# 18.155 LECTURE 14
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Abstract. Notes before and then after lecture.

Read:

## Before lecture

- Spectral projection. Using the functional calculus from last time we can define what is supposed to be the part of the space corresponding to spectrum in $(-\infty, t]$ for $t \in \mathbb{R}$ and a given $A = A^* \in \mathcal{B}(H)$. Namely

$$H_t = \bigcap \{\operatorname{Ran}(\chi(A))^{\perp}; \chi \in \mathcal{C}_{\mathrm{comp}}(t, \infty)\}. \tag{1}$$

Thus $H_t$ is the intersection of closed subspaces so is itself closed and we can let $E_t$ be the orthogonal projection onto $H_t$.
- $AE(t) = E(t)A$ since $A : H_t \longrightarrow H_t$.
- The $H_t$ get 'larger' as $t$ increases since the intersection in (1) shrinks with increasing $t$, thus

$$E_{t'}E_t = E_tE_{t'} = E_t \text{ if } t' \geq t. \tag{2}$$

- From (2) it follows that $\delta E = E_{t'} - E_s$ is a projection (possibily zero) if $t' \geq s$, and that

$$\chi(A)\delta E = \delta E \text{ if } \chi(t) = 1 \text{ on } [t', s] \tag{3}$$

where $\chi$ is continuous of course. This in turn means that

$$\|(A^j - s^j)\delta E\| \leq |(t')^j - s^j| \tag{4}$$

since $(A^j - s^j)\delta(E) = (A^j - s^j)\chi(A)\delta E$ with $\chi$ as in (3) and $(A^j - s^j)\chi(A) = \chi'(A)$, $\chi'(t) = (t^j - s^j)\chi(t)$.
- It follows that for any $u \in H$, $\langle E_t u, u\rangle = \|E_t u\|^2$ is an increasing function of $t$. If you worked through 'baby Rudin' you would know about the Riemann-Stieltjes integral and see that for any continuous function $f$ the right side of

$$\langle f(A)u, u\rangle = \int f(t) d\langle E_t u, u\rangle \tag{5}$$

is defined. To prove the identity it is enough to check it for polynomials using the definition of the functional calculus on the left and Stone-Weierstrass. The linearity of the integral means that one is reduced to the case $f = z^j$ whic is to say that

$$\langle A^j u, u\rangle = \int t^j d\langle E_t u, u\rangle.$$

The right side is the limit of a sequence of Riemann sums under sufficiently fine subdivison and these are

$$\sum_j t_j^j \delta_j E,\ \delta_j E = E(t_{j+1}) - E(t_j). \tag{6}$$

- This determines $f(A)$ as a bounded operator and with a bit more work one can think of $dP_\lambda$ itself as a measure 'with values in the projections' and write (**??**) in the form
$$f(A) = \int f dP_\lambda.$$
- Polar decomposition. I think I mentioned this before. First define a 'partial isometry' to be a bounded operator $V$ on $H$ such that $V^*V$ and $VV^*$ are both orthogonal projections. It follows that $V$ maps $\text{null}(V)^\perp$ isometrically onto $\text{Ran}(V)$ which are both closed subspaces – hence the name.
- Any bounded operator $B$ can be written $B = AV$ where $A$ is non-negative and self-adjoint and $V$ is a partial isometry. Define $C = (B^*B)^{\frac{1}{2}}$ using the functional calculus. Then set
$$Wh = W(Cu) = Bu\ \forall\ h = Cu \in \text{Ran}(C),\ Wh = 0 \text{ if } h \in \text{Ran}(C)^\perp. \tag{7}$$
Note that if $h \in \text{Ran}(C)$ then $h = Cu$ where $u$ is well-defined up to addition of $u' \in \text{Nul}(C)$ but $Cu' = 0$ implies $C^2u' = B^*Bu' = 0$ which implies $Bu' = 0$. Thus $B(u + u') = Bu$ and $W$ is well-defined on the range of $C$. Moreover, $\|Wh\|^2 = (Bu, B^*u) = \|Cu\|^2$ so $V$ is norm-preserving on $\text{Ran}(C)$. It follows that $W$ extends continuously to the closure of $\text{Ran}(C)$ to a partial isometry. Thus $B = WC$. The decomposition of $B = AV$ follows from the decomposition for $B^*$ so $A = (BB^*)^{\frac{1}{2}}$.
- If $A$ is compact then so is $AP_\lambda$ for each $\lambda$ and if $\lambda < 0$ then $AP_\lambda$ is invertible on $H_\lambda^-$ which must therefore be finite-dimensional. It follows that for $\lambda < 0$ $H_\lambda^-$ is the direct sum of one-dimensional eigenspaces for $A$. (Assuming you know the finite-dimensional theorem). The same is true for $A(\text{Id} - P_\lambda)$ when $\lambda > 0$ and $(\text{Id} - P_\epsilon)(\text{Id} - P_{-\epsilon})A$ has spectrum contained in $[-\epsilon, \epsilon]$ so tends to zero in norm as $\epsilon \downarrow 0$. From this it follows that any compact self-adjoint operator on a separable Hilbert space has a complete orthonormal basis of eigenfunctions (possibily including an infinite dimensional space of null vectors).
- One nice application of the polar decomposition is to see that there is a retraction:

