# Differential Analysis
# Lecture notes for 18.155 and 156

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# Contents

## Introduction

These notes are for the graduate analysis courses (18.155 and 18.156) at MIT. They are based on various earlier similar courses. In giving the lectures I usually cut many corners!

To thank:- Austin Frakt, Philip Dorrell, Jacob Bernstein....

CHAPTER 1

# Measure and Integration

A rather quick review of measure and integration.

## 1. Continuous functions

A the beginning I want to remind you of things I think you already know and then go on to show the direction the course will be taking. Let me first try to set the context.

One basic notion I assume you are reasonably familiar with is that of a *metric space* ([**6**] p.9). This consists of a set, $X$, and a distance function

$$d : X \times X = X^2 \longrightarrow [0, \infty)\,,$$

satisfying the following three axioms:

$$\begin{aligned} &i)\ d(x,y) = 0 \Leftrightarrow x = y,\ (\text{and } d(x,y) \geq 0)\\ &ii)\ d(x,y) = d(y,x)\ \forall\ x,y \in X\\ &iii)\ d(x,y) \leq d(x,z) + d(z,y)\ \forall\ x,y,z \in X. \end{aligned} \tag{1.1}$$

The basic theory of metric spaces deals with properties of subsets (open, closed, compact, connected), sequences (convergent, Cauchy) and maps (continuous) and the relationship between these notions. Let me just remind you of one such result.

Proposition 1.1. *A map $f : X \to Y$ between metric spaces is continuous if and only if one of the three following equivalent conditions holds*

(1) $f^{-1}(O) \subset X$ *is open* $\forall\ O \subset Y$ *open.*
(2) $f^{-1}(C) \subset X$ *is closed* $\forall\ C \subset Y$ *closed.*
(3) $\lim_{n\to\infty} f(x_n) = f(x)$ *in* $Y$ *if* $x_n \to x$ *in* $X$.

The basic example of a metric space is Euclidean space. Real $n$-dimensional Euclidean space, $\mathbb{R}^n$, is the set of ordered $n$-tuples of real numbers

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n\,,\ x_j \in \mathbb{R}\,,\ j = 1, \ldots, n\,.$$

It is also the basic example of a vector (or linear) space with the operations

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$cx = (cx_1, \ldots, cx_n)\,.$$

The metric is usually taken to be given by the Euclidean metric

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2} = (\sum_{j=1}^{n} x_j^2)^{1/2}\,,$$

in the sense that

$$d(x, y) = |x - y|\ .$$

Let us abstract this immediately to the notion of a normed vector space, or normed space. This is a vector space $V$ (over $\mathbb{R}$ or $\mathbb{C}$) equipped with a *norm*, which is to say a function

$$\|\ \| : V \longrightarrow [0, \infty)$$

satisfying

$$\begin{aligned} i)&\ \|v\| = 0 \Longleftrightarrow v = 0,\\ ii)&\ \|cv\| = |c|\,\|v\|\ \forall\ c \in \mathbb{K},\\ iii)&\ \|v + w\| \le \|v\| + \|w\|. \end{aligned} \tag{1.2}$$

This means that $(V, d)$, $d(v, w) = \|v - w\|$ is a vector space; I am also using $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$ as is appropriate.

The case of a finite dimensional normed space is not very interesting because, apart from the dimension, they are all "the same". We shall say (in general) that two norms $\|\bullet\|_1$ and $\|\bullet\|_2$ on $V$ are *equivalent* of there exists $C > 0$ such that

$$\frac{1}{C}\|v\|_1 \le \|v\|_2 \le C\|v\|_1\ \forall\ v \in V\,.$$

Proposition 1.2. *Any two norms on a finite dimensional vector space are equivalent.*

So, we are mainly interested in the infinite dimensional case. I will start the course, in a slightly unorthodox manner, by concentrating on one such normed space (really one class). Let $X$ be a metric space. The case of a continuous function, $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is a special case of Proposition 1.1 above. We then define

$$C(X) = \{f : X \to \mathbb{R},\ f \text{ bounded and continuous}\}\,.$$

In fact the same notation is generally used for the space of complex-valued functions. If we want to distinguish between these two possibilities we can use the more pedantic notation $C(X; \mathbb{R})$ and $C(X; \mathbb{C})$.

Now, the 'obvious' norm on this linear space is the supremum (or 'uniform') norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \,.$$

Here $X$ is an arbitrary metric space. For the moment $X$ is supposed to be a "physical" space, something like $\mathbb{R}^n$. Corresponding to the finite-dimensionality of $\mathbb{R}^n$ we often assume (or demand) that $X$ is *locally compact*. This just means that every point has a compact neighborhood, i.e., is in the interior of a compact set. Whether locally compact or not we can consider

$$\mathcal{C}_0(X) = \left\{ f \in \mathcal{C}(X);\, \forall\ \epsilon > 0\ \exists\ K \Subset X s.t. \sup_{x\notin K} |f(x)| \le \epsilon \right\} . \tag{1.3}$$

Here the notation $K \Subset X$ means '$K$ is a compact subset of $X$'.

If $V$ is a normed linear space we are particularly interested in the continuous linear functionals on $V$. Here 'functional' just means function but $V$ is allowed to be 'large' (not like $\mathbb{R}^n$) so 'functional' is used for historical reasons.

Proposition 1.3. *The following are equivalent conditions on a linear functional $u : V \longrightarrow \mathbb{R}$ on a normed space $V$.*

1. *$u$ is continuous.*
2. *$u$ is continuous at* $0$.
3. $\{u(f) \in \mathbb{R}\,;\ f \in V\,,\ \|f\| \le 1\}$ *is bounded.*
4. $\exists\ C$ *s.t.* $|u(f)| \le C\|f\|\ \forall\ f \in V$.

Proof. (1) $\Longrightarrow$ (2) by definition. Then (2) implies that $u^{-1}(-1,1)$ is a neighborhood of $0 \in V$, so for some $\epsilon > 0$, $u(\{f \in V; \|f\| < \epsilon\}) \subset (-1,1)$. By linearity of $u$, $u(\{f \in V; \|f\| < 1\}) \subset (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ is bounded, so (2) $\Longrightarrow$ (3). Then (3) implies that

$$|u(f)| \le C\ \forall\ f \in V, \|f\| \le 1$$

for some $C$. Again using linearity of $u$, if $f \ne 0$,

$$|u(f)| \le \|f\| u\left(\frac{f}{\|f\|}\right) \le C\|f\| \,,$$

giving (4). Finally, assuming (4),

$$|u(f) - u(g)| = |u(f-g)| \le C\|f-g\|$$

shows that $u$ is continuous at any point $g \in V$. □

In view of this identification, continuous linear functionals are often said to be *bounded*. One of the important ideas that we shall exploit later is that of 'duality'. In particular this suggests that it is a good idea to examine the totality of bounded linear functionals on $V$. The *dual* space is

$$V' = V^* = \{u : V \longrightarrow \mathbb{K},\ \text{linear and bounded}\}.$$

This is also a normed linear space where the linear operations are

$$\begin{aligned}(u+v)(f) &= u(f) + v(f) \\ (cu)(f) &= c(u(f))\end{aligned} \ \forall\ f \in V. \tag{1.4}$$

The natural norm on $V'$ is

$$\|u\| = \sup_{\|f\| \le 1} |u(f)|.$$

This is just the 'best constant' in the boundedness estimate,

$$\|u\| = \inf \{C; |u(f)| \le C\|f\| \ \forall\ f \subset V\}\ .$$

One of the basic questions I wish to pursue in the first part of the course is: What is the dual of $\mathcal{C}_0(X)$ for a locally compact metric space $X$? The answer is given by Riesz' representation theorem, in terms of (Borel) measures.

Let me give you a vague picture of 'regularity of functions' which is what this course is about, even though I have not introduced most of these spaces yet. Smooth functions (and small spaces) are towards the top. Duality flips up and down and as we shall see $L^2$, the space of Lebesgue square-integrable functions, is generally 'in the middle'. What I will discuss first is the right side of the diagramme, where we have the space of continuous functions on $\mathbb{R}^n$ which vanish at infinity and its dual space, $M_{\text{fin}}(\mathbb{R}^n)$, the space of finite Borel measures. There are many other spaces that you may encounter, here I only include test functions, Schwartz functions, Sobolev spaces and their duals; $k$ is a general positive integer.

$$
\begin{tikzcd}
\mathcal{S}(\mathbb{R}^n) \arrow[d, hook] \arrow[drr, hook] & & \\
H^k(\mathbb{R}^n) \arrow[d, hook] & \mathcal{C}_c(\mathbb{R}^n) \arrow[r, hook] \arrow[dl, hook] \arrow[dd, hook] & \mathcal{C}_0(\mathbb{R}^n) \\
L^2(\mathbb{R}^b) \arrow[d, hook] \arrow[dr, hook] & & \\
H^{-k}(\mathbb{R}^n) \arrow[d, hook] & M(\mathbb{R}^n) & M_{\text{fin}}(\mathbb{R}^n) \arrow[l, hook'] \arrow[dll, hook'] \\
\mathcal{S}'(\mathbb{R}^n). & &
\end{tikzcd}
\tag{1.5}
$$

I have set the goal of understanding the dual space $M_{\text{fin}}(\mathbb{R}^n)$ of $\mathcal{C}_0(X)$, where $X$ is a locally compact metric space. This will force me to go through the elements of measure theory and Lebesgue integration. It does require a little forcing!

The basic case of interest is $\mathbb{R}^n$. Then an obvious example of a continuous linear functional on $\mathcal{C}_0(\mathbb{R}^n)$ is given by Riemann integration, for instance over the unit cube $[0,1]^n$:

$$u(f) = \int_{[0,1]^n} f(x)\,dx\,.$$

In some sense we must show that *all* continuous linear functionals on $\mathcal{C}_0(X)$ are given by integration. However, we have to interpret integration somewhat widely since there are also *evaluation functionals*. If $z \in X$ consider the Dirac delta

$$\delta_z(f) = f(z)\,.$$

This is also called a *point mass* of $z$. So we need a theory of measure and integration wide enough to include both of these cases.

One special feature of $\mathcal{C}_0(X)$, compared to general normed spaces, is that there is a notion of positivity for its elements. Thus $f \geq 0$ just means $f(x) \geq 0 \ \forall\ x \in X$.

LEMMA 1.4. *Each $f \in \mathcal{C}_0(X)$ can be decomposed uniquely as the difference of its positive and negative parts*

$$f = f_+ - f_-\,,\ f_\pm \in \mathcal{C}_0(X)\,,\ f_\pm(x) \leq |f(x)|\ \ \forall\ x \in X\,. \tag{1.6}$$

PROOF. Simply define

$$f_{\pm}(x) = \begin{cases} \pm f(x) & \text{if} \quad \pm f(x) \geq 0 \\ 0 & \text{if} \quad \pm f(x) < 0 \end{cases}$$

for the same sign throughout. Then (3.8) holds. Observe that $f_+$ is continuous at each $y \in X$ since, with $U$ an appropriate neighborhood of $y$, in each case

$$f(y) > 0 \Longrightarrow f(x) > 0 \text{ for } x \in U \Longrightarrow f_+ = f \text{ in } U$$

$$f(y) < 0 \Longrightarrow f(x) < 0 \text{ for } x \in U \Longrightarrow f_+ = 0 \text{ in } U$$

$$f(y) = 0 \Longrightarrow \text{ given } \ \epsilon > 0 \ \exists \ U \text{ s.t. } \ |f(x)| < \epsilon \text{ in } U$$

$$\Longrightarrow |f_+(x)| < \epsilon \text{ in } U\,.$$

Thus $f_- = f - f_+ \in \mathcal{C}_0(X)$, since both $f_+$ and $f_-$ vanish at infinity. □

We can similarly split elements of the dual space into positive and negative parts although it is a little bit more delicate. We say that $u \in (\mathcal{C}_0(X))'$ is positive if

$$u(f) \geq 0 \ \forall \ 0 \leq f \in \mathcal{C}_0(X)\,. \tag{1.7}$$

For a general (real) $u \in (\mathcal{C}_0(X))'$ and for each $0 \leq f \in \mathcal{C}_0(X)$ set

$$u_+(f) = \sup\left\{u(g)\,;\, g \in \mathcal{C}_0(X)\,,\, 0 \leq g(x) \leq f(x) \ \forall \ x \in X\right\}\,. \tag{1.8}$$

This is certainly finite since $u(g) \leq C\|g\|_\infty \leq C\|f\|_\infty$. Moreover, if $0 < c \in \mathbb{R}$ then $u_+(cf) = cu_+(f)$ by inspection. Suppose $0 \leq f_i \in \mathcal{C}_0(X)$ for $i = 1, 2$. Then given $\epsilon > 0$ there exist $g_i \in \mathcal{C}_0(X)$ with $0 \leq g_i(x) \leq f_i(x)$ and

$$u_+(f_i) \leq u(g_i) + \epsilon\,.$$

It follows that $0 \leq g(x) \leq f_1(x) + f_2(x)$ if $g = g_1 + g_2$ so

$$u_+(f_1 + f_2) \geq u(g) = u(g_1) + u(g_2) \geq u_+(f_1) + u_+(f_2) - 2\epsilon\,.$$

Thus

$$u_+(f_1 + f_2) \geq u_+(f_1) + u_+(f_2).$$

Conversely, if $0 \leq g(x) \leq f_1(x) + f_2(x)$ set $g_1(x) = \min(g, f_1) \in \mathcal{C}_0(X)$ and $g_2 = g - g_1$. Then $0 \leq g_i \leq f_i$ and $u_+(f_1) + u_+(f_2) \geq u(g_1) + u(g_2) = u(g)$. Taking the supremum over $g$, $u_+(f_1 + f_2) \leq u_+(f_1) + u_+(f_2)$, so we find

$$u_+(f_1 + f_2) = u_+(f_1) + u_+(f_2)\,. \tag{1.9}$$

Having shown this effective linearity on the positive functions we can obtain a linear functional by setting

$$u_+(f) = u_+(f_+) - u_+(f_-) \ \forall \ f \in \mathcal{C}_0(X)\,. \tag{1.10}$$

Note that (1.9) shows that $u_+(f) = u_+(f_1) - u_+(f_2)$ for any decomposiiton of $f = f_1 - f_2$ with $f_i \in \mathcal{C}_0(X)$, both positive. [Since $f_1 + f_- = f_2 + f_+$ so $u_+(f_1) + u_+(f_-) = u_+(f_2) + u_+(f_+)$.] Moreover,

$$|u_+(f)| \le \max(u_+(f_+), u(f_-)) \le \|u\| \, \|f\|_\infty \\ \Longrightarrow \|u_+\| \le \|u\| \, .$$

The functional

$$u_- = u_+ - u$$

is also positive, since $u_+(f) \ge u(f)$ for all $0 \le f \in \mathcal{C}_0(x)$. Thus we have proved

Lemma 1.5. *Any element $u \in (\mathcal{C}_0(X))'$ can be decomposed,*

$$u = u_+ - u_-$$

*into the difference of positive elements with*

$$\|u_+\| \, , \, \|u_-\| \le \|u\| \, .$$

The idea behind the definition of $u_+$ is that $u$ itself is, more or less, "integration against a function" (even though we do *not* know how to interpret this yet). In defining $u_+$ from $u$ we are effectively throwing away the negative part of that 'function.' The next step is to show that a positive functional corresponds to a 'measure' meaning a function measuring the size of sets. To define this we really want to evaluate $u$ on the characteristic function of a set

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \, . \end{cases}$$

The problem is that $\chi_E$ is not continuous. Instead we use an idea similar to (15.9).

If $0 \le u \in (\mathcal{C}_0(X))'$ and $U \subset X$ is *open,* set[1]

$$\mu(U) = \sup \left\{ u(f) \, ; \, 0 \le f(x) \le 1, f \in \mathcal{C}_0(X) \, , \, \mathrm{supp}(f) \Subset U \right\} \, . \tag{1.11}$$

Here the support of $f$, $\mathrm{supp}(f)$, is the *closure* of the set of points where $f(x) \neq 0$. Thus $\mathrm{supp}(f)$ is always closed, in (15.4) we only admit $f$ if its support is a compact subset of $U$. The reason for this is that, only then do we 'really know' that $f \in \mathcal{C}_0(X)$.

Suppose we try to measure general sets in this way. We can do this by defining

$$\mu^*(E) = \inf \left\{ \mu(U) \, ; \, U \supset E \, , \, U \text{ open} \right\} \, . \tag{1.12}$$

Already with $\mu$ it may happen that $\mu(U) = \infty$, so we think of

$$\mu^* : \mathcal{P}(X) \to [0, \infty] \tag{1.13}$$

[1] See [**6**] starting p.42 or [**1**] starting p.206.

as defined on the *power set* of $X$ and taking values in the extended positive real numbers.

DEFINITION 1.6. *A positive extended function, $\mu^*$, defined on the power set of $X$ is called an* outer measure *if $\mu^*(\emptyset)=0$, $\mu^*(A)\le\mu^*(B)$ whenever $A\subset B$ and*

$$\mu^*(\bigcup_j A_j)\le\sum_j\mu(A_j)\ \forall\quad\{A_j\}_{j=1}^{\infty}\subset\mathcal{P}(X)\,. \tag{1.14}$$

LEMMA 1.7. *If $u$ is a positive continuous linear functional on $\mathcal{C}_0(X)$ then $\mu^*$, defined by* (15.4), (15.12) *is an outer measure.*

To prove this we need to find enough continuous functions. I have relegated the proof of the following result to Problem 2.

LEMMA 1.8. *Suppose $U_i$, $i=1,\dots,N$ is ,a finite collection of open sets in a locally compact metric space and $K\Subset\bigcup_{i=1}^{N}U_i$ is a compact subset, then there exist continuous functions $f_i\in C(X)$ with $0\le f_i\le 1$, $\operatorname{supp}(f_i)\Subset U_i$ and*

$$\sum_i f_i=1 \text{ in a neighborhood of } K\,. \tag{1.15}$$

PROOF OF LEMMA 15.8. We have to prove (15.6). Suppose first that the $A_i$ are open, then so is $A=\bigcup_i A_i$. If $f\in C(X)$ and $\operatorname{supp}(f)\Subset A$ then $\operatorname{supp}(f)$ is covered by a finite union of the $A_i$s. Applying Lemma 15.7 we can find $f_i$'s, all but a finite number identically zero, so $\operatorname{supp}(f_i)\Subset A_i$ and $\sum_i f_i=1$ in a neighborhood of $\operatorname{supp}(f)$.

