E_+(t) \to 0$ as $t \to 0$. We start with a formal computation

$$(g * E_+)(0,x) = \int_0^\infty \int_{\mathbf{R}^n} g(x-y,-s)E_+(s,y)\ dyds = \int_0^\infty \int_{\mathbf{R}^n} (0)E_+(s,y)\ dyds = 0,$$

and similarly with all derivatives. There are many ways to make this formal computation rigorous; for instance we can mollify E_+ to obtain approximations η_m and notice that $g * \eta_m \to g * E_+$ locally uniformly.

6. Finite Speed of Propagation and the Strong Huygens Principle

We now derive some useful properties of the representation formula above. The first is the famous finite speed of propagation property. Informally, it states that features of the solution to the wave equation propagate outwards with speed 1. Put another way, the behaviour of a solution u to the wave equation at a point (t,x) depends only the behaviour of u_0, u_1 in a ball of radius t centred at x and the behaviour of f in a cone with vertex at (t,x) and with base on $\{0\} \times \mathbf{R}^n$. Formally, we have

Proposition 6.1 (Finite Speed of Propagation). *Suppose u solves the wave equation*

$$\Box u = f$$

with initial data

$$(u, \partial_t u)|_{t=0} = (u_0, u_1).$$

Suppose futhermore that f vanishes on the cone $C_{(x,t)} := \{(y,s) \colon |x-y| \leq t-s\}$ and u_0, u_1 vanish on $\{|y-x| \leq t\}$. Then u vanishes on the cone $\{(y,s) \colon |x-y| \leq t-s\}$.

Proof. This follows from the representation formula (5.1), and the fact that $\operatorname{supp}(u * E_+) \subseteq \operatorname{supp}(u) + C$, where C is the forward cone. Indeed, if $(y, s) \notin C_{(x,t)}$ and $(z, r) \in C$, then $(y+z, r+s) \notin C_{(x,t)}$, either, since

$$|y + z| \geq |y| - |z| \geq |y| - r \geq t - (s + r).$$

□

If $n \geq 3$ is odd then we have a much stronger version of finite speed of propagation. Not only does u at (x, t) not depend on u_0, u_1 outside the ball $\{|y - x| \leq t\}$, but u only depends on the values of u_0, u_1 on the boundary of the ball. This is the Strong Huygens Principle.

Proposition 6.2 (Strong Huygens Principle)**.** *Suppose n is odd and u solves the wave equation*

$$\Box u = f$$

with initial data

$$(u, \partial_t u)|_{t=0} = (u_0, u_1).$$

Suppose futhermore that f vanishes on the cone $C_{(x,t)} := \{(y, s) \colon |x - y| \leq t - s\}$ and u_0, u_1 vanish on $\{|y - x| = t\}$. Then u vanishes on the cone $\{(y, s) \colon |x - y| \leq t - s\}$.

Proof. This will follow from (5.1) if we can show that E_+ is supported only on the boundary of the cone C, i.e. $\operatorname{supp} E_+ \subseteq \{(y, s) \colon |y| = s\}$. For this we return to examining the representations we had for the forwards fundamental solution. Recall that

$$E_+ = G((-n + 1)/2) = A'((-n + 1)/2)\langle \chi_+^{(-n+1)/2}, \widetilde{\varphi}\rangle.$$

Observe that $((-n + 1)/2)$ is an integer, so $\chi_+^{(-n+1)/2} = \delta^{(k)}(0)$ for some k. Thus, at least when acting on $C_c^\infty(\mathbf{R}^{n+1} \setminus \{0\})$, $\operatorname{supp}(E_+) \subseteq \{|y| = s\}$. If $\varphi \in C_c^\infty(\mathbf{R}^{n+1})$ is supported away from $\{(y, s) \colon |y| = s\}$, then $\varphi \in C_c^\infty(\mathbf{R}^{n+1} \setminus \{0\})$, and so this proves that $\langle E_+, \varphi\rangle = 0$, and we have the desired support condition.

Formally we write

$$E_+ = \frac{1}{2\pi^{(n-1)/2}} \delta^{(n-3)/2}(t^2 - |x|^2).$$

□

References

[1] F. G. Friedlander, *The wave equation on a curved spacetime.* Cambridge Univertsirt Press, Cambridge, 1975.
[2] S.-J. Oh, *Lectures on the wave equation.* https://math.berkeley.edu/~sjoh/pdfs/linearWave.pdf