**Proposition 1.** *There is a norm-continuous map*

$$I : \text{GL}(H) \times [0, 1] \longrightarrow \text{GL}(H),\ I(0) = \text{Id},\ I(1) : \text{GL}(H(=) \longrightarrow \text{U}(H). \tag{8}$$

*Proof.* The polar decomposition of an invertible operator is
$$A = (AA^*)^{\frac{1}{2}} U_A$$
where both factors are continuous in norm. This is not obvious from the spectral theorem itself but can be proved using the holomorphic functional calculus.

So the deformation can be defined by

$$I(A, t) = ((1 - t)(AA^*)^{\frac{1}{2}} + t\,\text{Id})U_A. \tag{9}$$

□

- The spectral theorem can also be used to show from this that $\mathrm{U}(H)$ is connected. Namely that any $U \in \mathrm{U}(H)$ is the endpoint of a (continuous) path starting at the identity. Using the retraction above it is enough to find a path in $\mathrm{GL}(H)$.

  Now, as for any operator, $U = A + iB$ with $A$ and $B$ self-adjoint but in this case $U^* = U^{-1}$ commute with $U$ so

  $$[A - iB, A + iB] = 2i[A.B] = 0,\ U^*U = A^2 + B^2 = \mathrm{Id}\,.$$

  Now, for a unitary operator,

  $$\mathrm{Spec}(U) \subset \mathbb{S} = \{z \in \mathbb{C}; |z| = 1\}. \tag{10}$$

  Since $\|U\| = 1$ we know that the spectrum is in the closed unit disk. Certainly $z \notin \mathrm{Spec}(U)$ since $U$ is invertible. More generally if $z \neq 0$ then

  $$(U - z) = -zU(z^{-1} - U^{-1})$$

  shows from the same argument that there is no point in the spectrum with $|z| < 1$.

  Now, if $a \in [-1, 1]$ but $a \notin \mathrm{Spec}(A)$ then $a + it \notin \mathrm{Spec}(U)$ for any $t$, although the only possibilities are $t = \pm(1 - a^2)^{\frac{1}{2}}$. This follows by writing

  $$U - (a + it) = i(A - a)((A - a)^{-1}(B - t)) - i\,\mathrm{Id})$$

  Since $A$ and $B$ commute $(A-a)^{-1}(B-t))$ is self-adjoint and so both factors are invertible.

  Using this observation we can use the spectral projection of $A$ at 0, $\Pi_0$, to write $A = A_- + A_+$ where $A_- = \Pi_0 A \leq 0$ and $A_+ = (\mathrm{Id} - \Pi_0) \geq 0$. From the functional calculus it follows that any operator, such as $B$ which commutes with $A$ commutes with any function of $A$. Looking at the definition of the spaces $H_0^{\pm}$ above it follows that $B$ acts on them, maps each into itself. So in fact $B$ commutes with $\Pi_0$ and hence with $A_\pm$. The two operators

  $$U_- = A_- + iB\Pi_0,\ U_+ = A_+ + iB(\mathrm{Id} - \Pi_0),\ U = U_- + U_+ \tag{11}$$

  are unitary operators on the range of $\Pi_0$ and its orthocomplement respectively. Moreover, $1 \notin \mathrm{Spec}(U_-)$ and $-1 \notin \mathrm{Spec}(U_+)$. We can rotate $U_-$ using the family $e^{is}U_-$ for $s \in [0, \pi]$ which rotates the spectrum.