Since $f=\sum_i f_if$ we conclude that

$$u(f)=\sum_i u(f_if)\Longrightarrow\mu^*(A)\le\sum_i\mu^*(A_i)$$

since $0\le f_if\le 1$ and $\operatorname{supp}(f_if)\Subset A_i$.

Thus (15.6) holds when the $A_i$ are open. In the general case if $A_i\subset B_i$ with the $B_i$ open then, from the definition,

$$\mu^*(\bigcup_i A_i)\le\mu^*(\bigcup_i B_i)\le\sum_i\mu^*(B_i)\,.$$

Taking the infimum over the $B_i$ gives (15.6) in general. □

## 2. Measures and $\sigma$-algebras

An outer measure such as $\mu^*$ is a rather crude object since, even if the $A_i$ are disjoint, there is generally strict inequality in (15.6). It turns out to be unreasonable to expect equality in (15.6), for disjoint unions, for a function defined on *all* subsets of $X$. We therefore restrict attention to smaller collections of subsets.

Definition 2.1. *A collection of subsets $\mathcal{M}$ of a set $X$ is a $\sigma$-algebra if*

(1) $\phi, X \in \mathcal{M}$
(2) $E \in \mathcal{M} \Longrightarrow E^C = X \backslash E \in \mathcal{M}$
(3) $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M} \Longrightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

For a general outer measure $\mu^*$ we define the notion of $\mu^*$-measurability of a set.

Definition 2.2. *A set $E \subset X$ is $\mu^*$-measurable (for an outer measure $\mu^*$ on $X$) if*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement}) \ \forall \ A \subset X \,. \tag{2.1}$$

Proposition 2.3. *The collection of $\mu^*$-measurable sets for any outer measure is a $\sigma$-algebra.*

Proof. Suppose $E$ is $\mu^*$-measurable, then $E^C$ is $\mu^*$-measurable by the symmetry of (3.9).

Suppose $A$, $E$ and $F$ are any three sets. Then

$$A \cap (E \cup F) = (A \cap E \cap F) \cup (A \cap E \cap F^C) \cup (A \cap E^C \cap F)$$
$$A \cap (E \cup F)^C = A \cap E^C \cap F^C \,.$$

From the subadditivity of $\mu^*$

$$\begin{aligned} \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C) \\ \leq \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cup F^C) \\ + \mu^*(A \cap E^C \cap F) + \mu^*(A \cap E^C \cap F^C). \end{aligned}$$

Now, if $E$ and $F$ are $\mu^*$-measurable then applying the definition twice, for any $A$,

$$\begin{aligned} \mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^C) \\ + \mu^*(A \cap E^C \cap F) + \mu^*(A \cap E^C \cap F^C) \\ \geq \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C) \,. \end{aligned}$$

The reverse inequality follows from the subadditivity of $\mu^*$, so $E \cup F$ is also $\mu^*$-measurable.

If $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint $\mu^*$-measurable sets, set $F_n = \bigcup_{i=1}^{n} E_i$ and $F = \bigcup_{i=1}^{\infty} E_i$. Then for any $A$,

$$\begin{aligned} \mu^*(A \cap F_n) &= \mu^*(A \cap F_n \cap E_n) + \mu^*(A \cap F_n \cap E_n^C) \\ &= \mu^*(A \cap E_n) + \mu^*(A \cap F_{n-1}) \,. \end{aligned}$$

Iterating this shows that

$$\mu^*(A \cap F_n) = \sum_{j=1}^{n} \mu^*(A \cap E_j)\,.$$

From the $\mu^*$-measurability of $F_n$ and the subadditivity of $\mu^*$,

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap F_n) + \mu^*(A \cap F_n^C) \\ &\geq \sum_{j=1}^{n} \mu^*(A \cap E_j) + \mu^*(A \cap F^C)\,.\end{aligned}$$

Taking the limit as $n \to \infty$ and using subadditivity,

$$\begin{aligned}\mu^*(A) &\geq \sum_{j=1}^{\infty} \mu^*(A \cap E_j) + \mu^*(A \cap F^C) \\ &\geq \mu^*(A \cap F) + \mu^*(A \cap F^C) \geq \mu^*(A)\end{aligned} \tag{2.2}$$

proves that inequalities are equalities, so $F$ is also $\mu^*$-measurable.

In general, for *any* countable union of $\mu^*$-measurable sets,

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \widetilde{A}_j\,,$$

$$\widetilde{A}_j = A_j \backslash \bigcup_{i=1}^{j-1} A_i = A_j \cap \left( \bigcup_{i=1}^{j-1} A_i \right)^C$$

is $\mu^*$-measurable since the $\widetilde{A}_j$ are disjoint. □

A *measure* (sometimes called a *positive measure*) is an extended function defined on the elements of a $\sigma$-algebra $\mathcal{M}$:

$$\mu : \mathcal{M} \to [0, \infty]$$

such that

$$\mu(\emptyset) = 0 \text{ and} \tag{2.3}$$

$$\mu\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i) \tag{2.4}$$

$$\text{if } \{A_i\}_{i=1}^{\infty} \subset \mathcal{M} \text{ and } A_i \cap A_j = \phi \ i \neq j.$$

The elements of $\mathcal{M}$ with measure zero, i.e., $E \in \mathcal{M}$, $\mu(E) = 0$, are supposed to be 'ignorable'. The measure $\mu$ is said to be *complete* if

$$E \subset X \text{ and } \exists\ F \in \mathcal{M}\,,\ \mu(F) = 0\,,\ E \subset F \Rightarrow E \in \mathcal{M}\,. \tag{2.5}$$

See Problem 4.

The first part of the following important result due to Caratheodory was shown above.

THEOREM 2.4. *If $\mu^*$ is an outer measure on $X$ then the collection of $\mu^*$-measurable subsets of $X$ is a $\sigma$-algebra and $\mu^*$ restricted to $\mathcal{M}$ is a complete measure.*

PROOF. We have already shown that the collection of $\mu^*$-measurable subsets of $X$ is a $\sigma$-algebra. To see the second part, observe that taking $A = F$ in (3.11) gives

$$\mu^*(F) = \sum_j \mu^*(E_j) \text{ if } F = \bigcup_{j=1}^{\infty} E_j$$

and the $E_j$ are disjoint elements of $\mathcal{M}$. This is (3.3).

Similarly if $\mu^*(E) = 0$ and $F \subset E$ then $\mu^*(F) = 0$. Thus it is enough to show that for any subset $E \subset X$, $\mu^*(E) = 0$ implies $E \in \mathcal{M}$. For any $A \subset X$, using the fact that $\mu^*(A \cap E) = 0$, and the 'increasing' property of $\mu^*$

$$\begin{aligned}\mu^*(A) &\le \mu^*(A \cap E) + \mu^*(A \cap E^C)\\ &= \mu^*(A \cap E^C) \le \mu^*(A)\end{aligned}$$

shows that these must always be equalities, so $E \in \mathcal{M}$ (i.e., is $\mu^*$-measurable). $\square$

Going back to our primary concern, recall that we constructed the outer measure $\mu^*$ from $0 \le u \in (\mathcal{C}_0(X))'$ using (15.4) and (15.12). For the measure whose existence follows from Caratheodory's theorem to be much use we need

PROPOSITION 2.5. *If $0 \le u \in (\mathcal{C}_0(X))'$, for $X$ a locally compact metric space, then each open subset of $X$ is $\mu^*$-measurable for the outer measure defined by* (15.4) *and* (15.12) *and $\mu$ in* (15.4) *is its measure.*

PROOF. Let $U \subset X$ be open. We only need to prove (3.9) for all $A \subset X$ with $\mu^*(A) < \infty$.[2]

Suppose first that $A \subset X$ is open and $\mu^*(A) < \infty$. Then $A \cap U$ is open, so given $\epsilon > 0$ there exists $f \in C(X)$ $\operatorname{supp}(f) \Subset A \cap U$ with $0 \le f \le 1$ and

$$\mu^*(A \cap U) = \mu(A \cap U) \le u(f) + \epsilon\,.$$

Now, $A \backslash \operatorname{supp}(f)$ is also open, so we can find $g \in C(X)$, $0 \le g \le 1$, $\operatorname{supp}(g) \Subset A \backslash \operatorname{supp}(f)$ with

$$\mu^*(A \backslash \operatorname{supp}(f)) = \mu(A \backslash \operatorname{supp}(f)) \le u(g) + \epsilon\,.$$

[2] Why?

Since

$$\begin{aligned} A\setminus \operatorname{supp}(f) \supset A \cap U^C\,,\ 0 \le f+g \le 1\,,\ \operatorname{supp}(f+g) \Subset A\,,\\ \mu(A) &\ge u(f+g) = u(f)+u(g)\\ &> \mu^*(A\cap U) + \mu^*(A\cap U^C) - 2\epsilon\\ &\ge \mu^*(A) - 2\epsilon \end{aligned}$$

using subadditivity of $\mu^*$. Letting $\epsilon \downarrow 0$ we conclude that

$$\mu^*(A) \le \mu^*(A\cap U) + \mu^*(A\cap U^C) \le \mu^*(A) = \mu(A)\,.$$

This gives (3.9) when $A$ is open.

In general, if $E \subset X$ and $\mu^*(E) < \infty$ then given $\epsilon > 0$ there exists $A \subset X$ open with $\mu^*(E) > \mu^*(A) - \epsilon$. Thus,

$$\begin{aligned} \mu^*(E) &\ge \mu^*(A\cap U) + \mu^*(A\cap U^C) - \epsilon\\ &\ge \mu^*(E\cap U) + \mu^*(E\cap U^C) - \epsilon\\ &\ge \mu^*(E) - \epsilon\,. \end{aligned}$$

This shows that (3.9) always holds, so $U$ is $\mu^*$-measurable if it is open. We have already observed that $\mu(U) = \mu^*(U)$ if $U$ is open. □

Thus we have shown that the $\sigma$-algebra given by Caratheodory's theorem contains all open sets. You showed in Problem 3 that the intersection of any collection of $\sigma$-algebras on a given set is a $\sigma$-algebra. Since $\mathcal{P}(X)$ is always a $\sigma$-algebra it follows that for *any* collection $\mathcal{E} \subset \mathcal{P}(X)$ there is always a smallest $\sigma$-algebra containing $\mathcal{E}$, namely

$$\mathcal{M}_{\mathcal{E}} = \bigcap \{\mathcal{M} \supset \mathcal{E}\,;\, \mathcal{M} \text{ is a } \sigma\text{-algebra}\,, \mathcal{M} \subset \mathcal{P}(X)\}\ .$$

The elements of the smallest $\sigma$-algebra containing the *open sets* are called 'Borel sets'. A measure defined on the $\sigma$-algebra of all Borel sets is called a *Borel measure*. This we have shown:

PROPOSITION 2.6. *The measure defined by* (15.4), (15.12) *from* $0 \le u \in (\mathcal{C}_0(X))'$ *by Caratheodory's theorem is a Borel measure.*

PROOF. This is what Proposition 3.14 says! See how easy proofs are. □

We can even continue in the same vein. A Borel measure is said to be *outer regular* on $E \subset X$ if

$$\mu(E) = \inf\{\mu(U)\,;\, U \supset E\,,\, U \text{ open}\}\ . \tag{2.6}$$

Thus the measure constructed in Proposition 3.14 is outer regular on all Borel sets! A Borel measure is *inner regular* on $E$ if

$$\mu(E) = \sup\{\mu(K)\,;\, K \subset E\,,\, K \text{ compact}\}\ . \tag{2.7}$$

Here we need to know that compact sets are Borel measurable. This is Problem 5.

Definition 2.7. *A* Radon measure *(on a metric space) is a Borel measure which is outer regular on all Borel sets, inner regular on open sets and finite on compact sets.*

Proposition 2.8. *The measure defined by* (15.4), (15.12) *from* $0 \le u \in (\mathcal{C}_0(X))'$ *using Caratheodory's theorem is a Radon measure.*

Proof. Suppose $K \subset X$ is compact. Let $\chi_K$ be the characteristic function of $K$, $\chi_K = 1$ on $K$, $\chi_K = 0$ on $K^C$. Suppose $f \in \mathcal{C}_0(X)$, $\operatorname{supp}(f) \Subset X$ and $f \ge \chi_K$. Set

$$U_\epsilon = \{x \in X\,;\, f(x) > 1 - \epsilon\}$$

where $\epsilon > 0$ is small. Thus $U_\epsilon$ is open, by the continuity of $f$ and contains $K$. Moreover, we can choose $g \in C(X)$, $\operatorname{supp}(g) \Subset U_\epsilon$, $0 \le g \le 1$ with $g = 1$ near[3] $K$. Thus, $g \le (1-\epsilon)^{-1} f$ and hence

$$\mu^*(K) \le u(g) = (1-\epsilon)^{-1} u(f)\,.$$

Letting $\epsilon \downarrow 0$, and using the measurability of $K$,

$$\mu(K) \le u(f)$$
$$\Rightarrow \mu(K) = \inf\{u(f)\,;\, f \in C(X)\,,\, \operatorname{supp}(f) \Subset X\ ,\, f \ge \chi_K\}\ .$$

In particular this implies that $\mu(K) < \infty$ if $K \Subset X$, but is also proves (3.17). □

Let me now review a little of what we have done. We used the positive functional $u$ to define an outer measure $\mu^*$, hence a measure $\mu$ and then checked the properties of the latter.

This is a pretty nice scheme; getting ahead of myself a little, let me suggest that we try it on something else.

Let us say that $Q \subset \mathbb{R}^n$ is 'rectangular' if it is a product of finite intervals (open, closed or half-open)

$$Q = \prod_{i=1}^{n} (\text{or}[a_i\,,\, b_i]\text{or})\ a_i \le b_i \tag{2.8}$$

we all agree on its standard volume:

$$v(Q) = \prod_{i=1}^{n} (b_i - a_i) \in [0, \infty)\,. \tag{2.9}$$

[3]Meaning in a neighborhood of $K$.

Clearly if we have two such sets, $Q_1 \subset Q_2$, then $v(Q_1) \le v(Q_2)$. Let us try to define an outer measure on subsets of $\mathbb{R}^n$ by

$$v^*(A) = \inf \left\{ \sum_{i=1}^{\infty} v(Q_i)\,;\ A \subset \bigcup_{i=1}^{\infty} Q_i,\ Q_i \text{ rectangular} \right\}. \tag{2.10}$$

We want to show that (3.22) does define an outer measure. This is pretty easy; certainly $v(\emptyset) = 0$. Similarly if $\{A_i\}_{i=1}^{\infty}$ are (disjoint) sets and $\{Q_{ij}\}_{i=1}^{\infty}$ is a covering of $A_i$ by open rectangles then all the $Q_{ij}$ together cover $A = \bigcup_i A_i$ and

$$\begin{gathered} v^*(A) \le \sum_i \sum_j v(Q_{ij}) \\ \Rightarrow v^*(A) \le \sum_i v^*(A_i)\,. \end{gathered}$$

So we have an outer measure. We also want

Lemma 2.9. *If $Q$ is rectangular then $v^*(Q) = v(Q)$.*

Assuming this, the measure defined from $v^*$ using Caratheodory's theorem is called Lebesgue measure.

Proposition 2.10. *Lebesgue measure is a Borel measure.*

To prove this we just need to show that (open) rectangular sets are $v^*$-measurable.

## 3. Measureability of functions

Suppose that $\mathcal{M}$ is a $\sigma$-algebra on a set $X$[4] and $\mathcal{N}$ is a $\sigma$-algebra on another set $Y$. A map $f : X \to Y$ is said to be *measurable* with respect to these given $\sigma$-algebras on $X$ and $Y$ if

$$f^{-1}(E) \in \mathcal{M} \ \forall\ E \in \mathcal{N}\,. \tag{3.1}$$

Notice how similar this is to one of the characterizations of continuity for maps between metric spaces in terms of open sets. Indeed this analogy yields a useful result.

Lemma 3.1. *If $G \subset \mathcal{N}$ generates $\mathcal{N}$, in the sense that*

$$\mathcal{N} = \bigcap \{\mathcal{N}'; \mathcal{N}' \supset G,\ \mathcal{N}' \ \textit{a } \sigma\textit{-algebra}\} \tag{3.2}$$

*then $f : X \longrightarrow Y$ is measurable iff $f^{-1}(A) \in \mathcal{M}$ for all $A \in G$.*

[4] Then $X$, or if you want to be pedantic $(X, \mathcal{M})$, is often said to be a *measure space* or even a *measurable space*.

Proof. The main point to note here is that $f^{-1}$ as a map on power sets, is very well behaved for *any* map. That is if $f : X \to Y$ then $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ satisfies:

$$
\begin{aligned}
&f^{-1}(E^C) = (f^{-1}(E))^C \\
&f^{-1}\left(\bigcup_{j=1}^{\infty} E_j\right) = \bigcup_{j=1}^{\infty} f^{-1}(E_j) \\
&f^{-1}\left(\bigcap_{j=1}^{\infty} E_j\right) = \bigcap_{j=1}^{\infty} f^{-1}(E_j) \\
&f^{-1}(\phi) = \phi\,,\ f^{-1}(Y) = X\,.
\end{aligned} \tag{3.3}
$$

Putting these things together one sees that if $\mathcal{M}$ is any $\sigma$-algebra on $X$ then

$$
f_*(\mathcal{M}) = \left\{E \subset Y; f^{-1}(E) \in \mathcal{M}\right\} \tag{3.4}
$$

is always a $\sigma$-algebra on $Y$.

