18.155 LECTURE 1, 8 SEPTEMBER 2016

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Abstract. Notes before and and then after lecture – if you have questions, ask!

Read: Notes Chapter 3, Sections 1 and 2.

1. Questions for you

- What do you hope to learn from this course?
- Do you know about $L^2(\mathbb{R}^n)$ and Riesz' Representation Theorem? [I do really plan to assume you know this.]
- Do you know about for instance $\mathcal{C}^0(\mathbb{B}^n)$ – continuous functions on the closed unit ball in $\mathbb{R}^n$ – as a Banach space and its dual as a space of measures, i.e. the 'other' Riesz' Representation Theorem. [I will not really use this.]
- Do you know the basic properties of smooth manifolds? [This is for the second half of the course.]

2. Intended lecture content

- Main aim today: Schwartz space of test functions.
- Continuous functions on open subsets of $\mathbb{R}^n$.
- Differentiable functions.
- Higer partial derivatives, symmetry, multli-index notation.
- Infinitely differentiable functions.
- The Gaussian.
- Schwartz' space, differential operators with polynomial coefficients.
- Norms and seminorms.
- Fréchet spaces, metrics.
- Tempered distributions defined.
- Continuous linear maps and functionals.
- Fourier transform – described only (and only maybe).

3. My notes

What I plan to cover.

(1) Functions and distributions, including Fourier transform, Sobolev spaces, conormal distributions, kernels.
(2) Operators on Hilbert space, spectral theorem; Fredholm, compact, unitary, trace class.
(3) Differential operators with constant coefficients. Fundamental solutions, elliptic regularity, hyperbolic operators.

Question: What do you know about?

Here is an outline of what I think you should understand at the beginning of this course. Note that if you are missing any of this the best idea, assuming you want to continue with the course, is to come and speak with me about how to 'pick it up'. The nice thing about measure and integration for instance is that you can fake it quite effectively!

(1) Metric spaces, continuity.
(2) Linear spaces, norms, completeness (Banach spaces), duals. Hilbert space, Riesz' representation theorem
(3) Continuous functions on metric spaces, including $\mathbb{R}^n$, supremum norm, measures as functionals – we do not really need this.
(4) Lebesgue integrability, measurable sets, completeness of L^2, local integrability.

Where I will start: Test functions, weak definitions and duality.
What to make sure you take away from today.

> Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the space of rapidly decreasing test functions on $\mathbb{R}^n$, is a Fréchet (and a Montel) space, so a complete metric space, with countably many norms; its dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered (also temperate, Schwartz) distributions on $\mathbb{R}^n$ consisting of the linear function(al)s continuous with respect to one of the norms.

Are there questions at this stage?
To work:

(1) The basic notion of a k times continuous differentiable function on an open subset $\Omega \subset \mathbb{R}^n$.

So the notion of a k times continuously differentiable function is defined iteratively starting from a continuous function $f : \Omega \longrightarrow \mathbb{C}$. This is continuously differentiable if all the difference quotients at all points exist so for each $j = 1, \dots\ n$, and all $x \in \Omega$,

$$\lim_{h\to 0} \frac{f(x + he_j) - f(x)}{h} = \partial_j f(x) \tag{1}$$

exists, so defining the right side, which is also required to be define a continuous function $\partial_j f : \Omega \longrightarrow \mathbb{C}$. Then a function is k times continuously differentiable for $k > 1$ if it is continuously differentiable and its partial derivatives are all $k - 1$ times continously differentiable.

Perhaps you should remind yourself of the proof that if f is twice continuously differentiable then $\partial_i(\partial_j f)(x) = \partial_j(\partial_i f)(x)$ for all $x \in \Omega$. This can be seen by applying the 1-d mean value theorem (carefully) to double difference quotient

$$\begin{gathered} D(s,t,x) = \frac{1}{st}\left[F(x + se_i) - F(x)\right] = \frac{1}{st}\left[G(x + te_j) - F(x)\right], \\ F(x) = f(x + te_j) - f(x),\ G(x) = f(x + se_i) - f(x). \end{gathered} \tag{2}$$

to prove that it has a limit as $s, t \to 0$.

(2) Then a function is inifinitely differentiable if it is k times continuously differentiable for all k.
(3) We want to make sure there are some, other than polynomials. Check that

$$f(x) = p(x) \exp(-|x|^2) \tag{3}$$

is ininitely diffferentiable for any polynomial p. We want to *really prove* things. Here a direct way is to prove by induction that $f(x)$ is continuously differentiable and that

$$\partial_i\left(p(x)\exp(-|x|^2)\right) = (\partial_i p - 2x_i p)\exp(-|x|^2) \tag{4}$$

is of the same form.

(4) Schwartz' space $\mathcal{S}(\mathbb{R}^n)$ consists of all the infinitely differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that

$$\sup_{\mathbb{R}^n} |q(x)\partial^\alpha f(x)| < \infty \tag{5}$$

for all polynomials q – the space of infinitely differentiable functions all of whose derivatives decary rapidly at infinity.