  At this point we have shown that $U$ can be connected by a path to a unitary operator which does not have $-1$ in its spectrum

  $$U = A + iB,\ -1 \notin \mathrm{Spec}(A).$$

  So look at the path

  $$B_t = ((1 - t)S + t\,\mathrm{Id}) + i((1 - t)B,\ B_t^* B_t = (((1 - t)S + t\,\mathrm{Id}))^2 + tB^2.$$

  This is invertible, hence so is $B_t$ and connects $U$ to the identity in $\mathrm{GL}(H)$ and so after applying the retraction, in $\mathrm{U}(H)$.
- Other ideals in the separable case – Hilbert-Schmidt and trace class.

- The Hilbert-Schmidt operators are those which satisfy

$$\mathrm{HS} = \{A; \sum_i \|Ae_i\|^2 < \infty\} \tag{12}$$

for one, and hence any orthonormal basis. Here we use the Fourier-Bessel expansion for any element $u = \sum_i (u, e_i)e_i$, and $\|u\|^2 = \sum_i |(u, e_i)|^2$ in terms of an orthonormal basis. Using this in (12),

$$\sum_i \|Ae_i\|^2 = \sum_{i,j} |(Ae_i, f_j)|^2 = \sum_j \|A^* f_j\|^2 \tag{13}$$

for any two orthonormal bases. Applying this twice shows that the norm

$$\|A\|^2_{\mathrm{HS}} = \sum_i \|Ae_i\|^2 \tag{14}$$

is independent of the orthonormal basis used to define it. In fact $\mathrm{HS}(H)$ is a Hilbert space with inner product

$$(A, B) = \sum_i (Ae_i, B^* e_i).$$

The Hilbert-Schmidt norm satisfies

$$\|A^*\|_{\mathrm{HS}} = \|A\|_{\mathrm{HS}},\ \|B_1 A B_2\|_{\mathrm{HS}} \le \|B_1\| \|A\|_{\mathrm{HS}} \|B_2\| \tag{15}$$

for any bounded operators $B_i$.

- An operator $T$ is of trace class if it is of the form $\sum\limits_{i=1}^{N} A_i B_i$ with $A_i$ and $B_i$ Hilbert-Schmidt. It follows that for any two orthonormal bases

$$\sum_j |(Te_j, f_j)| \le \sum_{i=1}^{N} \sum_i |(B_i e_j, A_i^* f_j)| \le \sum_i \|A_i\|_{\mathrm{HS}} \|B_i\|_{\mathrm{HS}}. \tag{16}$$

We define the trace norm to be

$$\|T\|_{\mathrm{TC}} = \sup \sum_j |(Te_j, f_j)| \tag{17}$$

with the supremum taken over pairs of orthonormal bases. Directly from the definition, if $T$ is trace class then so is $T^*$ and $ATB$ if $A$ and $B$ are bounded and

$$\|T^*\|_{\mathrm{TC}} = \|T\|_{\mathrm{TC}},\ \|ATB\|_{\mathrm{TC}} \le \|A\| \|T\|_{\mathrm{TC}} \|B\|. \tag{18}$$

- Conversely, if $T \in \mathcal{B}(H)$ is such that the trace norm is finite, then consider its polar decomposition $T = AV$. Take $f_j$ to be an orthonormal basis of eigenfunctions for $A$ where the $j \in I \subset \mathbb{N}$ correspond to non-zero eigenvalues. Then

$$(Te_j, f_j) = (Ve_j, Af_j) = \lambda_j (Ve_j, f_j) \tag{19}$$

In the polar decomposition, $V$ is an isometry onto $\mathrm{Nul}(A)^\perp$ so we can choose the orthonormal basis $e_j$ so that $Ve_j = f_j$ for $j \in I$. It follows that

$$\sum_j (Te_j, f_j) = \sum_{j \in I} \lambda_j \le \|T\|_{\mathrm{TC}}.$$

So in fact $\lambda_j^{\frac{1}{2}}$ is a sequence in $l^2$ from which it follows that $A^{\frac{1}{2}} \in \mathrm{HS}$ and hence $T = (VA^{\frac{1}{2}})A^{\frac{1}{2}}$ is the product of two Hilbert-Schmidt operators. Then

(20)
$$\|T\|_{\mathrm{TC}} = \sum_j \sigma_j^{\frac{1}{2}}$$

with $\sigma_j$ the eigenvalues (repeated with multiplicity) of $TT^*$.