In particular if $f^{-1}(A) \in \mathcal{M}$ for all $A \in G \subset \mathcal{N}$ then $f_*(\mathcal{M})$ is a $\sigma$-algebra containing $G$, hence containing $\mathcal{N}$ by the generating condition. Thus $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$ so $f$ is measurable. □

Proposition 3.2. *Any continuous map $f : X \to Y$ between metric spaces is measurable with respect to the Borel $\sigma$-algebras on $X$ and $Y$.*

Proof. The continuity of $f$ shows that $f^{-1}(E) \subset X$ is open if $E \subset Y$ is open. By definition, the open sets generate the Borel $\sigma$-algebra on $Y$ so the preceeding Lemma shows that $f$ is Borel measurable i.e.,

$$
f^{-1}(\mathcal{B}(Y)) \subset \mathcal{B}(X).
$$

□

We are mainly interested in functions on $X$. If $\mathcal{M}$ is a $\sigma$-algebra on $X$ then $f : X \to \mathbb{R}$ is *measurable* if it is measurable with respect to the Borel $\sigma$-algebra on $\mathbb{R}$ and $\mathcal{M}$ on $X$. More generally, for an extended function $f : X \to [-\infty, \infty]$ we take as the 'Borel' $\sigma$-algebra in $[-\infty, \infty]$ the smallest $\sigma$-algebra containing all open subsets of $\mathbb{R}$ and all sets $(a, \infty]$ and $[-\infty, b)$; in fact it is generated by the sets $(a, \infty]$. (See Problem 6.)

Our main task is to define the integral of a measurable function: we start with *simple functions*. Observe that the characteristic function of a set

$$
\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}
$$

is measurable if and only if $E \in \mathcal{M}$. More generally a simple function,

$$f = \sum_{i=1}^{N} a_i \chi_{E_i}, \ a_i \in \mathbb{R} \tag{3.5}$$

is measurable if the $E_i$ are measurable. The presentation, (3.5), of a simple function is not unique. We can make it so, getting the minimal presentation, by insisting that all the $a_i$ are non-zero and

$$E_i = \{x \in E\,;\ f(x) = a_i\}$$

then $f$ in (3.5) is measurable iff all the $E_i$ are measurable.

The Lebesgue integral is based on approximation of functions by simple functions, so it is important to show that this is possible.

Proposition 3.3. *For any non-negative $\mu$-measurable extended function $f : X \longrightarrow [0, \infty]$ there is an increasing sequence $f_n$ of simple measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$ and this limit is uniform on any measurable set on which $f$ is finite.*

Proof. Folland [**1**] page 45 has a nice proof. For each integer $n > 0$ and $0 \le k \le 2^{2n} - 1$, set

$$\begin{aligned} E_{n,k} &= \{x \in X; 2^{-n}k \le f(x) < 2^{-n}(k+1)\}, \\ E'_n &= \{x \in X; f(x) \ge 2^n\}. \end{aligned}$$

These are measurable sets. On increasing $n$ by one, the interval in the definition of $E_{n,k}$ is divided into two. It follows that the sequence of simple functions

$$f_n = \sum_k 2^{-n} k \chi_{E_{k,n}} + 2^n \chi_{E'_n} \tag{3.6}$$

is increasing and has limit $f$ and that this limit is uniform on any measurable set where $f$ is finite. □

## 4. Integration

The $(\mu)$-*integral* of a non-negative simple function is by definition

$$\int_Y f \, d\mu = \sum_i a_i \mu(Y \cap E_i)\,,\ Y \in \mathcal{M}\,. \tag{4.1}$$

Here the convention is that if $\mu(Y \cap E_i) = \infty$ but $a_i = 0$ then $a_i \cdot \mu(Y \cap E_i) = 0$. Clearly this integral takes values in $[0, \infty]$. More significantly, if $c \geq 0$ is a constant and $f$ and $g$ are two non-negative ($\mu$-measurable) simple functions then

$$\begin{aligned} \int_Y cf d\mu &= c \int_Y f d\mu \\ \int_Y (f+g) d\mu &= \int_Y f d\mu + \int_Y g d\mu \\ 0 \leq f \leq g \Rightarrow \int_Y f\, d\mu &\leq \int_Y g\, d\mu\,. \end{aligned} \tag{4.2}$$

(See [**1**] Proposition 2.13 on page 48.)

To see this, observe that (4.1) holds for *any* presentation (3.5) of $f$ with all $a_i \geq 0$. Indeed, by restriction to $E_i$ and division by $a_i$ (which can be assumed non-zero) it is enough to consider the special case

$$\chi_E = \sum_j b_j \chi_{F_j}.$$

The $F_j$ can always be written as the union of a finite number, $N'$, of disjoint measurable sets, $F_j = \cup_{l \in S_j} G_l$ where $j = 1, \ldots, N$ and $S_j \subset \{1, \ldots, N'\}$. Thus

$$\sum_j b_j \mu(F_j) = \sum_j b_j \sum_{l \in S_j} \mu(G_l) = \mu(E)$$

since $\sum_{\{j; l \in S_j\}} b_j = 1$ for each $j$.

From this all the statements follow easily.

Definition 4.1. *For a non-negative $\mu$-measurable extended function $f : X \longrightarrow [0, \infty]$ the integral (with respect to $\mu$) over any measurable set $E \subset X$ is*

$$\int_E f d\mu = \sup\{\int_E h d\mu;\ 0 \leq h \leq f,\ h \text{ simple and measurable.}\} \tag{4.3}$$

By taking suprema, $\int_E f d\mu$ has the first and last properties in (4.2). It also has the middle property, but this is less obvious. To see this, we shall prove the basic 'Monotone convergence theorem' (of Lebesgue). Before doing so however, note what the vanishing of the integral means.

Lemma 4.2. *If $f : X \longrightarrow [0, \infty]$ is measurable then $\int_E f d\mu = 0$ for a measurable set $E$ if and only if*

$$\{x \in E; f(x) > 0\} \text{ has measure zero.} \tag{4.4}$$

Proof. If (4.4) holds, then any positive simple function bounded above by $f$ must also vanish outside a set of measure zero, so its integral must be zero and hence $\int_E f d\mu = 0$. Conversely, observe that the set in (4.4) can be written as

$$E_n = \bigcup_n \{x \in E; f(x) > 1/n\}.$$

Since these sets increase with $n$, if (4.4) does not hold then one of these must have positive measure. In that case the simple function $n^{-1}\chi_{E_n}$ has positive integral so $\int_E f d\mu > 0$. □

Notice the fundamental difference in approach here between Riemann and Lebesgue integrals. The Lebesgue integral, (4.3), uses approximation by functions constant on possibly quite nasty measurable sets, not just intervals as in the Riemann lower and upper integrals.

THEOREM 4.3 (Monotone Convergence). *Let $f_n$ be an increasing sequence of non-negative measurable (extended) functions, then $f(x) = \lim_{n\to\infty} f_n(x)$ is measurable and*

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu \tag{4.5}$$

*for any measurable set $E \subset X$.*

PROOF. To see that $f$ is measurable, observe that

$$f^{-1}(a,\infty] = \bigcup_n f_n^{-1}(a,\infty]. \tag{4.6}$$

Since the sets $(a,\infty]$ generate the Borel $\sigma$-algebra this shows that $f$ is measurable.

So we proceed to prove the main part of the proposition, which is (4.5). Rudin has quite a nice proof of this, [**6**] page 21. Here I paraphrase it. We can easily see from (4.1) that

$$\alpha = \sup \int_E f_n d\mu = \lim_{n\to\infty} \int_E f_n d\mu \le \int_E f d\mu.$$

Given a simple measurable function $g$ with $0 \le g \le f$ and $0 < c < 1$ consider the sets $E_n = \{x \in E; f_n(x) \ge cg(x)\}$. These are measurable and increase with $n$. Moreover $E = \bigcup_n E_n$. It follows that

$$\int_E f_n d\mu \ge \int_{E_n} f_n d\mu \ge c\int_{E_n} g d\mu = \sum_i a_i \mu(E_n \cap F_i) \tag{4.7}$$

in terms of the natural presentation of $g = \sum_i a_i \chi_{F_i}$. Now, the fact that the $E_n$ are measurable and increase to $E$ shows that

$$\mu(E_n \cap F_i) \to \mu(E \cap F_i)$$

as $n \to \infty$. Thus the right side of (4.7) tends to $c \int_E g d\mu$ as $n \to \infty$. Hence $\alpha \geq c \int_E g d\mu$ for all $0 < c < 1$. Taking the supremum over $c$ and then over all such $g$ shows that

$$\alpha = \lim_{n\to\infty} \int_E f_n d\mu \geq \sup \int_E g d\mu = \int_E f d\mu.$$

They must therefore be equal. $\square$

Now for instance the additivity in (4.1) for $f \geq 0$ and $g \geq 0$ any measurable functions follows from Proposition 3.3. Thus if $f \geq 0$ is measurable and $f_n$ is an approximating sequence as in the Proposition then $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$. So if $f$ and $g$ are two non-negative measurable functions then $f_n(x) + g_n(x) \uparrow f + g(x)$ which shows not only that $f + g$ is measurable by also that

$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu.$$

As with the definition of $u_+$ long ago, this allows us to extend the definition of the integral to any *integrable* function.

Definition 4.4. *A measurable extended function $f : X \longrightarrow [-\infty, \infty]$ is said to be integrable on $E$ if its positive and negative parts both have finite integrals over $E$, and then*

$$\int_E f d\mu = \int_E f_+ d\mu - \int_E f_- d\mu.$$

Notice if $f$ is $\mu$-integrable then so is $|f|$. One of the objects we wish to study is the space of integrable functions. The fact that the integral of $|f|$ can vanish encourages us to look at what at first seems a much more complicated object. Namely we consider an equivalence relation between integrable functions

$$f_1 \equiv f_2 \Longleftrightarrow \mu(\{x \in X; f_1(x) \neq f_2(x)\}) = 0. \tag{4.8}$$

That is we identify two such functions if they are equal 'off a set of measure zero.' Clearly if $f_1 \equiv f_2$ in this sense then

$$\int_X |f_1| d\mu = \int_X |f_2| d\mu = 0, \ \int_X f_1 d\mu = \int_X f_2 d\mu.$$

A necessary condition for a measurable function $f \geq 0$ to be integrable is

$$\mu\{x \in X; f(x) = \infty\} = 0.$$

Let $E$ be the (necessarily measureable) set where $f = \infty$. Indeed, if this does not have measure zero, then the sequence of simple functions

$n\chi_E \le f$ has integral tending to infinity. It follows that each equivalence class under (4.8) has a representative which is an honest function, i.e. which is finite everywhere. Namely if $f$ is one representative then

$$f'(x) = \begin{cases} f(x) & x \notin E \\ 0 & x \in E \end{cases}$$

is also a representative.

We shall denote by $L^1(X, \mu)$ the space consisting of such equivalence classes of integrable functions. This is a normed linear space as I ask you to show in Problem 11.

The monotone convergence theorem often occurrs in the slightly disguised form of Fatou's Lemma.

LEMMA 4.5 (Fatou). *If $f_k$ is a sequence of non-negative integrable functions then*

$$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu \, .$$

PROOF. Set $F_k(x) = \inf_{n\ge k} f_n(x)$. Thus $F_k$ is an increasing sequence of non-negative functions with limiting function $\liminf_{n\to\infty} f_n$ and $F_k(x) \le f_n(x) \, \forall \; n \ge k$. By the monotone convergence theorem

$$\int \liminf_{n\to\infty} f_n \, d\mu = \lim_{k\to\infty} \int F_k(x) \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

$\square$

We further extend the integral to complex-valued functions, just saying that

$$f : X \to \mathbb{C}$$

is integrable if its real and imaginary parts are both integrable. Then, by definition,

$$\int_E f d\mu = \int_E \operatorname{Re} f d\mu + i \int_E \operatorname{Im} f d\mu$$

for any $E \subset X$ measurable. It follows that if $f$ is integrable then so is $|f|$. Furthermore

$$\left| \int_E f \, d\mu \right| \le \int_E |f| \, d\mu \, .$$

This is obvious if $\int_E f \, d\mu = 0$, and if not then

$$\int_E f \, d\mu = Re^{i\theta} \; R > 0 \, , \; \theta \subset [0, 2\pi) \, .$$

Then

$$\begin{aligned}
\left|\int_E f\,d\mu\right| &= e^{-i\theta}\int_E f\,d\mu \\
&= \int_E e^{-i\theta} f\,d\mu \\
&= \int_E \mathbb{R}e(e^{-i\theta} f)\,d\mu \\
&\le \int_E \left|\mathbb{R}e(e^{-i\theta} f)\right|\,d\mu \\
&\le \int_E \left|e^{-i\theta} f\right|\,d\mu = \int_E |f|\,d\mu\,.
\end{aligned}$$

The other important convergence result for integrals is Lebesgue's *Dominated convergence theorem*.

THEOREM 4.6. *If $f_n$ is a sequence of integrable functions, $f_k \to f$ a.e.*[5] *and $|f_n| \le g$ for some integrable $g$ then $f$ is integrable and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu\,.$$

PROOF. First we can make the sequence $f_n(x)$ converge by changing all the $f_n(x)$'s to zero on a set of measure zero outside which they converge. This does not change the conclusions. Moreover, it suffices to suppose that the $f_n$ are real-valued. Then consider

$$h_k = g - f_k \ge 0\,.$$

Now, $\liminf_{k\to\infty} h_k = g - f$ by the convergence of $f_n$; in particular $f$ is integrable. By monotone convergence and Fatou's lemma

$$\begin{aligned}
\int (g-f)d\mu = \int \liminf_{k\to\infty} h_k\,d\mu &\le \liminf_{k\to\infty} \int (g - f_k)\,d\mu \\
&= \int g\,d\mu - \limsup_{k\to\infty} \int f_k\,d\mu\,.
\end{aligned}$$

Similarly, if $H_k = g + f_k$ then

$$\int (g+f)d\mu = \int \liminf_{k\to\infty} H_k\,d\mu \le \int g\,d\mu + \liminf_{k\to\infty} \int f_k\,d\mu.$$

It follows that

$$\limsup_{k\to\infty} \int f_k\,d\mu \le \int f\,d\mu \le \liminf_{k\to\infty} \int f_k\,d\mu.$$

[5] Means on the complement of a set of measure zero.

Thus in fact

$$\int f_k \, d\mu \to \int f \, d\mu \, .$$

□

Having proved Lebesgue's theorem of dominated convergence, let me use it to show something important. As before, let $\mu$ be a positive measure on $X$. We have defined $L^1(X, \mu)$; let me consider the more general space $L^p(X, \mu)$. A measurable function

$$f : X \to \mathbb{C}$$

is said to be '$L^p$', for $1 \le p < \infty$, if $|f|^p$ is integrable[6], i.e.,

$$\int_X |f|^p \; d\mu < \infty \, .$$

As before we consider equivalence classes of such functions under the equivalence relation

$$f \sim g \Leftrightarrow \mu \{x; (f - g)(x) \neq 0\} = 0 \, . \tag{4.9}$$

We denote by $L^p(X, \mu)$ the space of such equivalence classes. It is a linear space and the function

$$\|f\|_p = \left( \int_X |f|^p \; d\mu \right)^{1/p} \tag{4.10}$$

is a norm (we always assume $1 \le p < \infty$, sometimes $p = 1$ is excluded but later $p = \infty$ is allowed). It is straightforward to check everything except the triangle inequality. For this we start with

Lemma 4.7. *If $a \ge 0$, $b \ge 0$ and $0 < \gamma < 1$ then*

$$a^\gamma b^{1-\gamma} \le \gamma a + (1 - \gamma) b \tag{4.11}$$

*with equality only when $a = b$.*

Proof. If $b = 0$ this is easy. So assume $b > 0$ and divide by $b$. Taking $t = a/b$ we must show

$$t^\gamma \le \gamma t + 1 - \gamma \, , \; 0 \le t \, , \; 0 < \gamma < 1 \, . \tag{4.12}$$

The function $f(t) = t^\gamma - \gamma t$ is differentiable for $t > 0$ with derivative $\gamma t^{\gamma - 1} - \gamma$, which is positive for $t < 1$ and negative for $t > 1$. Thus $f(t) \le f(1)$ with equality only for $t = 1$. Since $f(1) = 1 - \gamma$, this is (5.17), proving the lemma. □

We use this to prove Hölder's inequality

[6]Check that $|f|^p$ is automatically measurable.

LEMMA 4.8. *If $f$ and $g$ are measurable then*

$$\left|\int fg d\mu\right| \leq \|f\|_p \|g\|_q \tag{4.13}$$

*for any $1 < p < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$.*

PROOF. If $\|f\|_p = 0$ or $\|g\|_q = 0$ the result is trivial, as it is if either is infinite. Thus consider

$$a = \left|\frac{f(x)}{\|f\|_p}\right|^p \,,\, b = \left|\frac{g(x)}{\|g\|_q}\right|^q$$

and apply (5.16) with $\gamma = \frac{1}{p}$. This gives

$$\frac{|f(x)g(x)|}{\|f\|_p\|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}\,.$$

Integrating over $X$ we find

$$\begin{aligned}\frac{1}{\|f\|_p\|g\|_q} \int_X |f(x)g(x)|\ d\mu \\ \leq \frac{1}{p} + \frac{1}{q} = 1\,.\end{aligned}$$

Since $\left|\int_X fg\, d\mu\right| \leq \int_X |fg|\ d\mu$ this implies (5.18). □

The final inequality we need is *Minkowski's* inequality.

PROPOSITION 4.9. *If $1 < p < \infty$ and $f, g \in L^p(X, \mu)$ then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p\,. \tag{4.14}$$

PROOF. The case $p = 1$ you have already done. It is also obvious if $f + g = 0$ a.e.. If not we can write

$$|f + g|^p \leq (|f| + |g|)\,|f + g|^{p-1}$$

and apply Hölder's inequality, to the right side, expanded out,

$$\int |f + g|^p\ d\mu \leq (\|f\|_p + \|g\|_p)\ ,\ \left(\int |f + g|^{q(p-1)}\ d\mu\right)^{1/q}\,.$$

Since $q(p - 1) = p$ and $1 - \frac{1}{q} = 1/p$ this is just (5.20). □

So, now we know that $L^p(X, \mu)$ is a normed space for $1 \leq p < \infty$. In particular it is a metric space. One important additional property that a metric space may have is *completeness*, meaning that every Cauchy sequence is convergent.