(5) We need topology so we consider all the seminorms

$$\sup_{\mathbb{R}^n} |x^\beta \partial^\alpha f(x)| \text{ or better still the norms } \|f\|_{(k)} = \sum_{|\alpha|\le k, |\beta|\le k} \sup_{\mathbb{R}^n} |x^\beta \partial^\alpha f(x)|. \tag{6}$$

(6) The topology on $\mathcal{S}(\mathbb{R}^n)$ is the weakest topology with respect to which all these seminorms are continuous. This is actually a metric topology

$$d(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|f-g\|_{(k)}}{1+\|f-g\|_{(k)}} \tag{7}$$

with respect to which $\mathcal{S}(\mathbb{R}^n)$ is complete.

(7) A tempered distribution is a continuous linear functional

$$u : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}. \tag{8}$$

(8) The topology is such that a linear map (8) is continous iff there exists a k and a constant such that

$$|u(f)| \le C\|f\|_{(k)} \ \forall\ f \in \mathcal{S}(\mathbb{R}^n). \tag{9}$$

The linear space of all such tempered distributions is denoted $\mathcal{S}'(\mathbb{R}^n)$.

(9) Now, we just have to understand why and what it does for us. The important idea is an extension of duality for $L^2(\mathbb{R}^n)$, which is why I brought this up earlier. Namely, let's consider the space of locally integrable 'functions' of polynomial growth in the sense that

$$\{v \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and } \exists\ C, k \text{ s.t. } \int_{|z|<R} |v| \le C(1+R)^k\}. \tag{10}$$

For such a function the product

$$v(x)f(x) \in L^1(\mathbb{R}^n) \text{ if } f \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow U_v(f) = \int_{\mathbb{R}^n} vf \text{ defines } U_v \in \mathcal{S}'(\mathbb{R}^n). \tag{11}$$

See if you can prove this.

(10) In particular we have a map

$$\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n),\ \phi \longmapsto U_\phi,\ u_\phi(\psi) = \int_{\mathbb{R}^n} \phi\psi,\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{12}$$

This is an *injection* because it is linear and $U_\phi = 0$ means $\int \phi\psi = 0$ for all $\psi \in \mathcal{S}(\mathbb{R}^n)$ so taking $\psi = \overline{\phi}$ it follows that $\int |\phi|^2 = 0$ which implies $\phi \equiv 0$ (since it is continuous).

(11) This is how we define operations on distributions. We first define them on $\mathcal{S}(\mathbb{R}^n)$ and then transfer them to $\mathcal{S}'(\mathbb{R}^n)$. The most important of these operations is differentiation

$$\partial_j : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ \phi \in \mathcal{S}(\mathbb{R}^n) \Longrightarrow U_{\partial_j \phi}(\psi) = \int \frac{\partial \phi}{\partial x_j}\psi = -\int \phi \frac{\partial \psi}{\partial x_j} \tag{13}$$

where we have used integration by parts (and there are no boundary terms since everything is rapidly decaying at infinity. The last version here makes sense for any tempered distribution so we *define*

$$\partial_j : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n),\ (\partial_j u)(\psi) = -u(\partial_j \psi)\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{14}$$

This is then consistent with the standard meaning of differentiation on $\mathcal{S}(\mathbb{R}^n)$ – write a little commutative diagram!

4. After lecture

I really only covered as far as the metric on a Fréchet space but did not really do completeness.

To make sure that you understand the topology of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ it would be a good idea for you to flesh-out the following abbreviated proof of the estimate (which I stated but did not prove in class) that for some k,

$$F : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C},\ |F(u)| \le C\|u\|_k \tag{1}$$

is equivalent to continuity for a linear map from Schwartz space to the complex numbers. In words, a linear map is continuous if and only if it is bounded with respect to one of the norms.

Continuity of a map on a metric space is equivalent to the inverse image of each open set being open. For a linear map this reduces to

$$F^{-1}(\{|z| < 1, z \in \mathbb{C}\}) \supset \{u \in \mathcal{S}(\mathbb{R}^n); d(u,0) < \epsilon\} \text{ for some } \epsilon > 0.$$

That is, there is an open ball around the origin in $\mathcal{S}(\mathbb{R}^n)$ with image contained in the unit open ball in $\mathbb{C}$. This follows from the translation-invariance of the metric and the homogeneity of a linear map. So, for a continuous linear map there is such a ball of radius $\epsilon > 0$. Choose k so large that $2^{-k} < \epsilon/2$. Recall that the distance from the origin is

$$d(u,0) = \sum_{j=0}^{\infty} 2^{-j} \frac{\|u\|_j}{1+\|u\|_j} = \sum_{j\le k}^{\infty} 2^{-j} \frac{\|u\|_j}{1+\|u\|_j} + \sum_{j>k}^{\infty} 2^{-j} \frac{\|u\|_j}{1+\|u\|_j} \le 2\|u\|_k + \epsilon/2.$$

Here I have used the fact that the norms increase as j increases. It follows that if $\|u\|_k < \epsilon/4$ then $|F(u)| < 1$ from which the desired estimate follows by homogeneity with C determined by ϵ.

The converse follows from the continuity of the norms which I did do in class.

Now, you might want to do the more general case (you can do it for arbitrary Fréchet spaces if you want) that a linear map $L : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)$ is continuouse if and only if for each $l \in \mathbb{N}_0$ there exists C and k (depending on l) such that

$$\|L\phi\|_{(l)} \le C\|\phi\|_{(k)}. \tag{2}$$

Note that this is really an 'asymptotic' statement in the sense that once you know it for one l is automatically true for *smaller* l. Still, there are infinitely many estimates here.

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