- It follows easily that the trace class operators from a Banach space in which the finite rank operators are dense.
- For a trace class operator the trace

(21)
$$\mathrm{Tr}(T) = \sum_i (Te_i, e_i)$$

is a continuous linear functional with respect to the trace norm since

$$|\,\mathrm{Tr}(T)| \le \sum_i |(Te_i, e_i)| \le \|T\|_{\mathrm{Tr}}.$$

It is in fact independent of the choice of orthonormal basis but initially we can define $\mathrm{Tr}(T)$ with respect to some fixed choice of basis.

- If $T$ is trace class and $B$ is bounded then

$$\mathrm{Tr}([T, B]) = 0.$$

If $\Pi_N$ is the projection onto the span of the first $N$ elements of the orthonormal basis then $\Pi_N T \Pi_N \to T$ in trace norm. So it suffices to prove () with $T$ replace by $\Pi_N T \Pi_N$ that is to compute

$$\mathrm{Tr}([\Pi_N T \Pi_N, B]) = \sum_i \left((P_N T P_N B e_i, e_i) - (B \Pi_N T \Pi_N e_i, e_i)\right).$$

Both summands vanish if $i > N$ (since $P_N e_i = 0$ if $N > i$) so this reduces to

$$\sum_{i \le N} \left((P_N T P_N P_N B P_N e_i, e_i) - (P_N B P_N \Pi_N T \Pi_N e_i, e_i)\right)$$

which is the same computation for $N \times N$ matrices $P_N B P_N$ and $\Pi_N T \Pi_N$. Writing the matrices operators as matrices

$$\Pi_N T \Pi_N e_i = \sum_{j \le N} T_{ij} e_j,\ \Pi_N B \Pi_N e_i = \sum_{j \le N} B_{ij} e_j$$

the sum in () becomes

$$\sum_{i \le N} (T_{ij} B_{ji} - B_{ij} T_{ji}) = 0,$$

which proves ().

- Hence $\mathrm{Tr}(T) = \mathrm{Tr}(UTU^{-1})$ for any unitary (or invertible) $U$. Since any orthonormal bases $f_j$ and $e_j$ are related by a unitary operator, $Uf_j = e_j$ it follows that the formula (21) defining the trace functional is independent of the orthonormal basis used to define it.
- We define the Fredholm operators $\mathrm{Fr} \subset \mathcal{B}(H)$ to consist of those operators $F$ such that
  (1) The null space $\mathrm{Nul}(F)$ is finite-dimensional.
  (2) The range $\mathrm{Ran}(F)$ is closed.

(3) The orthocomplement of the range $\operatorname{Ran}(F)^\perp = \operatorname{Nul}(F^*)$ is finite dimensional.

- If $K \in \mathcal{K}(H)$ then $\operatorname{Id} - K$ is Fredholm. If $K$ is compact and self-adjoint this follows from the spectral theorem, since the spectrum of $\operatorname{Id} - K$ is discrete near 0 consisting of a finite dimensional null space and $\operatorname{Id} - K$ is invertible on its orthocomplement. On the unit ball in the null space of $\operatorname{Id} - K$, $u = Ku$ so it is compact and hence the null space is finite dimensional. Consider $(\operatorname{Id} - K)(\operatorname{Id} - K*) = \operatorname{Id} - K - K^* + KK^*$; so this is an isomorphism of the orthocomplement of its finite-dimensional null space to its. For the null space satisfies
$$(\operatorname{Id} - K)(\operatorname{Id} - K*)u = 0 \Longrightarrow ((\operatorname{Id} - K)(\operatorname{Id} - K*)u, u) = 0 \Longrightarrow (\operatorname{Id} - K^*)u = 0.$$
Thus $(\operatorname{Id} - K)(\operatorname{Id} - K^*)$ is an isomorphism on the closed space $\operatorname{Nul}(\operatorname{Id} - K^*)^\perp$ to itself, and hence this is the range of $\operatorname{Id} - K$.
- An operator is Fredholm if and only if it has a generalized inverse $G \in \mathcal{B}(H)$ satisfying
$$GF = \operatorname{Id} - P_1,\ FG = \operatorname{Id} - P_2 \tag{22}$$
where the $P_i$ are finite rank projections onto $\operatorname{Nul}(F)$ and $\operatorname{Nul}(F^*)$ respectively. Indeed from $F$ we can construct a reduced operator
$$\tilde{F} : \operatorname{Nul}(F)^\perp \longrightarrow \operatorname{Ran}(F)$$
which is a bijection, and so by the open mapping theorem has a bounded inverse
$$\tilde{G} : \operatorname{Ran}(F) \longrightarrow \operatorname{Nul}(F)^\perp.$$
We define $G$ by extending this as zero to $\operatorname{Nul}(F)$ and then (22) follows.
- An operator is Fredholm if and only if it has a parameterix modulo compact operators, so $F \in \mathcal{B}(H)$ is Fredholm iff there exists $Q \in \mathcal{B}(H)$ such that
$$QF = \operatorname{Id} - K_1,\ FQ = \operatorname{Id} - K_2,\ K_i \in \mathcal{K}. \tag{23}$$
This is equivalent to the existence of a parameterix modulo trace class operators and to the existence of a parametrix moduli finite rank operators.

