18.155 LECTURE 2, 13 SEPTEMBER 2016

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Abstract. Notes before and after lecture – if you have questions, ask!

Read: Notes Chapter 3 Section 3 from (3.28) and Section 4.

Before lecture

The main aim of this lecture is to prove:-

The Fourier transform is an isomorphism on Schwartz space of test functions and hence by duality is an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$.

- Make sure that you are on top of the topology of $\mathcal{S}(\mathbb{R}^n)$:

  A sequence converges in $\mathcal{S}(\mathbb{R}^n)$ iff it converges with respect to each norm $\|\cdot\|_k$. [Proof: We know that each norm is continuous, so $u_n \to u$ implies $u_n - u \to 0$ and $\|u_n - u\|_k \to 0$. Conversely, if $\|u_n - u\|_k \to 0$ for all k and $\epsilon > 0$ is given, choose k so that $2^{-k} < \epsilon/4$ and then N so that for $n > N$ $\|u - u_n\|_k < \epsilon/4$ and it follows that $d(u_n, u) < \epsilon$.]

  $F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)$ linear, is continuous iff for each j there exists $k = k(j)$ and C_k such that

$$\|F(u)\|'_j \le C_k \|u\|_k \tag{1}$$

  where $\|\cdot\|'_j$ are the norms on $\mathcal{S}(\mathbb{R}^m)$ (note that we would often use the same notation for the two sets of norms because you know which is which by what it is being evaluated on). [Proof: Use preceding result to see that this implies continuity. Consersely, given j consider the ball $d'(v, 0) < 2^{-j}/4$. This is contained in $\|v\|'_j < 1$ so the inverse image of the latter contains a ball around 0 and hence some norm ball $\|u\|_k < \delta$, $\delta > 0$ by earlier result. This gives the estimate (1).]
- From L1, the operators $\times x^\beta$ and ∂_x^α are continuous linear maps on $\mathcal{S}(\mathbb{R}^n)$.
- The Fourier transform $\mathcal{F}(u) = \hat{u}(\xi) = \int e^{-ix\cdot\xi} u(x)dx$ of $u \in L^1(\mathbb{R}^n)$.
- $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$.
- Continuity of FT
- Schwartz functions vanishing at zero
- Translations
- Inversion formula
- Fourier transform of tempered distributions

  I went through this rather quickly in lecture. Note that for ϕ, $\psi \in \mathcal{S}(\mathbb{R}^n)$ it follows by Fubini's theorem – the ability to reorder Lebesgue integrals (or the corresponding result for Riemann integrals) that

$$\int \hat{\phi}\psi = \int \phi\hat{\psi}. \tag{2}$$

So if we define, as we have done, the distribution corresponding to $\phi \in \mathcal{S}(\mathbb{R}^n)$ as

$$U_\phi(\psi) = \int \phi\psi$$

then

$$U_{\hat{\phi}}(\psi) = \int \hat{\phi}\psi = \int \phi\hat{\psi} = U_\phi(\hat{\psi}). \tag{3}$$

So if we *define*

$$\hat{u}(\psi) = u(\hat{\psi}),\ u \in \mathcal{S}'(\mathbb{R}^n),\ \psi \in \mathcal{S}(\mathbb{R}^n),\ \text{then } \hat{U}_\phi = U_{\hat{\phi}}. \tag{4}$$

You should check that this definition gives a linear bijection

$$\mathcal{S}'(\mathbb{R}^n) \ni u \longrightarrow \hat{u} \in \mathcal{S}'(\mathbb{R}^n). \tag{5}$$

When we get to the topology(ies) on $\mathcal{S}'(\mathbb{R}^n)$ you will see that it is continuous with a continuous image.

- I got to here.
- Very unlikely to get this far: Density of test functions in square-integrable functions
- Fourier transform of square-integrable functions
- Sobolev spaces

After lecture

I gave a different proof than is in the notes of the 'division result' for $\mathcal{S}(\mathbb{R}^n)$:

$$\mathcal{S}(\mathbb{R}^n) \ni \phi = \phi(0)\exp(-\frac{|x|^2}{2}) + \sum_{i=1}^n x_i\psi_i(x),\ \psi_i \in \mathcal{S}(\mathbb{R}^n). \tag{6}$$

For $n = 1$ this is straightforward since by Taylor's Theorem/FTC, for $x \neq 0$,

$$\phi(x) = \phi(0) + \int_0^x \phi'(s)ds = \phi(0) + x\int_0^1 \phi'(tx)dt \tag{7}$$

where we substitute $s = tx$. This shows that

$$\phi(x) - \phi(0) = x\mu(x),\ \mu(x) = \int_0^1 \phi'(tx)dt \in \mathcal{C}^\infty(\mathbb{R}). \tag{8}$$

