18.155 LECTURE 18
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ABSTRACT. Notes before and after lecture.

Read: Pseudodifferential operator notes

BEFORE LECTURE

- Harmonic oscillator on $\mathbb{R}^n$. Since you have done the work we can regard this as an unbounded self-adjoint operator

$$D(H) = \{u \in H^2(\mathbb{R}^n); |x|^2 u \in L^2(\mathbb{R}^n)\},\ H : D(H) \longrightarrow L^2(\mathbb{R}^n), \\ Hu = \sum_{j=1}^{n} (D_j^2 + x_j^2)u = \Delta u + |x|^2 u. \tag{1}$$

- In fact it is straightforward to describe the eigenfunctions of the harmonic oscillator 'algebraically'. I went through this in class since it is the simplest approach. What we really want is (8) below. You can get this more cheaply than I have shown here.
- We can show that H is an unbounded self-adjoint operator – I put this in the notes below. We can use the identity, now justified on the domain,

$$\|Hu\|_{L^2}^2 = \|\Delta u\|_{L^2}^2 + \||x|^2 u\|_{L^2}^2 + 2\sum_j \langle \partial_j u, |x|^2 \partial_j u\rangle + 4\sum_j \langle \partial_j u, x_j u\rangle \tag{2}$$

$$H = \sum_j (-\partial_j + x_j)(\partial_j + x_j)u + nu \Longrightarrow \langle Hu, u\rangle = \sum_j \|(\partial_j + x_j)u\|_{L^2}^2 + n\|u\|_{L^2}^2.$$

It follows that $H : D(H) \longrightarrow L^2(\mathbb{R}^n)$ is injective, from the computation of $D(A)^*$ below that it has dense range and then from (2) that it is a bijection.
- Thus $H^{-1} : L^2(\mathbb{R}^n) \longrightarrow D(H) \hookrightarrow L^2(\mathbb{R}^n)$ is bounded and self-adjoint. In fact it is compact, from the characterization of compactness in $L^2(\mathbb{R}^n)$ from a recent problem set. Thus there is an orthonormal basis of $L^2(\mathbb{R}^n)$ consisting of eigenfunctions of H^{-1}, and hence of H. These are the Hermite functions.
- Define spaces R^k by the condition that $R^0 = L^2(\mathbb{R}^n)$ and inductively $u \in R^k$ if and only if $u \in R^{k-1}$ and $Hu \in R^{k-1}$. By induction it follows that

$$u \in R^k \Longrightarrow x^\alpha D^\beta u \in L^2(\mathbb{R}^n) \text{ if } |\alpha| + |\beta| \le 2k. \tag{3}$$

Indeed we have seen this for $k = 1$ and the general case follows by a similar argument. So by Sobolev embedding

$$R^\infty = \bigcap_k R^k = \mathcal{S}(\mathbb{R}^n). \tag{4}$$

- It follows that all the eigenfunctions of H are in $\mathcal{S}(\mathbb{R}^n)$. The identities

$$(\partial_k + x_k)H = (H+2)(\partial_k + x_k) \tag{5}$$

show that the 'annihilation' operators map an eigenvector of eigenvalue λ to one with eigenvalue $\lambda - 2$. The positivity of H shows there must therefore be a 'ground state' which is annihilated by all these operators, it is the Gaussion

$$e_0 = c\exp(-|x|^2/2),\ He_0 = ne_0. \tag{6}$$

- Now all the eigenfunctions can be seen to be given by applying the 'creation operators' $(-\partial_j + x_j)$ to this Gaussian and so are parameterized by $\alpha \in \mathbb{N}_0^n$ with

$$He_\alpha = (n+2|\alpha|)e_\alpha,\ e_\alpha = p_\alpha(x)e_0,\ \|e_\alpha\|_{L^2} = 1. \tag{7}$$

- The Fourier-Bessel series for the Hermite basis of $L^2(\mathbb{R}^n)$ has the useful property that

$$u \in \mathcal{S}(\mathbb{R}^n)) \Longleftrightarrow u = \sum_\alpha \langle u, e_\alpha\rangle e_\alpha \text{ converges in } \mathcal{S}(\mathbb{R}^n). \tag{8}$$

- The Hermite basis also gives an eigenbasis for the Fourier transform itself!
- Now to pseudodifferential operators. These are supposed to include operators of the sort

$$\phi(x)b*,\ \phi \in \mathcal{S}(\mathbb{R}^n),\ \hat{b} \in S^m(\mathbb{R}^n). \tag{9}$$

I will define a space which is essentially the completion of the linear span of such operators. So, suppose we consider for m fixed

$$\mathcal{S}(\mathbb{R}^n; S^m(\mathbb{R}^n)) = \left\{a : \mathbb{R}^n \longrightarrow S^m(\mathbb{R}^n) \text{ s.t. } a \in \mathcal{C}^\infty(\mathbb{R}^{2n}),\ \|x^\beta D_x^\alpha a(x,\cdot)\|_{(k)}^{S^m} < \infty\right\}. \tag{10}$$

Here the norms on S^m are for instance

$$\|c(\xi)\|_{(k)}^{S^m} = \sup_{\xi\in\mathbb{R}^n, |\beta|\le k} (1+|\xi|)^{-m+|\beta|}|D^\beta c(\xi)|. \tag{11}$$

This is a Fréchet space (surprise, surprise) and

Lemma 1. *If $a \in \mathcal{S}(\mathbb{R}^n; S^m(\mathbb{R}^n))$ the Hermite series in the first variable*

$$a(x,\xi) = \sum_{\alpha\in\mathbb{N}_0^n} e_\alpha(x)\langle a(\cdot,\xi), e_\alpha\rangle \tag{12}$$

converges to a in $\mathcal{S}(\mathbb{R}^n; S^m(\mathbb{R}^n))$.