DEFINITION 4.10. *A normed space in which the underlying metric space is complete is called a* Banach *space.*

THEOREM 4.11. *For any measure space $(X, M, \mu)$ the spaces $L^p(X, \mu)$, $1 \leq p < \infty$, are Banach spaces.*

PROOF. We need to show that a given Cauchy sequence $\{f_n\}$ converges in $L^p(X, \mu)$. It suffices to show that it has a convergent subsequence. By the Cauchy property, for each $k$ $\exists$ $n = n(k)$ s.t.

$$\|f_n - f_\ell\|_p \leq 2^{-k} \ \forall \ \ell \geq n \, . \tag{4.15}$$

Consider the sequence

$$g_1 = f_1 \, , \ g_k = f_{n(k)} - f_{n(k-1)} \, , \ k > 1 \, .$$

By (5.3), $\|g_k\|_p \leq 2^{-k}$, for $k > 1$, so the series $\sum_k \|g_k\|_p$ converges, say to $B < \infty$. Now set

$$h_n(x) = \sum_{k=1}^{n} |g_k(x)| \ , \ n \geq 1 \, , \ h(x) = \sum_{k=1}^{\infty} g_k(x).$$

Then by the monotone convergence theorem

$$\int_X h^p \, d\mu = \lim_{n\to\infty} \int_X |h_n|^p \ d\mu \leq B^p \, ,$$

where we have also used Minkowski's inequality. Thus $h \in L^p(X, \mu)$, so the series

$$f(x) = \sum_{k=1}^{\infty} g_k(x)$$

converges (absolutely) almost everywhere. Since

$$|f(x)|^p = \lim_{n\to\infty} \left| \sum_{k=1}^{n} g_k \right|^p \leq h^p$$

with $h^p \in L'(X, \mu)$, the dominated convergence theorem applies and shows that $f \in L^p(X, \mu)$. Furthermore,

$$\sum_{k=1}^{\ell} g_k(x) = f_{n(\ell)}(x) \text{ and } \left|f(x) - f_{n(\ell)}(x)\right|^p \leq (2h(x))^p$$

so again by the dominated convergence theorem,

$$\int_X \left|f(x) - f_{n(\ell)}(x)\right|^p \to 0 \, .$$

Thus the subsequence $f_{n(\ell)} \to f$ in $L^p(X, \mu)$, proving its completeness.

$\square$

Next I want to return to our starting point and discuss the Riesz representation theorem. There are two important results in measure theory that I have not covered — I will get you to do most of them in the problems — namely the Hahn decomposition theorem and the Radon-Nikodym theorem. For the moment we can do without the latter, but I will use the former.

So, consider a locally compact metric space, $X$. By a Borel measure on $X$, or a signed Borel measure, we shall mean a function on Borel sets

$$\mu : \mathcal{B}(X) \to \mathbb{R}$$

which is given as the difference of two finite positive Borel measures

$$\mu(E) = \mu_1(E) - \mu_2(E)\,. \tag{4.16}$$

Similarly we shall say that $\mu$ is Radon, or a signed Radon measure, if it *can be written* as such a difference, with both $\mu_1$ and $\mu_2$ finite Radon measures. See the problems below for a discussion of this point.

Let $M_{\text{fin}}(X)$ denote the set of finite Radon measures on $X$. This is a normed space with

$$\|\mu\|_1 = \inf(\mu_1(X) + \mu_2(X)) \tag{4.17}$$

with the infimum over all Radon decompositions (4.16). Each signed Radon measure defines a continuous linear functional on $\mathcal{C}_0(X)$:

$$\int \cdot\, d\mu : \mathcal{C}_0(X) \ni f \longmapsto \int_X f \cdot d\mu\,. \tag{4.18}$$

THEOREM 4.12 (Riesz representation.). *If $X$ is a locally compact metric space then every continuous linear functional on $\mathcal{C}_0(X)$ is given by a unique finite Radon measure on $X$ through* (4.18).

Thus the dual space of $\mathcal{C}_0(X)$ *is* $M_{\text{fin}}(X)$ – at least this is how such a result is usually interpreted

$$(\mathcal{C}_0(X))' = M_{\text{fin}}(X), \tag{4.19}$$

see the remarks following the proof.

PROOF. We have done half of this already. Let me remind you of the steps.

We started with $u \in (\mathcal{C}_0(X))'$ and showed that $u = u_+ - u_-$ where $u_\pm$ are *positive* continuous linear functionals; this is Lemma 1.5. Then we showed that $u \geq 0$ defines a finite positive Radon measure $\mu$. Here $\mu$ is defined by (15.4) on open sets and $\mu(E) = \mu^*(E)$ is given by (15.12) on general Borel sets. It is finite because

$$\mu(X) = \sup\{u(f)\,;\, 0 \le f \le 1\,,\, \operatorname{supp} f \Subset X\,,\, f \in C(X)\} \le \|u\|\,. \tag{4.20}$$

From Proposition 3.19 we conclude that $\mu$ is a Radon measure. Since this argument applies to $u_\pm$ we get two positive finite Radon measures $\mu_\pm$ and hence a signed Radon measure

$$\mu = \mu_+ - \mu_- \in M_{\mathrm{fin}}(X). \tag{4.21}$$

In the problems you are supposed to prove the Hahn decomposition theorem, in particular in Problem 14 I ask you to show that (4.21) *is* the Hahn decomposition of $\mu$ — this means that there is a Borel set $E \subset X$ such that $\mu_-(E) = 0\,,\, \mu_+(X \setminus E) = 0$.

What we have defined is a linear map

$$(\mathcal{C}_0(X))' \to M(X),\ u \longmapsto \mu\,. \tag{4.22}$$

We want to show that this is an isomorphism, i.e., it is $1-1$ and onto.

We first show that it is $1-1$. That is, suppose $\mu = 0$. Given the uniqueness of the Hahn decomposition this implies that $\mu_+ = \mu_- = 0$. So we can suppose that $u \ge 0$ and $\mu = \mu_+ = 0$ and we have to show that $u = 0$; this is obvious since

$$\begin{aligned}\mu(X) = \sup\{u(f);\ \operatorname{supp} u \Subset X,\ 0 \le f \le 1\ f \in C(X)\} = 0\\ \Rightarrow u(f) = 0 \text{ for all such } f\,.\end{aligned} \tag{4.23}$$

If $0 \le f \in C(X)$ and $\operatorname{supp} f \Subset X$ then $f' = f/\|f\|_\infty$ is of this type so $u(f) = 0$ for every $0 \le f \in C(X)$ of compact support. From the decomposition of continuous functions into positive and negative parts it follows that $u(f) = 0$ for every $f$ of compact support. Now, if $f \in \mathcal{C}_o(X)$, then given $n \in \mathbb{N}$ there exists $K \Subset X$ such that $|f| < 1/n$ on $X \setminus K$. As you showed in the problems, there exists $\chi \in \mathcal{C}(X)$ with $\operatorname{supp}(\chi) \Subset X$ and $\chi = 1$ on $K$. Thus if $f_n = \chi f$ then $\operatorname{supp}(f_n) \Subset X$ and $\|f - f_n\| = \sup(|f - f_n| < 1/n$. This shows that $\mathcal{C}_0(X)$ is the closure of the subspace of continuous functions of compact support so by the assumed *continuity* of $u$, $u = 0$.

So it remains to show that *every* finite Radon measure on $X$ arises from (4.22). We do this by starting from $\mu$ and constructing $u$. Again we use the Hahn decomposition of $\mu$, as in (4.21)[7]. Thus we assume $\mu \ge 0$ and construct $u$. It is obvious what we want, namely

$$u(f) = \int_X f\,d\mu\,,\ f \in \mathcal{C}_c(X)\,. \tag{4.24}$$

[7] Actually we can just take any decomposition (4.21) into a difference of positive Radon measures.

Here we need to recall from Proposition 3.2 that continuous functions on $X$, a locally compact metric space, are (Borel) measurable. Furthermore, we know that there is an increasing sequence of simple functions with limit $f$, so

$$\left|\int_X f\,d\mu\right| \leq \mu(X)\cdot\|f\|_\infty\,. \tag{4.25}$$

This shows that $u$ in (4.24) *is* continuous and that its norm $\|u\| \leq \mu(X)$. In fact

$$\|u\| = \mu(X)\,. \tag{4.26}$$

Indeed, the inner regularity of $\mu$ implies that there is a compact set $K \Subset X$ with $\mu(K) \geq \mu(X) - \frac{1}{n}$; then there is $f \in \mathcal{C}_c(X)$ with $0 \leq f \leq 1$ and $f = 1$ on $K$. It follows that $\mu(f) \geq \mu(K) \geq \mu(X) - \frac{1}{n}$, for any $n$. This proves (4.26).

We still have to show that if $u$ is defined by (4.24), with $\mu$ a finite positive Radon measure, then the measure $\tilde{\mu}$ defined from $u$ via (4.24) is precisely $\mu$ itself.

This is easy provided we keep things clear. Starting from $\mu \geq 0$ a finite Radon measure, define $u$ by (4.24) and, for $U \subset X$ open

$$\tilde{\mu}(U) = \sup\left\{\int_X f d\mu,\ 0 \leq f \leq 1,\ f \in C(X),\ \operatorname{supp}(f) \Subset U\right\}. \tag{4.27}$$

By the properties of the integral, $\tilde{\mu}(U) \leq \mu(U)$. Conversely if $K \Subset U$ there exists an element $f \in \mathcal{C}_c(X)$, $0 \leq f \leq 1$, $f = 1$ on $K$ and $\operatorname{supp}(f) \subset U$. Then we know that

$$\tilde{\mu}(U) \geq \int_X f d\mu \geq \mu(K). \tag{4.28}$$

By the inner regularity of $\mu$, we can choose $K \Subset U$ such that $\mu(K) \geq \mu(U) - \epsilon$, given $\epsilon > 0$. Thus $\tilde{\mu}(U) = \mu(U)$.

This proves the Riesz representation theorem, modulo the decomposition of the measure - which I will do in class if the demand is there! In my view this is quite enough measure theory. □

Notice that we have in fact proved something stronger than the statement of the theorem. Namely we have shown that under the correspondence $u \longleftrightarrow \mu$,

$$\|u\| = |\mu|\,(X) =: \|\mu\|_1\,. \tag{4.29}$$

Thus the map is an *isometry*.

CHAPTER 2

# Hilbert spaces and operators

## 1. Hilbert space

We have shown that $L^p(X,\mu)$ is a Banach space – a complete normed space. I shall next discuss the class of Hilbert spaces, a special class of Banach spaces, of which $L^2(X,\mu)$ is a standard example, in which the norm arises from an inner product, just as it does in Euclidean space.

An inner product on a vector space $V$ over $\mathbb{C}$ (one can do the real case too, not much changes) is a *sesquilinear* form

$$V \times V \to \mathbb{C}$$

written $(u,v)$, if $u,v \in V$. The 'sesqui-' part is just linearity in the first variable

$$(a_1u_1 + a_2u_2\,,\ v) = a_1(u_1,v) + a_2(u_2,v), \tag{1.1}$$

anti-linearly in the second

$$(u, a_1v_1 + a_2v_2) = \overline{a}_1(u,v_1) + \overline{a}_2(u,v_2) \tag{1.2}$$

and the conjugacy condition

$$(u,v) = \overline{(v,u)}\,. \tag{1.3}$$

Notice that (1.2) follows from (1.1) and (1.3). If we assume in addition the positivity condition[1]

$$(u,u) \geq 0\,,\ (u,u) = 0 \Rightarrow u = 0\,, \tag{1.4}$$

then

$$\|u\| = (u,u)^{1/2} \tag{1.5}$$

is a *norm* on $V$, as we shall see.

Suppose that $u,v \in V$ have $\|u\| = \|v\| = 1$. Then $(u,v) = e^{i\theta}\,|(u,v)|$ for some $\theta \in \mathbb{R}$. By choice of $\theta$, $e^{-i\theta}(u,v) = |(u,v)|$ is

[1]Notice that $(u,u)$ is real by (1.3).

real, so expanding out using linearity for $s \in \mathbb{R}$,

$$\begin{aligned} 0 &\le (e^{-i\theta}u - sv\,,\, e^{-i\theta}u - sv) \\ &= \|u\|^2 - 2s\,\mathrm{Re}\; e^{-i\theta}(u,v) + s^2\|v\|^2 = 1 - 2s|(u,v)| + s^2. \end{aligned}$$

The minimum of this occurs when $s = |(u,v)|$ and this is negative unless $|(u,v)| \le 1$. Using linearity, and checking the trivial cases $u =$ or $v = 0$ shows that

$$|(u,v)| \le \|u\|\,\|v\|,\ \forall\ u, v \in V\,. \tag{1.6}$$

This is called Schwarz'[2] inequality.

Using Schwarz' inequality

$$\begin{aligned} \|u+v\|^2 &= \|u\|^2 + (u,v) + (v,u) + \|v\|^2 \\ &\le (\|u\| + \|v\|)^2 \\ &\Longrightarrow \|u+v\| \le \|u\| + \|v\|\ \forall\ u, v \in V \end{aligned}$$

which is the triangle inequality.

Definition 1.1. *A Hilbert space is a vector space $V$ with an inner product satisfying* (1.1) - (1.4) *which is complete as a normed space (i.e., is a Banach space).*

Thus we have already shown $L^2(X,\mu)$ to be a Hilbert space for any positive measure $\mu$. The inner product is

$$(f,g) = \int_X f\overline{g}\,d\mu\,, \tag{1.7}$$

since then (1.3) gives $\|f\|_2$.

Another important identity valid in any inner product spaces is the parallelogram law:

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2\,. \tag{1.8}$$

This can be used to prove the basic 'existence theorem' in Hilbert space theory.

Lemma 1.2. *Let $C \subset H$, in a Hilbert space, be closed and convex (i.e., $su + (1-s)v \in C$ if $u, v \in C$ and $0 < s < 1$). Then $C$ contains a unique element of smallest norm.*

Proof. We can certainly choose a sequence $u_n \in C$ such that

$$\|u_n\| \to \delta = \inf\left\{\|v\|\,;\, v \in C\right\}\,.$$

[2]No 't' in this Schwarz.

By the parallelogram law,

$$\begin{aligned}\|u_n - u_m\|^2 &= 2\|u_n\|^2 + 2\|u_m\|^2 - \|u_n + u_m\|^2 \\ &\le 2(\|u_n\|^2 + \|u_m\|^2) - 4\delta^2\end{aligned}$$

where we use the fact that $(u_n + u_m)/2 \in C$ so must have norm at least $\delta$. Thus $\{u_n\}$ is a Cauchy sequence, hence convergent by the assumed completeness of $H$. Thus $\lim u_n = u \in C$ (since it is assumed closed) and by the triangle inequality

$$|\|u_n\| - \|u\|| \le \|u_n - u\| \to 0$$

So $\|u\| = \delta$. Uniqueness of $u$ follows again from the parallelogram law which shows that if $\|u'\| = \delta$ then

$$\|u - u'\| \le 2\delta^2 - 4\|(u + u')/2\|^2 \le 0\,.$$

□

The fundamental fact about a Hilbert space is that each element $v \in H$ defines a continuous linear functional by

$$H \ni u \longmapsto (u, v) \in \mathbb{C}$$

and conversely *every* continuous linear functional arises this way. This is also called the Riesz representation theorem.

Proposition 1.3. *If $L : H \to \mathbb{C}$ is a continuous linear functional on a Hilbert space then this is a unique element $v \in H$ such that*

$$Lu = (u, v)\ \forall\ u \in H\,, \tag{1.9}$$

Proof. Consider the linear space

$$M = \{u \in H\,;\ Lu = 0\}$$

the null space of $L$, a continuous linear functional on $H$. By the assumed continuity, $M$ is closed. We can suppose that $L$ is *not* identically zero (since then $v = 0$ in (1.9)). Thus there exists $w \notin M$. Consider

$$w + M = \{v \in H\,;\ v = w + u\,,\ u \in M\}\ .$$

This is a closed convex subset of $H$. Applying Lemma 1.2 it has a unique smallest element, $v \in w + M$. Since $v$ minimizes the norm on $w + M$,

$$\|v + su\|^2 = \|v\|^2 + 2\,\mathrm{Re}(su, v) + \|s\|^2\|u\|^2$$

is stationary at $s = 0$. Thus $\mathrm{Re}(u, v) = 0\ \forall\ u \in M$, and the same argument with $s$ replaced by $is$ shows that $(v, u) = 0\ \forall\ u \in M$.

Now $v \in w + M$, so $Lv = Lw \neq 0$. Consider the element $w' = w/Lw \in H$. Since $Lw' = 1$, for any $u \in H$

$$L(u - (Lu)w') = Lu - Lu = 0\,.$$

It follows that $u - (Lu)w' \in M$ so if $w'' = w'/\|w'\|^2$

$$(u, w'') = ((Lu)w', w'') = Lu\frac{(w', w')}{\|w'\|^2} = Lu\,.$$

The uniqueness of $v$ follows from the positivity of the norm. $\square$

COROLLARY 1.4. *For any positive measure* $\mu$, *any continuous linear functional*

$$L : L^2(X, \mu) \to \mathbb{C}$$

*is of the form*

$$Lf = \int_X f\overline{g}\,d\mu\,,\ \ g \in L^2(X, \mu)\,.$$

Notice the apparent power of 'abstract reasoning' here! Although we seem to have constructed $g$ out of nowhere, its existence follows from the *completeness* of $L^2(X, \mu)$, but it is very convenient to express the argument abstractly for a general Hilbert space.

## 2. Spectral theorem

For a bounded operator $T$ on a Hilbert space we define the spectrum as the set

$$\operatorname{spec}(T) = \{z \in \mathbb{C}; T - z\operatorname{Id} \text{ is not invertible}\}. \tag{2.1}$$

PROPOSITION 2.1. *For any bounded linear operator on a Hilbert space* $\operatorname{spec}(T) \subset \mathbb{C}$ *is a compact subset of* $\{|z| \le \|T\|\}$.