  Clearly the generalized inverse of a Fredholm operator is a parametrix in the sense of (23). Conversely if $Q$ is such an operator then
$$\operatorname{Ran}(F) \supset \operatorname{Ran}(FQ) = \operatorname{Ran}(\operatorname{Id} - K_2),\ \operatorname{Nul}(F) \subset \operatorname{Nul}(FQ) = \operatorname{Nul}(\operatorname{Id} - K_1).$$
From the discussion above of $\operatorname{Id} - K$, the null space is contained in a finite dimensional space, so is finite dimensional. The range contains a closed subspace of finite codimension so it follows that it is the sum of a closed and of finite dimensional space so it too is closed of finite codimension.
- The Fredholm operators form an open set and if $P$ is Fredholm and $K$ is compact then $P - K$ is Fredholm.

  If $Q$ is a parametrix for $F \in \operatorname{Fr}(H)$ and $B \in \mathcal{B}(H)$ has $\|B\| < 1/\|Q\|$ then $\|QB\| < 1$ and

(24)
$$Q(F+B) = QF+QB = (\operatorname{Id} + QB) - K_1 \Longrightarrow Q_R(F+B) = \operatorname{Id} - K_1',\ K_1' = (\operatorname{Id} + QB)^{-1}K_1,\ Q_R = (\operatorname{Id} + QB)^{-1}$$

  Similarly,

(25)
$$(F+B)Q = FQ+BQ = (\operatorname{Id} + BQ) - K_2 \Longrightarrow (F+B)Q_R = \operatorname{Id} - K_2',\ K_2' = K_2(\operatorname{Id} + BQ)^{-1},\ Q_L = Q(\operatorname{Id} + BQ)^{-}$$

These are respectively a 'left' and a 'right' parametrix modulo compact operators, but it follows that either is a parametrix since using these identities

$$Q_L = Q_L(\mathrm{Id} - K_2') + Q_L K_2' = Q_L F Q_R + Q_L K_2' = Q_R - K_1' Q_R + Q_L K_2',$$

so the difference is $K_1' Q_R - Q_L K_2'$ itself compact.

- The index

$$\operatorname{ind}(P) = \dim \operatorname{Nul}(P) - \dim \operatorname{Nul}(P^*) \tag{26}$$

is locally constant on the Fredholm operators.

- One nice way to prove this is to use Calderón's formula. Namely if $Q$ is a parameterix for $P$ modulo trace class operators then

$$\operatorname{ind}(P) = \operatorname{Tr}([P, Q]). \tag{27}$$

Note the 'irony' here that if $P$ or $Q$ were trace class (which they cannot be except in the finite-dimensional case) then the trace would vanish. Really this is an 'anomaly' or 'trace-defect' formula.