Applying the same result to $\phi(0)\exp(-\frac{x^2}{2}) \in \mathcal{S}(\mathbb{R})$ and taking the difference shows that

$$\phi(x) - \phi(0)\exp(-\frac{x^2}{2}) = x\psi(x),\ \psi \in \mathcal{C}^\infty(\mathbb{R}) \tag{9}$$

and we just need to show that $\psi \in \mathcal{S}(\mathbb{R})$ (note that μ above is *not* in $\mathcal{S}(\mathbb{R})$ unless $\phi(0) = 0$.) This follows from the fact that it is uniquely determined by division:

$$\psi(x) = \frac{\tilde{\phi}(x)}{x},\ x \neq 0,\ \tilde{\phi} \in \mathcal{S}(\mathbb{R}). \tag{10}$$

Differentiating this shows that $x^k\frac{d^p\psi}{dx^p}$ is bounded in $|x| > 1$ for all k, p and combined with (9) this shows $\psi \in \mathcal{S}(\mathbb{R})$.

Now the general case is a bit tricky precisely because the ψ_i satisfying (6), for $n > 1$, are *not* unique. Still, we can make consistent choices.

For $n > 1$ proceed by induction, assuming the result to be true for $n-1$. So immediately, writing $x = (x', x_n)$ we have

$$\phi(x',0) = \phi(0)\exp(-\frac{|x'|^2}{2}) + \sum_{i=1}^{n-1} x_i\psi_i(x'). \tag{11}$$

Define $\psi_i(x) = \psi_i(x')\exp(-\frac{x_n^2}{2}) \in \mathcal{S}(\mathbb{R}^n)$ for $i = 1, \ldots, n-1$ and multiply the identity by this Gaussian factor so

$$\phi(x',0)\exp(-\frac{x_n^2}{2}) = \phi(0)\exp(-\frac{|x|^2}{2}) + \sum_{i=1}^{n-1} x_i\psi_i(x). \tag{12}$$

Subtracting the left side from $\phi(x)$ we are reduced to the case that $\phi(x',0) \equiv 0$. Then we can integrate in x_n as in the one-dimensional case, but now with parameters, and see that

$$\phi(x) - \phi(x',0)\exp(-\frac{x_n^2}{2}) = x\psi_n(x),\ \psi_n \in \mathcal{S}(\mathbb{R}) \tag{13}$$

where you should check the estimates for the last part carefully.

This completes the proof of (6).

So, now we can reduce the Fourier inversion theorem on $\mathcal{S}(\mathbb{R}^n)$ to a computation in two steps:

(1) If $\phi \in \mathcal{S}(\mathbb{R}^n)$ then

$$\int \hat{\phi} = C_n\phi(0) \tag{14}$$

for a dimension-dependent constant.

(2) If $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $y \in \mathcal{S}(\mathbb{R}^n)$ then setting $\Phi(x) = \phi(x+y) \in \mathcal{S}(\mathbb{R}^n)$ we can see directly that

(15)

$$\hat{\Phi}(\xi) = \int e^{-ix\cdot\xi}\phi(x+y)dx = e^{iy\cdot\xi}\hat{\phi}(\xi) \Longrightarrow C\phi(y) = C\Phi(0) = \int\hat{\Phi} = \int e^{iy\cdot\xi}\hat{\phi}(\xi)d\xi.$$

This is the Fourier inversion formula provied $C_n = (2\pi)^n$,

$$\phi(y) = (2\pi)^{-n}\int e^{iy\cdot\xi}\hat{\phi}(\xi) = \mathcal{G}\hat{\phi}. \tag{16}$$

Note that the inverse is equal to the Fourier tranform itself, except for the change of sign and the constant, so it is immediately clear that

$$\mathcal{G} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ \mathcal{G}\circ\mathcal{F} = \mathcal{F}\circ\mathcal{G} = \mathrm{Id}\,. \tag{17}$$

The second identity follows by changing signs and moving the constant.

(3) So it only remains to compute the constant and we can do that for one function, namely the Gaussian:-

$$\int\exp(-\frac{x^2}{2})dx = \sqrt{2\pi},\ \mathcal{F}(\exp(-\frac{x^2}{2})(\xi) = \sqrt{2\pi}\exp(-\frac{\xi^2}{2}) \Longrightarrow C = (2\pi)^n. \tag{18}$$

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