PROOF. We show that the set $\mathbb{C} \setminus \operatorname{spec}(T)$ (generally called the resolvent set of $T$) is open and contains the complement of a sufficiently large ball. This is based on the convergence of the Neumann series. Namely if $T$ is bounded and $\|T\| < 1$ then

$$(\operatorname{Id} - T)^{-1} = \sum_{j=0}^{\infty} T^j \tag{2.2}$$

converges to a bounded operator which is a two-sided inverse of $\operatorname{Id} - T$. Indeed, $\|T^j\| \le \|T\|^j$ so the series is convergent and composing with $\operatorname{Id} - T$ on either side gives a telescoping series reducing to the identity.

Applying this result, we first see that

$$(T - z) = -z(\operatorname{Id} - T/z) \tag{2.3}$$

is invertible if $|z| > \|T\|$. Similarly, if $(T - z_0)^{-1}$ exists for some $z_0 \in \mathbb{C}$ then

$$(T - z) = (T - z_0) - (z - z_0) = (T - z_0)^{-1}(\operatorname{Id} - (z - z_0)(T - z_0)^{-1}) \tag{2.4}$$

exists for $|z - z_0|\|(T - z_0)^{-1}\| < 1$. $\square$

In general it is rather difficult to precisely locate $\operatorname{spec}(T)$.

However for a bounded self-adjoint operator it is easier. One sign of this is the the norm of the operator has an alternative, simple, characterization. Namely

$$\text{if } A^* = A \text{ then } \sup_{\|\phi\|=1} \langle A\phi, \phi\rangle| = \|A\|. \tag{2.5}$$

If $a$ is this supermum, then clearly $a \leq \|A\|$. To see the converse, choose any $\phi$, $\psi \in H$ with norm 1 and then replace $\psi$ by $e^{i\theta}\psi$ with $\theta$ chosen so that $\langle A\phi, \psi\rangle$ is real. Then use the polarization identity to write

$$\begin{aligned} 4\langle A\phi, \psi\rangle = \langle A(\phi+\psi), (\phi+\psi)\rangle - \langle A(\phi-\psi), (\phi-\psi)\rangle \\ + i\langle A(\phi+i\psi), (\phi+i\psi)\rangle - i\langle A(\phi-i\psi), (\phi-i\psi)\rangle. \end{aligned} \tag{2.6}$$

Now, by the assumed reality we may drop the last two terms and see that

$$4|\langle A\phi, \psi\rangle| \leq a(\|\phi+\psi\|^2 + \|\phi-\psi\|^2) = 2a(\|\phi\|^2 + \|\psi\|^2) = 4a. \tag{2.7}$$

Thus indeed $\|A\| = \sup_{\|\phi\|=\|\psi\|=1} |\langle A\phi, \psi\rangle| = a$.

We can always subtract a real constant from $A$ so that $A' = A - t$ satisfies

$$-\inf_{\|\phi\|=1} \langle A'\phi, \phi\rangle = \sup_{\|\phi\|=1} \langle A'\phi, \phi\rangle = \|A'\|. \tag{2.8}$$

Then, it follows that $A' \pm \|A'\|$ is not invertible. Indeed, there exists a sequence $\phi_n$, with $\|\phi_n\| = 1$ such that $\langle (A' - \|A'\|)\phi_n, \phi_n\rangle \to 0$. Thus

$$\|(A' - \|A'\|)\phi_n\|^2 = -2\langle A'\phi_n, \phi_n\rangle + \|A'\phi_n\|^2 + \|A'\|^2 \leq -2\langle A'\phi_n, \phi_n\rangle + 2\|A'\|^2 \to 0. \tag{2.9}$$

This shows that $A' - \|A'\|$ cannot be invertible and the same argument works for $A' + \|A'\|$. For the original operator $A$ if we set

$$m = \inf_{\|\phi\|=1} \langle A\phi, \phi\rangle \ M = \sup_{\|\phi\|=1} \langle A\phi, \phi\rangle \tag{2.10}$$

then we conclude that neither $A - m\,\mathrm{Id}$ nor $A - M\,\mathrm{Id}$ is invertible and $\|A\| = \max(-m, M)$.

Proposition 2.2. *If $A$ is a bounded self-adjoint operator then, with $m$ and $M$ defined by* (2.10),

$$\{m\} \cup \{M\} \subset \operatorname{spec}(A) \subset [m, M]. \tag{2.11}$$

Proof. We have already shown the first part, that $m$ and $M$ are in the spectrum so it remains to show that $A - z$ is invertible for all $z \in \mathbb{C} \setminus [m, M]$.

Using the self-adjointness

$$\operatorname{Im}\langle (A - z)\phi, \phi\rangle = -\operatorname{Im} z\|\phi\|^2. \tag{2.12}$$

This implies that $A-z$ is invertible if $z \in \mathbb{C} \setminus \mathbb{R}$. First it shows that $(A-z)\phi = 0$ implies $\phi = 0$, so $A-z$ is injective. Secondly, the range is closed. Indeed, if $(A-z)\phi_n \to \psi$ then applying (2.12) directly shows that $\|\phi_n\|$ is bounded and so can be replaced by a weakly convergent subsequence. Applying (2.12) again to $\phi_n - \phi_m$ shows that the sequence is actually Cauchy, hence convergens to $\phi$ so $(A-z)\phi = \psi$ is in the range. Finally, the orthocomplement to this range is the null space of $A^* - \bar{z}$, which is also trivial, so $A-z$ is an isomorphism and (2.12) also shows that the inverse is bounded, in fact

$$\|(A-z)^{-1}\| \le \frac{1}{|\operatorname{Im} z|}. \tag{2.13}$$

When $z \in \mathbb{R}$ we can replace $A$ by $A'$ satisfying (2.8). Then we have to show that $A' - z$ is inverible for $|z| > \|A\|$, but that is shown in the proof of Proposition 2.1. □

The basic estimate leading to the spectral theorem is:

PROPOSITION 2.3. *If $A$ is a bounded self-adjoint operator and $p$ is a real polynomial in one variable,*

$$p(t) = \sum_{i=0}^{N} c_i t^i, \ c_N \ne 0, \tag{2.14}$$

*then $p(A) = \sum\limits_{i=0}^{N} c_i A^i$ satisfies*

$$\|p(A)\| \le \sup_{t \in [m,M]} |p(t)|. \tag{2.15}$$

PROOF. Clearly, $p(A)$ is a bounded self-adjoint operator. If $s \notin p([m,M])$ then $p(A) - s$ is invertible. Indeed, the roots of $p(t) - s$ must cannot lie in $[m.M]$, since otherwise $s \in p([m,M])$. Thus, factorizing $p(s) - t$ we have

(2.16)

$$p(t) - s = c_N \prod_{i=1}^{N} (t - t_i(s)), \ t_i(s) \notin [m,M] \Longrightarrow (p(A) - s)^{-1} \text{ exists}$$

since $p(A) = c_N \sum\limits_i (A - t_i(s))$ and each of the factors is invertible. Thus $\operatorname{spec}(p(A)) \subset p([m,M])$, which is an interval (or a point), and from Proposition 2.3 we conclude that $\|p(A)\| \le \sup p([m,M])$ which is (2.15). □

Now, reinterpreting (2.15) we have a linear map

$$\mathcal{P}(\mathbb{R}) \ni p \longmapsto p(A) \in \mathcal{B}(H) \tag{2.17}$$

from the real polynomials to the bounded self-adjoint operators which is continuous with respect to the supremum norm on $[m, M]$. Since polynomials are dense in continuous functions on finite intervals, we see that (2.17) extends by continuity to a linear map

$$\mathcal{C}([m, M]) \ni f \longmapsto f(A) \in \mathcal{B}(H),\ \|f(A)\| \leq \|f\|_{[m,M]},\ fg(A) = f(A)g(A) \tag{2.18}$$

where the multiplicativity follows by continuity together with the fact that it is true for polynomials.

Now, consider any two elements $\phi, \psi \in H$. Evaluating $f(A)$ on $\phi$ and pairing with $\psi$ gives a linear map

$$\mathcal{C}([m, M]) \ni f \longmapsto \langle f(A)\phi, \psi\rangle \in \mathbb{C}. \tag{2.19}$$

This is a linear functional on $\mathcal{C}([m, M])$ to which we can apply the Riesz representatin theorem and conclude that it is defined by integration against a unique Radon measure $\mu_{\phi,\psi}$ :

$$\langle f(A)\phi, \psi\rangle = \int_{[m,M]} f d\mu_{\phi,\psi}. \tag{2.20}$$

The total mass $|\mu_{\phi,\psi}|$ of this measure is the norm of the functional. Since it is a Borel measure, we can take the integral on $-\infty, b]$ for any $b \in \mathbb{R}$ ad, with the uniqueness, this shows that we have a continuous sesquilinear map

$$P_b(\phi, \psi) : H \times H \ni (\phi, \psi) \longmapsto \int_{[m,b]} d\mu_{\phi,\psi} \in \mathbb{R},\ |P_b(\phi, \psi)| \leq \|A\|\|\phi\|\|\psi\|. \tag{2.21}$$

From the Hilbert space Riesz representation theorem it follows that this sesquilinear form defines, and is determined by, a bounded linear operator

$$P_b(\phi, \psi) = \langle P_b\phi, \psi\rangle,\ \|P_b\| \leq \|A\|. \tag{2.22}$$

In fact, from the functional calculus (the multiplicativity in (2.18)) we see that

$$P_b^* = P_b,\ P_b^2 = P_b,\ \|P_b\| \leq 1, \tag{2.23}$$

so $P_b$ is a projection.

Thus the spectral theorem gives us an increasing (with $b$) family of commuting self-adjoint projections such that $\mu_{\phi,\psi}((-\infty, b]) = \langle P_b\phi, \psi\rangle$ determines the Radon measure for which (2.20) holds. One can go further and think of $P_b$ itself as determining a measure

$$\mu((-\infty, b]) = P_b \tag{2.24}$$

which takes values in the projections on $H$ and which allows the functions of $A$ to be written as integrals in the form

$$f(A) = \int_{[m,M]} f d\mu \tag{2.25}$$

of which (2.20) becomes the 'weak form'. To do so one needs to develop the theory of such measures and the corresponding integrals. This is not so hard but I shall not do it.

CHAPTER 3

# Distributions

## 1. Test functions

So far we have largely been dealing with integration. One thing we have seen is that, by considering dual spaces, we can think of functions as functionals. Let me briefly review this idea.

Consider the unit ball in $\mathbb{R}^n$,

$$\overline{\mathbb{B}}^n = \{x \in \mathbb{R}^n\,;\, |x| \le 1\}\ .$$

I take the *closed* unit ball because I want to deal with a compact metric space. We have dealt with several Banach spaces of functions on $\overline{\mathbb{B}^n}$, for example

$$C(\overline{\mathbb{B}^n}) = \big\{u : \overline{\mathbb{B}^n} \to \mathbb{C}\,;\, u \text{ continuous}\big\}$$
$$L^2(\overline{\mathbb{B}^n}) = \left\{u : \overline{\mathbb{B}^n} \to \mathbb{C}; \text{Borel measurable with } \int |u|^2\, dx < \infty\right\}.$$

Here, as always below, $dx$ is Lebesgue measure and functions are identified if they are equal almost everywhere.

Since $\overline{\mathbb{B}^n}$ is compact we have a natural inclusion

$$C(\overline{\mathbb{B}^n}) \hookrightarrow L^2(\overline{\mathbb{B}^n})\,. \tag{1.1}$$

This is also a topological inclusion, i.e., is a bounded linear map, since

$$\|u\|_{L^2} \le C\|u\|_\infty \tag{1.2}$$

where $C^2$ is the volume of the unit ball.

In general if we have such a set up then

Lemma 1.1. *If $V \hookrightarrow U$ is a subspace with a stronger norm,*

$$\|\varphi\|_U \le C\|\varphi\|_V\ \forall\ \varphi \in V$$

*then restriction gives a continuous linear map*

$$U' \to V',\ U' \ni L \longmapsto \tilde{L} = L|_V \in V',\ \|\tilde{L}\|_{V'} \le C\|L\|_{U'}\,. \tag{1.3}$$

*If $V$ is dense in $U$ then the map* (6.9) *is injective.*

PROOF. By definition of the dual norm

$$
\begin{aligned}
\|\tilde{L}\|_{V'} &= \sup\left\{\left|\tilde{L}(v)\right| \;;\; \|v\|_V \le 1\,,\ v \in V\right\} \\
&\le \sup\left\{\left|\tilde{L}(v)\right| \;;\; \|v\|_U \le C\,,\ v \in V\right\} \\
&\le \sup\left\{|L(u)| \;;\; \|u\|_U \le C\,,\ u \in U\right\} \\
&= C\|L\|_{U'}\,.
\end{aligned}
$$

If $V \subset U$ is dense then the vanishing of $L : U \to \mathbb{C}$ on $V$ implies its vanishing on $U$.

□

Going back to the particular case (6.8) we do indeed get a continuous map between the dual spaces

$$L^2(\overline{\mathbb{B}^n}) \cong (L^2(\overline{\mathbb{B}^n}))' \to (C(\overline{\mathbb{B}^n}))' = M(\overline{\mathbb{B}^n})\,.$$

Here we use the Riesz representation theorem and duality for Hilbert spaces. The map use here is supposed to be *linear* not antilinear, i.e.,

$$L^2(\overline{\mathbb{B}^n}) \ni g \longmapsto \int \cdot g\,dx \in (C(\overline{\mathbb{B}^n}))'\,. \tag{1.4}$$

So the idea is to make the space of 'test functions' as small as reasonably possible, while still retaining *density* in reasonable spaces.

Recall that a function $u : \mathbb{R}^n \to \mathbb{C}$ is *differentiable* at $\overline{x} \in \mathbb{R}^n$ if there exists $a \in \mathbb{C}^n$ such that

$$|u(x) - u(\overline{x}) - a \cdot (x - \overline{x})| = o(|x - \overline{x}|)\,. \tag{1.5}$$

The 'little oh' notation here means that given $\epsilon > 0$ there exists $\delta > 0$ s.t.

$$|x - \overline{x}| < \delta \Rightarrow |u(x) - u(\overline{x}) - a(x - \overline{x})| < \epsilon\,|x - \overline{x}|\,.$$

The coefficients of $a = (a_1, \dots, a_n)$ are the partial derivations of $u$ at $\overline{x}$,

$$a_i = \frac{\partial u}{\partial x_j}(\overline{x})$$

since

$$a_i = \lim_{t\to 0} \frac{u(\overline{x} + te_i) - u(\overline{x})}{t}\,, \tag{1.6}$$

$e_i = (0, \dots, 1, 0, \dots, 0)$ being the $i$th basis vector. The function $u$ is said to be *continuously differentiable* on $\mathbb{R}^n$ if it is differentiable at *each* point $\overline{x} \in \mathbb{R}^n$ and each of the $n$ partial derivatives are continuous,

$$\frac{\partial u}{\partial x_j} : \mathbb{R}^n \to \mathbb{C}\,. \tag{1.7}$$

Definition 1.2. *Let $\mathcal{C}_0^1(\mathbb{R}^n)$ be the subspace of $\mathcal{C}_0(\mathbb{R}^n) = \mathcal{C}_0^0(\mathbb{R}^n)$ such that each element $u \in \mathcal{C}_0^1(\mathbb{R}^n)$ is continuously differentiable and $\frac{\partial u}{\partial x_j} \in \mathcal{C}_0(\mathbb{R}^n)$, $j = 1, \dots, n$.*

Proposition 1.3. *The function*

$$\|u\|_{\mathcal{C}^1} = \|u\|_\infty + \sum_{i=1}^n \|\frac{\partial u}{\partial x_1}\|_\infty$$

*is a norm on $\mathcal{C}_0^1(\mathbb{R}^n)$ with respect to which it is a Banach space.*

Proof. That $\| \ \|_{\mathcal{C}^1}$ is a norm follows from the properties of $\| \ \|_\infty$. Namely $\|u\|_{\mathcal{C}^1} = 0$ certainly implies $u = 0$, $\|au\|_{\mathcal{C}^1} = |a| \ \|u\|_{\mathcal{C}^1}$ and the triangle inequality follows from the same inequality for $\| \ \|_\infty$.

Similarly, the main part of the completeness of $\mathcal{C}_0^1(\mathbb{R}^n)$ follows from the completeness of $\mathcal{C}_0^0(\mathbb{R}^n)$. If $\{u_n\}$ is a Cauchy sequence in $\mathcal{C}_0^1(\mathbb{R}^n)$ then $u_n$ and the $\frac{\partial u_n}{\partial x_j}$ are Cauchy in $\mathcal{C}_0^0(\mathbb{R}^n)$. It follows that there are limits of these sequences,

$$u_n \to v \, , \ \frac{\partial u_n}{\partial x_j} \to v_j \in \mathcal{C}_0^0(\mathbb{R}^n) \, .$$

However we do have to check that $v$ is continuously differentiable and that $\frac{\partial v}{\partial x_j} = v_j$.

One way to do this is to use the Fundamental Theorem of Calculus in each variable. Thus

$$u_n(\overline{x} + te_i) = \int_0^t \frac{\partial u_n}{\partial x_j}(\overline{x} + se_i)\, ds + u_n(\overline{x}) \, .$$

As $n \to \infty$ all terms converge and so, by the continuity of the integral,

$$u(\overline{x} + te_i) = \int_0^t v_j(\overline{x} + se_i)\, ds + u(\overline{x}) \, .$$

This shows that the limit in (6.20) exists, so $v_i(\overline{x})$ is the partial derivation of $u$ with respect to $x_i$. It remains only to show that $u$ is indeed differentiable at each point and I leave this to you in Problem 17.

□

So, almost by definition, we have an example of Lemma 6.17,

$$\mathcal{C}_0^1(\mathbb{R}^n) \hookrightarrow \mathcal{C}_0^0(\mathbb{R}^n).$$

It is in fact dense but I will not bother showing this (yet). So we know that

$$(\mathcal{C}_0^0(\mathbb{R}^n))' \to (\mathcal{C}_0^1(\mathbb{R}^n))'$$

and we expect it to be injective. Thus there are *more* functionals on $\mathcal{C}_0^1(\mathbb{R}^n)$ including things that are ‘more singular than measures’.