To prove (27) observe that it is true if $Q$ is the generalized inverse of $P$ as in (22). Then

$$[P, Q] = PQ - QP = (\mathrm{Id} - R_2) - (\mathrm{Id} - R_1) = R_1 - R_2 \Longrightarrow \operatorname{Tr}([P, Q]) = \operatorname{Tr} P_1 - \operatorname{Tr} P_2 = \operatorname{ind}(P). \tag{28}$$

In general if $Q$ and $Q'$ are both parameterices modulo trace class errors then so is $Q_t = (1-t)Q + tQ'$ for $t \in [0, 1]$ since

$$Q_t P = (1-t)(\mathrm{Id} - R_1) + t(\mathrm{Id} - R_1') = \mathrm{Id} - ((1-t)R_1 + tR_1')$$
$$PQ_t = (1-t)(\mathrm{Id} - R_2) + t(\mathrm{Id} - R_2') = \mathrm{Id} - ((1-t)R_2 + tR_1')$$

in terms of the 'error' terms for $Q$ and $Q'$. Thus in fact $[P, Q_t] = [P, Q] + t[P, Q - Q']$. The argument in () above shows that $Q - Q'$ is trace class so $\operatorname{Tr}([P, Q - Q']) = 0$ and

$$\operatorname{Tr}([P, Q_t]) = \operatorname{Tr}([P, Q]) + t \operatorname{Tr}([P, Q - Q']) \text{ is constant.}$$

So Calderón's formula (27) holds for any parametrix modulo trace class errors.

- Calderón's formula also easily shows that

**Lemma 1.** *The product of two Fredholm operators is Fredholm with index the sum of the indexes.*

*Proof.* If $F$ and $F'$ are the Fredholm operators with generalized inverses $G$ and $G'$ then $G'G$ is a parameterix for $FF'$ modulo trace class operators:

$$\begin{aligned} G'GFF' &= G'(\mathrm{Id} - P_1)F' = \mathrm{Id} - P_1' - G'P_1F', \\ FF'G'G &= F(\mathrm{Id} - P_2')G = \mathrm{Id} - P_2 - FP_2'G \end{aligned} \tag{29}$$

where the errors are indeed trace class. Thus $FF'$ is Fredholm and using Calderón's formula

$$\begin{aligned} \operatorname{ind}(FF') &= \operatorname{Tr}(P_1' + G'P_1F' - P_2 - FP_2G) \\ &= \operatorname{Tr}(P_1' + P_1F'G' - P_2 - P_2GF) = \operatorname{Tr}(P_1' + P_1(\mathrm{Id} - P_2) - P_2 - P_2(\mathrm{Id} - P_1)) \\ &= \operatorname{Tr}(P_1' + P_1 - P_2 - P_2') = \operatorname{ind}(F) + \operatorname{ind}(F') \end{aligned}$$

where the trace identity has been used several times. ∎

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**Proposition 2.** *The index is constant on components of* Fr *and labels these components, i.e. if two Fredholm operators have the same index then they can be connected by a path in* Fr.

The constancy of the index reduces to its local constancy. So consider the proof above that the Fredholm operators are open in the bounded opeators. If we consider $F + tB$ where $F$ is Fredholm and has generalized inverse $G$ it follows from the discussion above that $Q_t = (\mathrm{Id} + tGB)^{-1}G$ is a continuous family of parameterices modulo a continuous family of errors, meaning continuous in trace class. Indeed

$$\begin{aligned} Q_t(F + tB) &= \mathrm{Id} - (\mathrm{Id} + tGB)^{-1}P_1, \\ (F + tB)Q_t &= \mathrm{Id} - P_2(\mathrm{Id} + tBG)^{-1} + (F + tB)[(\mathrm{Id} + tGB)^{-1}G - G(\mathrm{Id} + tBG)^{-1}]. \end{aligned}$$

From Calderón's formula it follows that the index is continuous, but since it is valued in the integers it is locally constant.

Suppose $F$ has index zero. Then it follows that its generalized inverse is an invertible map from $\mathrm{Nul}(F)^\perp$ to $\mathrm{Nul}(F^*)^\perp$ where the two finite-dimensional spaces have the same dimension. So if we can find a finite rank operator $B$ which maps $\mathrm{Nul}(F)$ to $\mathrm{Nul}(F^*)$ and as such is invertible. Then $F + tB$ is invertible for $t \neq 0$. Thus the Fredholm operators or index zero may be path connected to elements of $\mathrm{GL}(H)$. This is shown somewhere else to be connected.

