18.155 LECTURE 13
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Abstract. Notes before then and after lecture.

Read: Notes Chapter 2, section 2.

1. Before lecture

- Spectrum and resolvent set of a bounded operator.
- Resolvent set is open and the resolvent is holomorphic.
- Spectrum of a compact operator
- Self-adjoint operators have real spectrum
- If $A^* = A$ then $\|A\| = \sup_{\|u\|=1} |\langle Au, u\rangle|$.
- If $A^* = A$ then

$$\{\alpha\} \cup \{\beta\} \subset \operatorname{Spec}(A) \subset [\alpha, \beta]$$
$$\alpha = \inf_{\|u\|=1} \langle Au, u\rangle,\ \beta = \sup_{\|u\|=1} \langle Au, u\rangle.$$

  Proof: Replace A by $A - \frac{1}{2}(\alpha + \beta)$ which is self-adjoint and has $\beta = -\alpha = \|A\|$. There is a sequence u_n, $\|u_n\| = 1$, $\langle Au_n, u_n\rangle \to -\|A\|$. Then
  $$\|(A + \|A\| \operatorname{Id})u_n\| = \|Au_n\|^2 + 2\|A\|(Au_n, u_n) + \|A\|^2\|u_n\| \to 0$$
  which implies that $(A + \|A\|)^{-1}$ cannot exist and similarly for $A - \|A\|$.
- Functional calculus via Stone-Weierstrass
- Polar decomposition
- I got to about here
- Spectral projection and measure. Riemann-Stieltjes
- Spectral decomposition for a compact self-adjoint operator
- Hilbert-Schmidt and trace class operators.

2. After lecture

I think I skipped a bit of the proof of the polar decomposition of a general bounded operator $B \in \mathcal{B}(H)$ which says

$$B = UA,\ A = A^* \ge 0,\ U^*U = \operatorname{Id} - \Pi_{(\operatorname{Ran} B)^\perp},\ UU^* = \Pi_{(\operatorname{Ran} A)^\perp} \tag{1}$$

so U is a 'partial isometry', which is to say an inner-product preserving linear map in this case between $\overline{\operatorname{Ran}(A)}$ and $\overline{\operatorname{Ran}(B)}$ which is zero on $\operatorname{Ran}(A)^\perp$.

The main step is to define $A = (B^*B)^{\frac{1}{2}}$ using the fact that $\operatorname{Spec}(B^*B) \subset [0, \infty)$ on which $z^{\frac{1}{2}}$ is a continuous function. Then define $U : \operatorname{Ran}(A) \longrightarrow \operatorname{Ran}(B)$ by

$$Ug = Bf \text{ if } g = Af. \tag{2}$$

We first need to check that this makes sense, of course if $g \in \operatorname{Ran}(A)$ then f exists but the problem is that it may not be unique. However if f' is another choice then

$$A(f'-f)=0 \Longrightarrow (B^*B)^{\frac{1}{2}}(f'-f)=0 \\ \Longrightarrow (B^*B)(f'-f)=0 \Longrightarrow \|B(f'-f)\|=0 \Longrightarrow Bf'=Bf$$

so U is well-defined. Also for $g \in \operatorname{Ran}(A)$,

$$\|Ug\|^2 = \langle Bf, Bf\rangle = \langle B^*Bf, f\rangle = \|Af\|^2$$

so U is norm-preserving from $\operatorname{Ran}(A)$ to $\operatorname{Ran}(B)$. It can then be extended by continuity to $\overline{\operatorname{Ran}(A)}$ and defined to be zero on $\operatorname{Ran}(A)^{\perp}$. It follows that U is a partial isometry and that $B = UA$.

If $B \in \operatorname{GL}(H)$ is invertible then B^*B has spectrum in $(0,\infty)$ and the same is true of A which is therefore invertible and U is actually unitary since $\operatorname{Ran}(A) = \operatorname{Ran}(B) = H$. This allows one to show that $\operatorname{GL}(H)$ is connected to $\operatorname{U}(H)$ by the curve

$$B_t = U(t(B^*B)^{\frac{1}{2}} + (1-t)\operatorname{Id}),\ t \in [0,1],\ B_1 = B,\ B_0 = U \tag{3}$$

lying in $\operatorname{GL}(H)$. In fact this gives a retraction $\operatorname{GL}(H) \ni B \longrightarrow U = B(B^*B)^{-\frac{1}{2}}$ continuous in the norm topology. To see this we need to show that $B \longmapsto (B^*B)^{\frac{1}{2}}$ is continuous in the norm topology. That is most easily done using the holomorphic functional calculus which I will mention next time.

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