An example is related to the Dirac delta

$$\delta(\overline{x})(u) = u(\overline{x})\,,\ u \in \mathcal{C}^0_0(\mathbb{R}^n)\,,$$

namely

$$\mathcal{C}^1_0(\mathbb{R}^n) \ni u \longmapsto \frac{\partial u}{\partial x_j}(\overline{x}) \in \mathbb{C}\,.$$

This is clearly a continuous linear functional which it is only just to denote $\frac{\partial}{\partial x_j}\delta(\overline{x})$.

Of course, why stop at one derivative?

Definition 1.4. *The space $\mathcal{C}^k_0(\mathbb{R}^n) \subset \mathcal{C}^1_0(\mathbb{R}^n)$ $k \geq 1$ is defined inductively by requiring that*

$$\frac{\partial u}{\partial x_j} \in \mathcal{C}^{k-1}_0(\mathbb{R}^n)\,,\ j = 1, \dots, n\,.$$

*The norm on $\mathcal{C}^k_0(\mathbb{R}^n)$ is taken to be*

$$\|u\|_{\mathcal{C}^k} = \|u\|_{\mathcal{C}^{k-1}} + \sum_{j=1}^{n} \|\frac{\partial u}{\partial x_j}\|_{\mathcal{C}^{k-1}}\,. \tag{1.8}$$

These are all Banach spaces, since if $\{u_n\}$ is Cauchy in $\mathcal{C}^k_0(\mathbb{R}^n)$, it is Cauchy and hence convergent in $\mathcal{C}^{k-1}_0(\mathbb{R}^n)$, as is $\partial u_n/\partial x_j$, $j = 1, \dots, n-1$. Furthermore the limits of the $\partial u_n/\partial x_j$ are the derivatives of the limits by Proposition 1.3.

This gives us a sequence of spaces getting 'smoother and smoother'

$$\mathcal{C}^0_0(\mathbb{R}^n) \supset \mathcal{C}^1_0(\mathbb{R}^n) \supset \cdots \supset \mathcal{C}^k_0(\mathbb{R}^n) \supset \cdots\,,$$

with norms getting larger and larger. The duals can also be expected to get larger and larger as $k$ increases.

As well as looking at functions getting smoother and smoother, we need to think about 'infinity', since $\mathbb{R}^n$ is not compact. Observe that an element $g \in L^1(\mathbb{R}^n)$ (with respect to Lebesgue measure by default) defines a functional on $\mathcal{C}^0_0(\mathbb{R}^n)$ — and hence *all* the $\mathcal{C}^k_0(\mathbb{R}^n)$s. However a function such as the constant function 1 is *not* integrable on $\mathbb{R}^n$. Since we certainly want to talk about this, and polynomials, we consider a second condition of *smallness at infinity.* Let us set

$$\langle x \rangle = (1 + |x|^2)^{1/2} \tag{1.9}$$

a function which is the size of $|x|$ for $|x|$ large, but has the virtue of being smooth[1]

[1] See Problem 18.

DEFINITION 1.5. *For any $k,\ l \in \mathbb{N} = \{1, 2, \cdots\}$ set*

$$\langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) = \left\{u \in \mathcal{C}_0^k(\mathbb{R}^n)\,;\, u = \langle x\rangle^{-l}v\,,\ v \in \mathcal{C}_0^k(\mathbb{R}^n)\right\}\ ,$$

*with norm, $\|u\|_{k,l} = \|v\|_{\mathcal{C}^k}\,,\ v = \langle x\rangle^l u$.*

Notice that the definition just says that $u = \langle x\rangle^{-l}v$, with $v \in \mathcal{C}_0^k(\mathbb{R}^n)$. It follows immediately that $\langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n)$ is a Banach space with this norm.

DEFINITION 1.6. *Schwartz' space*[2] *of test functions on $\mathbb{R}^n$ is*

$$\mathcal{S}(\mathbb{R}^n) = \left\{u : \mathbb{R}^n \to \mathbb{C}; u \in \langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) \textit{ for all } k \textit{ and } l \in \mathbb{N}\right\}.$$

It is not immediately apparent that this space is non-empty (well 0 is in there but...); that

$$P(x)\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n) \tag{1.10}$$

for any polynomial $P$ is Problem 19.

COROLLARY 1.7. *$\mathcal{S}(\mathbb{R}^n)$ is infinite-dimensional.*

In fact the linear space in (1.10) turns out to be dense in $\mathcal{S}(\mathbb{R}^n)$ when we sort out the topology – so it will be separable.

Schwartz' idea is that the dual of $\mathcal{S}(\mathbb{R}^n)$ should contain all the 'interesting' objects, at least those of 'polynomial growth'. The problem is that we do *not* have a good norm on $\mathcal{S}(\mathbb{R}^n)$. Rather we have a *lot* of them. Observe that

$$\langle x\rangle^{-l}\mathcal{C}_0^k(\mathbb{R}^n) \subset \langle x\rangle^{-l'}\mathcal{C}_0^{k'}(\mathbb{R}^n) \text{ if } l \geq l' \text{ and } k \geq k'\,.$$

Thus we see that as a linear space

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n). \tag{1.11}$$

Since these spaces are getting smaller, we have a countably infinite number of norms. For this reason $\mathcal{S}(\mathbb{R}^n)$ is called a *countably normed* space.

PROPOSITION 1.8. *For $u \in \mathcal{S}(\mathbb{R}^n)$, set*

$$\|u\|_{(k)} = \|\langle x\rangle^k u\|_{\mathcal{C}^k} \tag{1.12}$$

*and define*

$$d(u, v) = \sum_{k=0}^{\infty} 2^{-k}\frac{\|u - v\|_{(k)}}{1 + \|u - v\|_{(k)}}, \tag{1.13}$$

*then $d$ is a distance function in $\mathcal{S}(\mathbb{R}^n)$ with respect to which it is a complete metric space.*

[2] Laurent Schwartz – this one with a 't'.

Proof. The series in (1.13) certainly converges, since
$$\frac{\|u-v\|_{(k)}}{1+\|u-v\|_{(k)}} \le 1.$$
The first two conditions on a metric are clear,
$$d(u,v)=0 \Rightarrow \|u-v\|_{\mathcal{C}_0}=0 \Rightarrow u=v,$$
and symmetry is immediate. The triangle inequality is perhaps more mysterious!

Certainly it is enough to show that
$$\tilde{d}(u,v)=\frac{\|u-v\|}{1+\|u-v\|} \tag{1.14}$$
is a metric on any normed space, since then we may sum over $k$. Thus we consider
$$\begin{aligned}\frac{\|u-v\|}{1+\|u-v\|}&+\frac{\|v-w\|}{1+\|v-w\|}\\ &=\frac{\|u-v\|(1+\|v-w\|)+\|v-w\|(1+\|u-v\|)}{(1+\|u-v\|)(1+\|v-w\|)}.\end{aligned}$$
Comparing this to $\tilde{d}(v,w)$ we must show that
$$\begin{aligned}&(1+\|u-v\|)(1+\|v-w\|)\|u-w\|\\ &\quad\le(\|u-v\|(1+\|v-w\|)+\|v-w\|(1+\|u-v\|))(1+\|u-w\|).\end{aligned}$$
Starting from the LHS and using the triangle inequality,
$$\begin{aligned}\text{LHS } &\le \|u-w\|+(\|u-v\|+\|v-w\|+\|u-v\|\|v-w\|)\|u-w\|\\ &\le(\|u-v\|+\|v-w\|+\|u-v\|\|v-w\|)(1+\|u-w\|)\\ &\le \text{RHS}.\end{aligned}$$
Thus, $d$ is a metric.

Suppose $u_n$ is a Cauchy sequence. Thus, $d(u_n,u_m)\to 0$ as $n,m\to\infty$. In particular, given
$$\begin{gathered}\epsilon>0\ \exists\ N \text{ s.t. } n,m>N \text{ implies}\\ d(u_n,u_m)<\epsilon 2^{-k}\ \forall\ n,m>N.\end{gathered}$$
The terms in (1.13) are all positive, so this implies
$$\frac{\|u_n-u_m\|_{(k)}}{1+\|u_n-u_m\|_{(k)}}<\epsilon\ \forall\ n,m>N.$$
If $\epsilon<1/2$ this in turn implies that
$$\|u_n-u_m\|_{(k)}<2\epsilon,$$

so the sequence is Cauchy in $\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)$ for each $k$. From the completeness of these spaces it follows that $u_n \to u$ in $\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)_j$ for each $k$. Given $\epsilon > 0$ choose $k$ so large that $2^{-k} < \epsilon/2$. Then $\exists\ N$ s.t. $n > N$

$$\Rightarrow \|u - u_n\|_{(j)} < \epsilon/2 \ n > N,\ j \le k.$$

Hence

$$d(u_n, u) = \sum_{j\le k} 2^{-j} \frac{\|u-u_n\|_{(j)}}{1+\|u-u_n\|_{(j)}} + \sum_{j>k} 2^{-j} \frac{\|u-u_n\|_{(j)}}{1+\|u-u_n\|_{(j)}} \le \epsilon/4 + 2^{-k} < \epsilon.$$

This $u_n \to u$ in $\mathcal{S}(\mathbb{R}^n)$. □

As well as the Schwartz space, $\mathcal{S}(\mathbb{R}^n)$, of functions of rapid decrease with all derivatives, there is a smaller 'standard' space of test functions, namely

$$\mathcal{C}_c^\infty(\mathbb{R}^n) = \{u \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(u) \Subset \mathbb{R}^n\}\,, \tag{1.15}$$

the space of smooth functions of compact support. Again, it is not quite obvious that this has any non-trivial elements, but it does as we shall see. If we fix a compact subset of $\mathbb{R}^n$ and look at functions with support in that set, for instance the closed ball of radius $R > 0$, then we get a closed subspace of $\mathcal{S}(\mathbb{R}^n)$, hence a complete metric space. One 'problem' with $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is that it does not have a complete metric topology which restricts to this topology on the subsets. Rather we must use an *inductive limit* procedure to get a decent topology.

Just to show that this is not really hard, I will discuss it briefly here, but it is not used in the sequel. In particular I will not do this in the lectures themselves. By definition our space $\mathcal{C}_c^\infty(\mathbb{R}^n)$ (denoted traditionally as $\mathcal{D}(\mathbb{R}^n)$) is a countable union of subspaces

$$\mathcal{C}_c^\infty(\mathbb{R}^n) = \bigcup_{n\in\mathbb{N}} \dot{\mathcal{C}}_c^\infty(B(n)),\ \dot{\mathcal{C}}_c^\infty(B(n)) = \{u \in \mathcal{S}(\mathbb{R}^n); u = 0 \text{ in } |x| > n\}. \tag{1.16}$$

Consider

$$\mathcal{T} = \{U \subset \mathcal{C}_c^\infty(\mathbb{R}^n); U \cap \dot{\mathcal{C}}_c^\infty(B(n)) \text{ is open in } \dot{\mathcal{C}}^\infty(B(n)) \text{ for each } n\}. \tag{1.17}$$

This is a topology on $\mathcal{C}_c^\infty(\mathbb{R}^n)$ – contains the empty set and the whole space and is closed under finite intersections and arbitrary unions –

simply because the same is true for the open sets in $\dot{\mathcal{C}}^\infty(B(n))$ for each $n$. This is in fact the inductive limit topology. One obvious question is:- what does it mean for a linear functional $u : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathbb{C}$ to be continuous? This just means that $u^{-1}(O)$ is open for each open set in $\mathbb{C}$. Directly from the definition this in turn means that $u^{-1}(O)\cap\dot{\mathcal{C}}^\infty(B(n))$ should be open in $\dot{\mathcal{C}}^\infty(B(n))$ for each $n$. This however just means that, restricted to each of these subspaces $u$ is continuous. If you now go forwards to Lemma 2.3 you can see what this means; see Problem 74.

Of course there is a lot more to be said about these spaces; you can find plenty of it in the references.

## 2. Tempered distributions

A good first reference for distributions is [**2**], [**5**] gives a more exhaustive treatment.

The complete metric topology on $\mathcal{S}(\mathbb{R}^n)$ is described above. Next I want to try to convice you that elements of its dual space $\mathcal{S}'(\mathbb{R}^n)$, have enough of the properties of functions that we can work with them as 'generalized functions'.

First let me develop some notation. A differentiable function $\varphi : \mathbb{R}^n \to \mathbb{C}$ has partial derivatives which we have denoted $\partial\varphi/\partial x_j : \mathbb{R}^n \to \mathbb{C}$. For reasons that will become clear later, we put a $\sqrt{-1}$ into the definition and write

$$D_j\varphi = \frac{1}{i}\frac{\partial\varphi}{\partial x_j}\,. \tag{2.1}$$

We say $\varphi$ is once continuously differentiable if each of these $D_j\varphi$ is continuous. Then we defined $k$ times continuous differentiability inductively by saying that $\varphi$ and the $D_j\varphi$ are $(k-1)$-times continuously differentiable. For $k = 2$ this means that

$$D_jD_k\varphi \text{ are continuous for } j,k = 1,\cdots,n\,.$$

Now, recall that, if continuous, these second derivatives are symmetric:

$$D_jD_k\varphi = D_kD_j\varphi\,. \tag{2.2}$$

This means we can use a compact notation for higher derivatives. Put $\mathbb{N}_0 = \{0,1,\ldots\}$; we call an element $\alpha \in \mathbb{N}_0^n$ a 'multi-index' and if $\varphi$ is at least $k$ times continuously differentiable, we set[3]

$$D^\alpha\varphi = \frac{1}{i^{|\alpha|}}\frac{\partial^{\alpha_1}}{\partial x_1}\cdots\frac{\partial^{\alpha_n}}{\partial x_n}\varphi \text{ whenever } |\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n \le k. \tag{2.3}$$

[3] Periodically there is the possibility of confusion between the two meanings of $|\alpha|$ but it seldom arises.

In fact we will use a closely related notation of powers of a variable. Namely if $\alpha$ is a multi-index we shall also write

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}. \tag{2.4}$$

Now we have *defined* the spaces.

$$\mathcal{C}_0^k(\mathbb{R}^n) = \left\{\varphi : \mathbb{R}^n \to \mathbb{C}\,;\ D^\alpha\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| \le k\right\}\ . \tag{2.5}$$

Notice the convention is that $D^\alpha\varphi$ is asserted to exist if it is required to be continuous! Using $\langle x\rangle = (1+|x|^2)$ we defined

$$\langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n) = \left\{\varphi : \mathbb{R}^n \to \mathbb{C}\,;\ \langle x\rangle^k\varphi \in \mathcal{C}_0^k(\mathbb{R}^n)\right\}\ , \tag{2.6}$$

and then our space of test functions is

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x\rangle^{-k}\mathcal{C}_0^k(\mathbb{R}^n)\,.$$

Thus,

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \Leftrightarrow D^\alpha(\langle x\rangle^k\varphi) \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| \le k \text{ and all } k\,. \tag{2.7}$$

Lemma 2.1. *The condition $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written*

$$\langle x\rangle^k D^\alpha\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| \le k\,,\ \forall\ k\,.$$

Proof. We first check that

$$\begin{aligned} &\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ D_j(\langle x\rangle\varphi) \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ j = 1,\cdots,n \\ \Leftrightarrow\ &\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ \langle x\rangle D_j\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\,,\ j = 1,\cdots,n\,. \end{aligned}$$

Since

$$D_j\langle x\rangle\varphi = \langle x\rangle D_j\varphi + (D_j\langle x\rangle)\varphi$$

and $D_j\langle x\rangle = \frac{1}{i}x_j\langle x\rangle^{-1}$ is a bounded continuous function, this is clear. Then consider the same thing for a larger $k$:

$$\begin{aligned} &D^\alpha\langle x\rangle^p\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| = p\,,\ 0 \le p \le k \\ \Leftrightarrow\ &\langle x\rangle^p D^\alpha\varphi \in \mathcal{C}_0^0(\mathbb{R}^n)\ \forall\ |\alpha| = p\,,\ 0 \le p \le k\,. \end{aligned} \tag{2.8}$$

□

I leave you to check this as Problem 2.1.

Corollary 2.2. *For any $k \in \mathbb{N}$ the norms*

$$\|\langle x\rangle^k\varphi\|_{\mathcal{C}^k} \text{ and } \sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \|x^\alpha D_x^\beta\varphi\|_\infty$$

*are equivalent.*

PROOF. Any reasonable proof of (2.2) shows that the norms

$$\|\langle x\rangle^k\varphi\|_{\mathcal{C}^k} \text{ and } \sum_{|\beta|\le k}\|\langle x\rangle^k D^\beta\varphi\|_\infty$$

are equivalent. Since there are positive constants such that

$$C_1\left(1+\sum_{|\alpha|\le k}|x^\alpha|\right)\le\langle x\rangle^k\le C_2\left(1+\sum_{|\alpha|\le k}|x^\alpha|\right)$$

the equivalent of the norms follows.

□

PROPOSITION 2.3. *A linear functional* $u:\mathcal{S}(\mathbb{R}^n)\to\mathbb{C}$ *is continuous if and only if there exist* $C$, $k$ *such that*

$$|u(\varphi)|\le C\sum_{\substack{|\alpha|\le k,\\|\beta|\le k}}\sup_{\mathbb{R}^n}\left|x^\alpha D_x^\beta\varphi\right|.$$

PROOF. This is just the equivalence of the norms, since we showed that $u\in\mathcal{S}'(\mathbb{R}^n)$ if and only if

$$|u(\varphi)|\le C\|\langle x\rangle^k\varphi\|_{\mathcal{C}^k}$$

for some $k$.

□

LEMMA 2.4. *A linear* map

$$T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$$

*is continuous if and only if for* each $k$ *there exist* $C$ *and* $j$ *such that if* $|\alpha|\le k$ *and* $|\beta|\le k$

$$\sup\left|x^\alpha D^\beta T\varphi\right|\le C\sum_{|\alpha'|\le j,\ |\beta'|\le j}\sup_{\mathbb{R}^n}\left|x^{\alpha'}D^{\beta'}\varphi\right|\ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^n). \tag{2.9}$$

PROOF. This is Problem 2.2. □

All this messing about with norms shows that

$$x_j:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n) \text{ and } D_j:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}(\mathbb{R}^n)$$

are continuous.