I have written it this way round since $\langle a(\cdot,\xi), e_\alpha\rangle \in S^m(\mathbb{R}^n)$.

- 'Quantizing' each element in the series (12) to the operator (9) where $\phi = e_\alpha(x)$ and $\hat{b} = \langle a(\cdot,\xi), e_\alpha\rangle$ gives a convergent sequence of operators $H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n)$ for any s which restrict to the same operator

$$a(x, D_x) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{13}$$

After lecture

Here are some more details on the self-adjointness and regularity of the harmonic oscillator.

A) Symmetry is clear since if $u,\ v \in d = D(H)$ then $u, v \in H^2(\mathbb{R}^n)$ and $|x|^2u$, $|x|^2v \in L^2(\mathbb{R}^n)$ so in terms of the (sesquilinear) L^2 pairing

$$\langle Hu, v\rangle = \langle \Delta u, v\rangle + \langle |x|^2u, v\rangle = \langle u, Hv\rangle. \tag{14}$$

B) The adjoint condition is not quite so obvious – set

$$D(H)^* = \left\{v \in L^2(\mathbb{R}^n); D(H) \ni u \longrightarrow \langle Du, v\rangle \text{ extends to be continuous on } L^2(\mathbb{R}^n)\right\}. \tag{15}$$

The notation comes from the idea that this is the domain of the adjoint. Since $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset D(H)$ this certainly implies that $Hv \in L^2(\mathbb{R}^n)$ in the distributional ('weak') sense. So $v \in D(H)^*$ implies that $Hv \in L^2(\mathbb{R}^n)$ and so be elliptic regularity that $u \in H^2_{\mathrm{loc}}(\mathbb{R}^n)$.

C) To go further we need some sort of approximation argument. Take $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ with $\mu(t) = 1$ in $|t| < 1$ and set

$$\chi(x) = \mu(x_1)\dots\mu(x_n),\ \chi_\epsilon(x) = \chi(\epsilon x).$$

So we know that $\chi^2_\epsilon v \in H^2(\mathbb{R}^n)$ and $\chi^2_\epsilon v \to v$ in $L^2(\mathbb{R}^n)$ as $\epsilon \downarrow 0$. So we can compute the L^2 inner product

$$\langle Hv, \chi_\epsilon v\rangle = \sum_j \left(\|\chi_\epsilon D_j v\|^2_{L^2} + \|\chi_\epsilon x_j v\|^2_{L^2}\right) + 2\sum_j \langle \chi_\epsilon \partial_j v, (\partial_j \chi_\epsilon)v\rangle. \tag{16}$$

The derivatives of $\chi_\epsilon(x)$ are uniformly bounded by $C\epsilon$ so by Cauchy-Schwartz the last sum is bounded by

$$C\epsilon \sum_j \|\chi_\epsilon D_j v\|\|v\|_{L^2}. \tag{17}$$

Now the left side of (16) converges and it follows that $\chi_\epsilon D_j v$ and $\chi_\epsilon v$ are bounded in $L^2(\mathbb{R}^n)$ (since they dominate (17)). Using weak=strong as usual, it follows that

$$v \in D(H)^* \Longrightarrow v \in H^1(\mathbb{R}^n) \text{ and } x_j v \in L^2(\mathbb{R}^n). \tag{18}$$

D) Proceeding now in the same sort of way

$$\|\chi_\epsilon Hu\|^2_{L^2} = \|\chi_\epsilon \Delta u\|^2_{L^2} + \|\chi_\epsilon |x|^2 u\|^2_{L^2} + 2\sum_j \langle \chi^2_\epsilon \partial_j u, |x|^2 \partial_j u\rangle$$
$$+ 4\sum_j \langle \chi_\epsilon \partial_j u, \chi_\epsilon x_j u\rangle + 4\sum_j \langle (\partial_j \chi_\epsilon)\partial_j u, \chi_\epsilon |x|^2 u\rangle.$$

All three terms on the first line are positive and the left side converges. The first term on the second line we already know converges by (18) and the second one is bounded by $\epsilon C\|u\|_{H^1}\|\chi|x|^2u\|_{L^2}$ so is dominated by the second term on the right. Thus it follows that Δu, $|x|^2u$ are in L^2 and hence that $D(A)^* = D(A)$ from which self-adjointness follows.

E) We can also see by integrating by parts that

$$\|\chi_\epsilon x_k D_j u\|^2_{L^2} = \langle \chi_\epsilon D^2_j u, \chi_\epsilon x^2_k\rangle - \langle \chi_\epsilon D_j u, (D_j\chi_\epsilon)x^2_k u\rangle \tag{19}$$

from which it follows that

(20)
$$u \in D(H) \Longleftrightarrow u \in H^2(\mathbb{R}^n),\ |x|^2 u \in L^2(\mathbb{R}^n),\ x_k D_j u \in L^2(\mathbb{R}^n),\ x_j u \in L^2(\mathbb{R}^n).$$

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