If $F$ and $F'$ are both Fredholm with index $k \neq 0$ and $G$ is the generalized inverse of $F$ then $G$ is Fredholm of index $-k$ and $GF'$ is Fredholm of index 0. Thus, assuming the case of index 0, $GF'$ can be connected to the identity through Fredholm operators of index 0 by a continuous path $B_t$, $B_0 = GF'$, $B_1 = \mathrm{Id}$. Now $FB_t$ is a path from $FGF'$ to $F$ in the Fredholms of index $k$ and $FGF' = F' - P_2F'$ can be connected to $F'$ by the path $F' - sP_2F'$ since $P_2F'$ has finite rank.

## After lecture

I added the discussion on polar decomposition, and expanded the parts on Hilbert-Schmidt, trace class operators and Fredholm operators.

You might also like to check the proof that two-sided ideal containing an non-compact operator much be the whole of $\mathcal{B}(H)$. Let $A$ be a non-compact operator in the ideal, then so is $A^*A$. It is certainly in the ideal and if it were compact then restricted to the orthonomal complement $W$ of a sufficiently large finite dimensional space (spanned by eigenvalues) it has small norm

$$(\pi_W A^* A \pi_W u, u) \le \epsilon^2 \Longrightarrow \|A\pi_W\| < \epsilon,$$

But this shows that $A = A\Pi_W + A(\mathrm{Id} - \Pi_W)$ can be norm approximated by finite rank operators.

Since $A^*A$ is not compact it must be the case that the spectral projection $P$ for $[\epsilon^2, \|A\|^2]$ has infinite rank for some $\epsilon > 0$ – this is the same argument. But then $PA^*AP$ is invertible on the range of $A$. Composing with the inverse it follows that $P$ itself is in the ideal. The orthocomplement of $P$ is either finite or infinite dimensional. In any case there is a partial isometry from a subspace of $\mathrm{Ran}(P)$ to $\mathrm{Ran}(P)^\perp$ and this is of the form $BP$ so is in the ideal. But then $BPPB^*$ is also in the ideal and this is the projection onto $\mathrm{Ran}(P)^\perp$. Adding these the identity is in the ideal.

You might also like to observe that all these spaces $\mathcal{R}$, $\mathcal{K}$, HS, TC and Fr have the property that the are mapped into themselves by composition with an element of $\mathrm{GL}(H)$ on either side. This means that if $H_1$ and $H_2$ are two separable Hilbert spaces then we have perfectly reasonable definitionts of

$$(30)\quad \mathcal{R}(H_1,H_2),\ \mathcal{K}(H_1,H_2),\ \mathrm{HS}(H_1,H_2),\ \mathrm{TC}(H_1,H_2),\ \mathrm{Fr}(H_1,H_2)\subset\mathcal{B}(H_1,H_2).$$

Namely chose an isomorphism between $H_1$ and $H_2$ and use composition with this to identify each of the spaces with the corresponding one on $H_2$. The result does not depend on the choice and the first four are two-sided modules – are closed under composition on the right or left with bounded operators on the appropriate space. The Fredholm operators are closed under composition with Fredholm operators on left or right. The index of a Fredholm operator makes sense in the case but there is no trace functional.

### 0.1. **Semi-Fredholm opeators.**

I also said but did not prove that the semi-Fredholm operators are dense in all bounded operators. An operator is semi-Fredholm if it satisfies the closed range condition of and one of the two other conditions, so *either* the null space is finite dimensional *or* the null space of the adjoint is finite dimensional. Despite what I may have said at one point in the lecture

**Lemma 2.** *The semi-Fredholm operators are open and dense in $\mathcal{B}(H)$.*

*Proof.* The polar decomposition shows that $A=(AA^*)^{\frac12}V$ where $V$ is a partial isometry and $(AA^*)^{\frac12}$ is non-negative and on the range of $V$. So we can add $\epsilon P_{\mathrm{Ran}(V)}$ to make it an invertible map from one closed subspace of $H$ to another. Now, if one of the orthicomplements of these subspaces is finite dimensional then we have a semi-Fredholm operator. If they are both infinite dimensional then there is an isomorphism between them, adding a small multiple of this gives an (overall) small perturbation in norm to an invertible operator. Thus the semi-Fredholm operators are dense.

Now to see that they are open consider an operator $F$ with finite dimensional null space and closed range. Then $F^*F$ is self-adjoint and has finite dimensional null space, it is also Fredholm.

□

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