So now we have some idea of what $u\in\mathcal{S}'(\mathbb{R}^n)$ means. Let's notice that $u\in\mathcal{S}'(\mathbb{R}^n)$ implies

$$x_j u\in\mathcal{S}'(\mathbb{R}^n)\ \forall\ j=1,\cdots,n \tag{2.10}$$

$$D_j u\in\mathcal{S}'(\mathbb{R}^n)\ \forall\ j=1,\cdots,n \tag{2.11}$$

$$\varphi u\in\mathcal{S}'(\mathbb{R}^n)\ \forall\ \varphi\in\mathcal{S}(\mathbb{R}^n) \tag{2.12}$$

where we have to *define* these things in a reasonable way. Remember that $u \in \mathcal{S}'(\mathbb{R}^n)$ is "supposed" to be like an integral against a "generalized function"

$$u(\psi) = \int_{\mathbb{R}^n} u(x)\psi(x)\,dx \ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{2.13}$$

Since it would be true if $u$ *were* a function we *define*

$$x_j u(\psi) = u(x_j\psi) \ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{2.14}$$

Then we check that $x_j u \in \mathcal{S}'(\mathbb{R}^n)$:

$$\begin{aligned}|x_j u(\psi)| &= |u(x_j\psi)| \\ &\le C \sum_{|\alpha|\le k,\ |\beta|\le k} \sup_{\mathbb{R}^n} \left|x^\alpha D^\beta(x_j\psi)\right| \\ &\le C' \sum_{|\alpha|\le k+1,\ |\beta|\le k} \sup_{\mathbb{R}^n} \left|x^\alpha D^\beta \psi\right| .\end{aligned}$$

Similarly we can define the partial *derivatives* by using the standard integration by parts formula

$$\int_{\mathbb{R}^n} (D_j u)(x)\varphi(x)\,dx = -\int_{\mathbb{R}^n} u(x)(D_j\varphi(x))\,dx \tag{2.15}$$

if $u \in \mathcal{C}^1_0(\mathbb{R}^n)$. Thus if $u \in \mathcal{S}'(\mathbb{R}^n)$ again we *define*

$$D_j u(\psi) = -u(D_j\psi) \ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$$

Then it is clear that $D_j u \in \mathcal{S}'(\mathbb{R}^n)$.

Iterating these definition we find that $D^\alpha$, for any multi-index $\alpha$, defines a linear map

$$D^\alpha : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,. \tag{2.16}$$

In general a linear differential operator with constant coefficients is a sum of such "monomials". For example Laplace's operator is

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2} = D_1^2 + D_2^2 + \cdots + D_n^2\,.$$

We will be interested in trying to solve differential equations such as

$$\Delta u = f \in \mathcal{S}'(\mathbb{R}^n)\,.$$

We can also multiply $u \in \mathcal{S}'(\mathbb{R}^n)$ by $\varphi \in \mathcal{S}(\mathbb{R}^n)$, simply defining

$$\varphi u(\psi) = u(\varphi\psi) \ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{2.17}$$

For this to make sense it suffices to check that

$$\sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \sup_{\mathbb{R}^n} \left|x^\alpha D^\beta(\varphi\psi)\right| \le C \sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \sup_{\mathbb{R}^n} \left|x^\alpha D^\beta\psi\right| . \tag{2.18}$$

This follows easily from Leibniz' formula.

Now, to start thinking of $u \in \mathcal{S}'(\mathbb{R}^n)$ as a generalized function we first define its *support*. Recall that

$$\operatorname{supp}(\psi) = \operatorname{clos}\{x \in \mathbb{R}^n; \psi(x) \ne 0\} . \tag{2.19}$$

We can write this in another 'weak' way which is easier to generalize. Namely

$$p \notin \operatorname{supp}(u) \Leftrightarrow \ \exists \varphi \in \mathcal{S}(\mathbb{R}^n) \, , \, \varphi(p) \ne 0 \, , \, \varphi u = 0 \, . \tag{2.20}$$

In fact this definition makes sense for *any* $u \in \mathcal{S}'(\mathbb{R}^n)$.

Lemma 2.5. *The set* $\operatorname{supp}(u)$ *defined by* (2.20) *is a closed subset of* $\mathbb{R}^n$ *and reduces to* (2.19) *if* $u \in \mathcal{S}(\mathbb{R}^n)$.

Proof. The set defined by (2.20) is closed, since

$$\operatorname{supp}(u)^{\complement} = \{p \in \mathbb{R}^n ; \ \exists \ \varphi \in \mathcal{S}(\mathbb{R}^n), \ \varphi(p) \ne 0, \ \varphi u = 0\} \tag{2.21}$$

is clearly open — the same $\varphi$ works for nearby points. If $\psi \in \mathcal{S}(\mathbb{R}^n)$ we define $u_\psi \in \mathcal{S}'(\mathbb{R}^n)$, which we will again identify with $\psi$, by

$$u_\psi(\varphi) = \int \varphi(x)\psi(x)\, dx \, . \tag{2.22}$$

Obviously $u_\psi = 0 \Longrightarrow \psi = 0$, simply set $\varphi = \overline{\psi}$ in (2.22). Thus the map

$$\mathcal{S}(\mathbb{R}^n) \ni \psi \longmapsto u_\psi \in \mathcal{S}'(\mathbb{R}^n) \tag{2.23}$$

is injective. We want to show that

$$\operatorname{supp}(u_\psi) = \operatorname{supp}(\psi) \tag{2.24}$$

on the left given by (2.20) and on the right by (2.19). We show first that

$$\operatorname{supp}(u_\psi) \subset \operatorname{supp}(\psi).$$

Thus, we need to see that $p \notin \operatorname{supp}(\psi) \Rightarrow p \notin \operatorname{supp}(u_\psi)$. The first condition is that $\psi(x) = 0$ in a neighbourhood, $U$ of $p$, hence there is a $\mathcal{C}^\infty$ function $\varphi$ with support in $U$ and $\varphi(p) \ne 0$. Then $\varphi\psi \equiv 0$. Conversely suppose $p \notin \operatorname{supp}(u_\psi)$. Then there exists $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(p) \ne 0$ and $\varphi u_\psi = 0$, i.e., $\varphi u_\psi(\eta) = 0 \ \forall \ \eta \in \mathcal{S}(\mathbb{R}^n)$. By the injectivity of $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ this means $\varphi\psi = 0$, so $\psi \equiv 0$ in a neighborhood of $p$ and $p \notin \operatorname{supp}(\psi)$. $\square$

Consider the simplest examples of distribution which are not functions, namely those with support at a given point $p$. The obvious one is the Dirac delta 'function'

$$\delta_p(\varphi) = \varphi(p)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{2.25}$$

We can make many more, because $D^\alpha$ is *local*

$$\operatorname{supp}(D^\alpha u) \subset \operatorname{supp}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{2.26}$$

Indeed, $p \notin \operatorname{supp}(u) \Rightarrow \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n)$, $\varphi u \equiv 0$, $\varphi(p) \neq 0$. Thus each of the distributions $D^\alpha \delta_p$ also has support contained in $\{p\}$. In fact none of them vanish, and they are all linearly independent.

## 3. Convolution and density

We have defined an inclusion map

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n),\ u_\varphi(\psi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x)\,dx\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{3.1}$$

This allows us to 'think of' $\mathcal{S}(\mathbb{R}^n)$ as a subspace of $\mathcal{S}'(\mathbb{R}^n)$; that is we habitually identify $u_\varphi$ with $\varphi$. We can do this because we know (3.1) to be injective. We can extend the map (3.1) to include bigger spaces

$$\begin{gathered} \mathcal{C}^0_0(\mathbb{R}^n) \ni \varphi \longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n) \\ L^p(\mathbb{R}^n) \ni \varphi \longmapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n) \\ M(\mathbb{R}^n) \ni \mu \longmapsto u_\mu \in \mathcal{S}'(\mathbb{R}^n) \\ u_\mu(\psi) = \int_{\mathbb{R}^n} \psi\,d\mu\,, \end{gathered} \tag{3.2}$$

but we need to know that these maps are injective before we can forget about them.

We can see this using *convolution*. This is a sort of 'product' of functions. To begin with, suppose $v \in \mathcal{C}^0_0(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$. We define a new function by 'averaging $v$ with respect to $\psi$:'

$$v * \psi(x) = \int_{\mathbb{R}^n} v(x-y)\psi(y)\,dy\,. \tag{3.3}$$

The integral converges by dominated convergence, namely $\psi(y)$ is integrable and $v$ is bounded,

$$|v(x-y)\psi(y)| \leq \|v\|_{\mathcal{C}^0_0}\ |\psi(y)|\ .$$

We can use the same sort of estimates to show that $v*\psi$ is continuous. Fix $x \in \mathbb{R}^n$,

$$\text{(3.4)} \quad v*\psi(x+x') - v*\psi(x) \\ = \int (v(x+x'-y) - v(x-y))\psi(y)\,dy\,.$$

To see that this is small for $x'$ small, we split the integral into two pieces. Since $\psi$ is very small near infinity, given $\epsilon > 0$ we can choose $R$ so large that

$$\text{(3.5)} \qquad \|v\|_\infty \cdot \int_{|y]|\geq R} |\psi(y)|\ dy \leq \epsilon/4\,.$$

The set $|y| \leq R$ is compact and if $|x| \leq R'$, $|x'| \leq 1$ then $|x+x'-y| \leq R+R'+1$. A continuous function is *uniformly continuous* on any compact set, so we can chose $\delta > 0$ such that

$$\text{(3.6)} \qquad \sup_{\substack{|x'|<\delta \\ |y|\leq R}} |v(x+x'-y) - v(x-y)| \cdot \int_{|y|\leq R} |\psi(y)|\ dy\ < \epsilon/2\,.$$

Combining (3.5) and (3.6) we conclude that $v*\psi$ is continuous. Finally, we conclude that

$$\text{(3.7)} \qquad v \in \mathcal{C}^0_0(\mathbb{R}^n) \Rightarrow v*\psi \in \mathcal{C}^0_0(\mathbb{R}^n)\,.$$

For this we need to show that $v*\psi$ is small at infinity, which follows from the fact that $v$ is small at infinity. Namely given $\epsilon > 0$ there exists $R > 0$ such that $|v(y)| \leq \epsilon$ if $|y| \geq R$. Divide the integral defining the convolution into two

$$|v*\psi(x)| \leq \int_{|y|>R} u(y)\psi(x-y)dy + \int_{y<R} |u(y)\psi(x-y)|dy \\ \leq \epsilon/2\|\psi\|_\infty + \|u\|_\infty \sup_{B(x,R)} |\psi|.$$

Since $\psi \in \mathcal{S}(\mathbb{R}^n)$ the last constant tends to 0 as $|x| \to \infty$.

We can do much better than this! Assuming $|x'| \leq 1$ we can use Taylor's formula with remainder to write

$$\text{(3.8)} \qquad \psi(z+x') - \psi(z) = \int_0^{\prime} \frac{d}{dt}\psi(z+tx')\,dt = \sum_{j=1}^n x_j \cdot \tilde{\psi}_j(z,x')\,.$$

As Problem 23 I ask you to check carefully that

$$\text{(3.9)} \qquad \psi_j(z;x') \in \mathcal{S}(\mathbb{R}^n) \text{ depends continuously on } x' \text{ in } |x'| \leq 1\,.$$

Going back to (3.3))we can use the translation and reflection-invariance of Lebesgue measure to rewrite the integral (by changing variable) as

$$v * \psi(x) = \int_{\mathbb{R}^n} v(y)\psi(x-y)\,dy\,. \tag{3.10}$$

This reverses the role of $v$ and $\psi$ and shows that if *both* $v$ and $\psi$ are in $\mathcal{S}(\mathbb{R}^n)$ then $v * \psi = \psi * v$.

Using this formula on (3.4) we find

(3.11)

$$\begin{aligned} v * \psi(x+x') - v * \psi(x) &= \int v(y)(\psi(x+x'-y) - \psi(x-y))\,dy \\ &= \sum_{j=1}^{n} x_j \int_{\mathbb{R}^n} v(y)\tilde{\psi}_j(x-y,x')\,dy = \sum_{j=1}^{n} x_j(v * \psi_j(\cdot;x')(x)\,. \end{aligned}$$

From (3.9) and what we have already shown, $v * \psi(\cdot;x')$ is continuous in both variables, and is in $\mathcal{C}^0_0(\mathbb{R}^n)$ in the first. Thus

$$v \in \mathcal{C}^0_0(\mathbb{R}^n)\,,\ \psi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow v * \psi \in \mathcal{C}^1_0(\mathbb{R}^n)\,. \tag{3.12}$$

In fact we also see that

$$\frac{\partial}{\partial x_j} v * \psi = v * \frac{\partial \psi}{\partial x_j}\,. \tag{3.13}$$

Thus $v * \psi$ inherits its regularity from $\psi$.

Proposition 3.1. *If $v \in \mathcal{C}^0_0(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$v * \psi \in \mathcal{C}^\infty_0(\mathbb{R}^n) = \bigcap_{k \ge 0} \mathcal{C}^k_0(\mathbb{R}^n)\,. \tag{3.14}$$

Proof. This follows from (3.12), (3.13) and induction. □

Now, let us make a more special choice of $\psi$. We have shown the existence of

$$\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)\,,\ \varphi \ge 0\,,\ \operatorname{supp}(\varphi) \subset \{|x| \le 1\}\ . \tag{3.15}$$

We can also assume $\int_{\mathbb{R}^n} \varphi\,dx = 1$, by multiplying by a positive constant. Now consider

$$\varphi_t(x) = t^{-n}\varphi\left(\frac{x}{t}\right)\ 1 \ge t > 0\,. \tag{3.16}$$

This has all the same properties, except that

$$\operatorname{supp}\varphi_t \subset \{|x| \le t\}\ ,\ \int \varphi_t\,dx = 1\,. \tag{3.17}$$

Proposition 3.2. *If $v \in \mathcal{C}^0_0(\mathbb{R}^n)$ then as $t \to 0$, $v_t = v * \varphi_t \to v$ in $\mathcal{C}^0_0(\mathbb{R}^n)$.*

PROOF. using (3.17) we can write the difference as

$$
\begin{aligned}
(3.18)\quad |v_t(x) - v(x)| = |\int_{\mathbb{R}^n} (v(x-y) - v(x))\varphi_t(y)\,dy| \\
\le \sup_{|y|\le t} |v(x-y) - v(x)| \to 0.
\end{aligned}
$$

Here we have used the fact that $\varphi_t \ge 0$ has support in $|y| \le t$ and has integral 1. Thus $v_t \to v$ uniformly on any set on which $v$ is uniformly continuous, namel $\mathbb{R}^n$! □

COROLLARY 3.3. *$\mathcal{C}_0^k(\mathbb{R}^n)$ is dense in $\mathcal{C}_0^p(\mathbb{R}^n)$ for any $k \ge p$.*

PROPOSITION 3.4. *$\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{C}_0^k(\mathbb{R}^n)$ for any $k \ge 0$.*

PROOF. Take $k = 0$ first. The subspace $\mathcal{C}_c^0(\mathbb{R}^n)$ is dense in $\mathcal{C}_0^0(\mathbb{R}^n)$, by cutting off outside a large ball. If $v \in \mathcal{C}_c^0(\mathbb{R}^n)$ has support in $\{|x| \le R\}$ then

$$v * \varphi_t \in \mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$$

has support in $\{|x| \le R + 1\}$. Since $v * \varphi_t \to v$ the result follows for $k = 0$.

For $k \ge 1$ the same argument works, since $D^\alpha(v * \varphi_t) = (D^\alpha V) * \varphi_t$. □

COROLLARY 3.5. *The map from finite Radon measures*

$$M_{fin}(\mathbb{R}^n) \ni \mu \longmapsto u_\mu \in \mathcal{S}'(\mathbb{R}^n) \tag{3.19}$$

*is injective.*

Now, we want the same result for $L^2(\mathbb{R}^n)$ (and maybe for $L^p(\mathbb{R}^n)$, $1 \le p < \infty$). I leave the measure-theoretic part of the argument to you.

PROPOSITION 3.6. *Elements of $L^2(\mathbb{R}^n)$ are "continuous in the mean" i.e.,*

$$\lim_{|t|\to 0} \int_{\mathbb{R}^n} |u(x+t) - u(x)|^2 \, dx = 0\,. \tag{3.20}$$

This is Problem 24.

Using this we conclude that

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n) \text{ is dense} \tag{3.21}$$

as before. First observe that the space of $L^2$ functions of compact support is dense in $L^2(\mathbb{R}^n)$, since

$$\lim_{R\to\infty} \int_{|x|\ge R} |u(x)|^2 \, dx = 0 \;\forall\; u \in L^2(\mathbb{R}^n)\,.$$

Then look back at the discussion of $v * \varphi$, now $v$ is replaced by $u \in L^2_c(\mathbb{R}^n)$. The compactness of the support means that $u \in L^1(\mathbb{R}^n)$ so in

$$u * \varphi(x) = \int_{\mathbb{R}^n} u(x-y)\varphi(y)dy \tag{3.22}$$

the integral is absolutely convergent. Moreover

$$\begin{aligned}
&|u * \varphi(x+x') - u * \varphi(x)| \\
&\qquad = \left| \int u(y)(\varphi(x+x'-y) - \varphi(x-y))\, dy \right| \\
&\qquad\qquad \leq C\|u\| \sup_{|y|\leq R} |\varphi(x+x'-y) - \varphi(x-y)| \to 0
\end{aligned}$$

when $\{|x| \leq R\}$ large enough. Thus $u * \varphi$ is continuous and the same argument as before shows that

$$u * \varphi_t \in \mathcal{S}(\mathbb{R}^n)\,.$$

Now to see that $u * \varphi_t \to u$, assuming $u$ has compact support (or not) we estimate the integral

$$\begin{aligned}
|u * \varphi_t(x) - u(x)| &= \left| \int (u(x-y) - u(x))\varphi_t(y)\, dy \right| \\
&\leq \int |u(x-y) - u(x)|\, \varphi_t(y)\, dy\,.
\end{aligned}$$

Using the same argument twice

$$\begin{aligned}
&\int |u * \varphi_t(x) - u(x)|^2\, dx \\
&\quad \leq \iiint |u(x-y) - u(x)|\, \varphi_t(y)\, |u(x-y') - u(x)|\, \varphi_t(y')\, dx\, dy\, dy' \\
&\qquad \leq \left( \int |u(x-y) - u(x)|^2\, \varphi_t(y)\varphi_t(y') dx\, dy\, dy' \right) \\
&\qquad\qquad \leq \sup_{|y|\leq t} \int |u(x-y) - u(x)|^2\, dx\,.
\end{aligned}$$

Note that at the second step here I have used Schwarz's inequality with the integrand written as the product

$$|u(x-y) - u(x)|\, \varphi_t^{1/2}(y)\varphi_t^{1/2}(y') \cdot |u(x-y') - u(x)|\, \varphi_t^{1/2}(y)\varphi_t^{1/2}(y')\,.$$

Thus we now know that

$$L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \text{ is injective.}$$

This means that all our usual spaces of functions 'sit inside' $\mathcal{S}'(\mathbb{R}^n)$.

Finally we can use convolution with $\varphi_t$ to show the existence of *smooth* partitions of unity. If $K \Subset U \subset \mathbb{R}^n$ is a compact set in an open set then we have shown the existence of $\xi \in \mathcal{C}^0_c(\mathbb{R}^n)$, with $\xi = 1$ in some neighborhood of $K$ and $\xi = 1$ in some neighborhood of $K$ and $\mathrm{supp}(\xi) \Subset U$.

Then consider $\xi * \varphi_t$ for $t$ small. In fact

$$\mathrm{supp}(\xi * \varphi_t) \subset \{p \in \mathbb{R}^n\,;\, \mathrm{dist}(p, \mathrm{supp}\,\xi) \leq 2t\}$$

and similarly, $0 \leq \xi * \varphi_t \leq 1$ and

$$\xi * \varphi_t = 1 \text{ at } p \text{ if } \xi = 1 \text{ on } B(p, 2t)\,.$$

Using this we get:

PROPOSITION 3.7. *If $U_a \subset \mathbb{R}^n$ are open for $a \in A$ and $K \Subset \bigcup_{a\in A} U_a$ then there exist finitely many $\varphi_i \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, with $0 \leq \varphi_i \leq 1$, $\mathrm{supp}(\varphi_i) \subset U_{a_i}$ such that $\sum\limits_i \varphi_i = 1$ in a neighbourhood of $K$.*

PROOF. By the compactness of $K$ we may choose a finite open subcover. Using Lemma 15.7 we may choose a continuous partition, $\phi'_i$, of unity subordinate to this cover. Using the convolution argument above we can replace $\phi'_i$ by $\phi'_i * \varphi_t$ for $t > 0$. If $t$ is sufficiently small then this is again a partition of unity subordinate to the cover, but now smooth. □

Next we can make a simple 'cut off argument' to show

LEMMA 3.8. *The space $\mathcal{C}^\infty_c(\mathbb{R}^n)$ of $\mathcal{C}^\infty$ functions of compact support is dense in $\mathcal{S}(\mathbb{R}^n)$.*

PROOF. Choose $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\varphi(x) = 1$ in $|x| \leq 1$. Then given $\psi \in \mathcal{S}(\mathbb{R}^n)$ consider the sequence

$$\psi_n(x) = \varphi(x/n)\psi(x)\,.$$

Clearly $\psi_n = \psi$ on $|x| \leq n$, so if it converges in $\mathcal{S}(\mathbb{R}^n)$ it must converge to $\psi$. Suppose $m \geq n$ then by Leibniz's formula[4]

$$D^\alpha_x(\psi_n(x) - \psi_m(x)) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} D^\beta_x \left(\varphi(\frac{x}{n}) - \varphi(\frac{x}{m})\right) \cdot D^{\alpha-\beta}_x \psi(x)\,.$$

All derivatives of $\varphi(x/n)$ are bounded, independent of $n$ and $\psi_n = \psi_m$ in $|x| \leq n$ so for any $p$

$$|D^\alpha_x(\psi_n(x) - \psi_m(x))| \leq \begin{cases} 0 & |x| \leq n \\ C_{\alpha,p}\langle x\rangle^{-2p} & |x| \geq n \end{cases}\,.$$

[4]Problem 25.

Hence $\psi_n$ is Cauchy in $\mathcal{S}(\mathbb{R}^n)$. $\square$

Thus every element of $\mathcal{S}'(\mathbb{R}^n)$ is determined by its restriction to $\mathcal{C}_c^\infty(\mathbb{R}^n)$. The support of a tempered distribution was defined above to be

$$\operatorname{supp}(u) = \{x \in \mathbb{R}^n;\ \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi(x) \neq 0\,,\ \varphi u = 0\}^{\complement}\,. \tag{3.23}$$

Using the preceding lemma and the construction of smooth partitions of unity we find

PROPOSITION 3.9. *f* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\operatorname{supp}(u) = \emptyset$ *then* $u = 0$.

PROOF. From (3.23), if $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}(\psi u) \subset \operatorname{supp}(u)$. If $x \ni \operatorname{supp}(u)$ then, by definition, $\varphi u = 0$ for some $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(x) \neq 0$. Thus $\varphi \neq 0$ on $B(x, \epsilon)$ for $\epsilon > 0$ sufficiently small. If $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has support in $B(x, \epsilon)$ then $\psi u = \tilde{\psi}\varphi u = 0$, where $\tilde{\psi} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$:

$$\tilde{\psi} = \begin{cases} \psi/\varphi & \text{in } B(x,\epsilon) \\ 0 & \text{elsewhere}\,. \end{cases}$$

Thus, given $K \Subset \mathbb{R}^n$ we can find $\varphi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, supported in such balls, so that $\sum_j \varphi_j \equiv 1$ on $K$ but $\varphi_j u = 0$. For given $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ apply this to $\operatorname{supp}(\mu)$. Then

$$\mu = \sum_j \varphi_j \mu \Rightarrow u(\mu) = \sum_j (\phi_j u)(\mu) = 0\,.$$

Thus $u = 0$ on $\mathcal{C}_c^\infty(\mathbb{R}^n)$, so $u = 0$. $\square$

The linear space of distributions of compact support will be denoted $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$; it is often written $\mathcal{E}'(\mathbb{R}^n)$.

Now let us give a characterization of the 'delta function'

$$\delta(\varphi) = \varphi(0)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,,$$

or at least the one-dimensional subspace of $\mathcal{S}'(\mathbb{R}^n)$ it spans. This is based on the simple observation that $(x_j\varphi)(0) = 0$ if $\varphi \in \mathcal{S}(\mathbb{R}^n)$!

PROPOSITION 3.10. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *satisfies* $x_j u = 0$, $j = 1, \cdots, n$ *then* $u = c\delta$.

PROOF. The main work is in characterizing the null space of $\delta$ as a linear functional, namely in showing that

$$\mathcal{H} = \{\varphi \in \mathcal{S}(\mathbb{R}^n);\ \varphi(0) = 0\} \tag{3.24}$$

can also be written as

$$\mathcal{H} = \left\{\varphi \in \mathcal{S}(\mathbb{R}^n);\ \varphi = \sum_{j=1}^n x_j\psi_j\,,\ \varphi_j \in \mathcal{S}(\mathbb{R}^n)\right\}\,. \tag{3.25}$$

Clearly the right side of (3.25) is contained in the left. To see the converse, suppose first that

$$\varphi \in \mathcal{S}(\mathbb{R}^n)\,,\ \varphi = 0 \text{ in } |x| < 1\,. \tag{3.26}$$

Then define

$$\psi = \begin{cases} 0 & |x| < 1 \\ \varphi/\left|x\right|^2 & |x| \geq 1\,. \end{cases}$$

All the derivatives of $1/\left|x\right|^2$ are bounded in $|x| \geq 1$, so from Leibniz's formula it follows that $\psi \in \mathcal{S}(\mathbb{R}^n)$. Since

$$\varphi = \sum_j x_j(x_j\psi)$$

this shows that $\varphi$ of the form (3.26) is in the right side of (3.25). In general suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then

$$\begin{aligned} \varphi(x) - \varphi(0) &= \int_0^t \frac{d}{dt}\varphi(tx)\,dt \\ &= \sum_{j=1}^n x_j \int_0^t \frac{\partial\varphi}{\partial x_j}(tx)\,dt\,. \end{aligned} \tag{3.27}$$

Certainly these integrals are $\mathcal{C}^\infty$, but they may not decay rapidly at infinity. However, choose $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\mu = 1$ in $|x| \leq 1$. Then (3.27) becomes, if $\varphi(0) = 0$,

$$\begin{aligned} \varphi &= \mu\varphi + (1-\mu)\varphi \\ &= \sum_{j=1}^n x_j\psi_j + (1-\mu)\varphi\,,\ \psi_j = \mu\int_0^t \frac{\partial\varphi}{\partial x_j}(tx)\,dt \in \mathcal{S}(\mathbb{R}^n)\,. \end{aligned}$$

Since $(1-\mu)\varphi$ is of the form (3.26), this proves (3.25).

Our assumption on $u$ is that $x_ju = 0$, thus

$$u(\varphi) = 0 \ \forall\ \varphi \in \mathcal{H}$$

by (3.25). Choosing $\mu$ as above, a general $\varphi \in \mathcal{S}(\mathbb{R}^n)$ can be written

$$\varphi = \varphi(0)\cdot\mu + \varphi'\,,\ \varphi' \in \mathcal{H}\,.$$

Then

$$u(\varphi) = \varphi(0)u(\mu) \Rightarrow u = c\delta\,,\ c = u(\mu)\,.$$

$\square$

This result is quite powerful, as we shall soon see. The Fourier transform of an element $\varphi \in \mathcal{S}(\mathbb{R}^n)$ is[5]

$$\hat{\varphi}(\xi) = \int e^{-ix\cdot\xi}\varphi(x)\,dx\,,\ \xi \in \mathbb{R}^n\,. \tag{3.28}$$

The integral certainly converges, since $|\varphi| \le C\langle x\rangle^{-n-1}$. In fact it follows easily that $\hat{\varphi}$ is continuous, since

$$\begin{aligned}|\hat{\varphi}(\xi) - \hat{\varphi}(\xi')| \in \int \left|e^{ix-\xi} - e^{-x\cdot\xi'}\right| |\varphi|\ dx \\ \to 0 \text{ as } \xi' \to \xi\,.\end{aligned}$$

In fact

Proposition 3.11. *Fourier transformation,* (3.28)*, defines a continuous linear map*

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,\ \mathcal{F}\varphi = \hat{\varphi}\,. \tag{3.29}$$

Proof. Differentiating under the integral[6] sign shows that

$$\partial_{\xi_j}\hat{\varphi}(\xi) = -i\int e^{-ix\cdot\xi}x_j\varphi(x)\,dx\,.$$

Since the integral on the right *is* absolutely convergent that shows that (remember the $i$'s)

$$D_{\xi_j}\hat{\varphi} = -\widehat{x_j\varphi}\,,\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{3.30}$$

Similarly, if we multiply by $\xi_j$ and observe that $\xi_j e^{-ix\cdot\xi} = i\frac{\partial}{\partial x_j}e^{-ix\cdot\xi}$ then integration by parts shows

$$\begin{aligned}\xi_j\hat{\varphi} &= i\int(\frac{\partial}{\partial x_j}e^{-ix\cdot\xi})\varphi(x)\,dx \\ &= -i\int e^{-ix\cdot\xi}\frac{\partial\varphi}{\partial x_j}\,dx \\ \widehat{D_j\varphi} &= \xi_j\hat{\varphi}\,,\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,.\end{aligned} \tag{3.31}$$

Since $x_j\varphi$, $D_j\varphi \in \mathcal{S}(\mathbb{R}^n)$ these results can be iterated, showing that

$$\xi^\alpha D_\xi^\beta\hat{\varphi} = \mathcal{F}\left((-1)^{|\beta|}D^\alpha{}_x x^\beta\varphi\right)\,. \tag{3.32}$$

Thus $\left|\xi^\alpha D_\xi^\beta\hat{\varphi}\right| \le C_{\alpha\beta}\sup\left|\langle x\rangle^{+n+1}D^\alpha{}_x x^\beta\varphi\right| \le C\|\langle x\rangle^{n+1+|\beta|}\varphi\|_{\mathcal{C}^{|\alpha|}}$, which shows that $\mathcal{F}$ is continuous as a map (3.32).

□

[5] Normalizations vary, but it doesn't matter much.

[6] See [**6**]

Suppose $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Since $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ we can consider the distribution $u \in \mathcal{S}'(\mathbb{R}^n)$

$$u(\varphi) = \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\xi\,. \tag{3.33}$$

The continuity of $u$ follows from the fact that integration is continuous and (3.29). Now observe that

$$\begin{aligned} u(x_j\varphi) &= \int_{\mathbb{R}^n} \widehat{x_j\varphi}(\xi)\, d\xi \\ &= -\int_{\mathbb{R}^n} D_{\xi_j}\hat{\varphi}\, d\xi = 0 \end{aligned}$$

where we use (3.30). Applying Proposition 3.10 we conclude that $u = c\delta$ for some (universal) constant $c$. By definition this means

$$\int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\xi = c\varphi(0)\,. \tag{3.34}$$

So what is the constant? To find it we need to work out an example. The simplest one is

$$\varphi = \exp(-|x|^2/2)\,.$$

Lemma 3.12. *The Fourier transform of the Gaussian* $\exp(-|x|^2/2)$ *is the Gaussian* $(2\pi)^{n/2}\exp(-|\xi|^2/2)$.

Proof. There are two obvious methods — one uses complex analysis (Cauchy's theorem) the other, which I shall follow, uses the uniqueness of solutions to ordinary differential equations.

First observe that $\exp(-|x|^2/2) = \prod_j \exp(-x_j^2/2)$. Thus[7]

$$\hat{\varphi}(\xi) = \prod_{j=1}^{n} \hat{\psi}(\xi_j)\,,\ \psi(x) = e^{-x^2/2}\,,$$

being a function of one variable. Now $\psi$ satisfies the differential equation

$$(\partial_x + x)\,\psi = 0\,,$$

and is the *only* solution of this equation up to a constant multiple. By (3.30) and (3.31) its Fourier transform satisfies

$$\widehat{\partial_x\psi} + \widehat{x\psi} = i\xi\hat{\psi} + i\frac{d}{d\xi}\hat{\varphi} = 0\,.$$

[7]Really by Fubini's theorem, but here one can use Riemann integrals.

This is the same equation, but in the $\xi$ variable. Thus $\hat{\psi} = ce^{-|\xi|^2/2}$. Again we need to find the constant. However,

$$\hat{\psi}(0) = c = \int e^{-x^2/2}\, dx = (2\pi)^{1/2}$$

by the standard use of polar coordinates:

$$c^2 = \int_{\mathbb{R}^n} e^{-(x^2+y^2)/2}\, dx\, dy = \int_0^\infty \int_0^{2\pi} e^{-r^2/2} r\, dr\, d\theta = 2\pi\,.$$

This proves the lemma.

□

Thus we have shown that for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\xi = (2\pi)^n \varphi(0)\,. \tag{3.35}$$

Since this is true for $\varphi = \exp(-|x|^2/2)$. The identity allows us to *invert* the Fourier transform.

## 4. Fourier inversion

It is shown above that the Fourier transform satisfies the identity

$$\varphi(0) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{\varphi}(\xi)\, d\xi \;\forall\; \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{4.1}$$

If $y \in \mathbb{R}^n$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ set $\psi(x) = \varphi(x+y)$. The translation-invariance of Lebesgue measure shows that

$$\begin{aligned}\hat{\psi}(\xi) &= \int e^{-ix\cdot\xi} \varphi(x+y)\, dx \\ &= e^{iy\cdot\xi}\hat{\varphi}(\xi)\,.\end{aligned}$$

Applied to $\psi$ the inversion formula (4.1) becomes

$$\begin{aligned}\varphi(y) = \psi(0) &= (2\pi)^{-n} \int \hat{\psi}(\xi)\, d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\cdot\xi}\hat{\varphi}(\xi)\, d\xi\,.\end{aligned} \tag{4.2}$$

THEOREM 4.1. *Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is an isomorphism with inverse*

$$\mathcal{G} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,,\; \mathcal{G}\psi(y) = (2\pi)^{-n} \int e^{iy\cdot\xi}\psi(\xi)\, d\xi\,. \tag{4.3}$$

PROOF. The identity (4.2) shows that $\mathcal{F}$ is $1-1$, i.e., injective, since we can remove $\varphi$ from $\hat{\varphi}$. Moreover,

$$\mathcal{G}\psi(y) = (2\pi)^{-n}\mathcal{F}\psi(-y) \tag{4.4}$$

So $\mathcal{G}$ is also a continuous linear map, $\mathcal{G} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$. Indeed the argument above shows that $\mathcal{G} \circ \mathcal{F} = Id$ and the same argument, with some changes of sign, shows that $\mathcal{F} \cdot \mathcal{G} = Id$. Thus $F$ and $\mathcal{G}$ are isomorphisms.

□

LEMMA 4.2. *For all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, Paseval's identity holds:*

$$\int_{\mathbb{R}^n} \varphi\overline{\psi}\,dx = (2\pi)^{-n}\int_{\mathbb{R}^n} \hat{\varphi}\overline{\hat{\psi}}\,d\xi\,. \tag{4.5}$$

PROOF. Using the inversion formula on $\varphi$,

$$\begin{aligned}\int \varphi\overline{\psi}\,dx &= (2\pi)^{-n}\int \left(e^{ix\cdot\xi}\hat{\varphi}(\xi)\,d\xi\right)\overline{\psi}(x)\,dx \\ &= (2\pi)^{-n}\int \hat{\varphi}(\xi)\overline{\int e^{-ix\cdot\xi}\psi(x)\,dx}\,d\xi \\ &= (2\pi)^{-n}\int \hat{\varphi}(\xi)\overline{\hat{\varphi}}(\xi)\,d\xi\,.\end{aligned}$$

Here the integrals are absolutely convergent, justifying the exchange of orders.

□

PROPOSITION 4.3. *Fourier transform extends to an isomorphism*

$$\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\,. \tag{4.6}$$

PROOF. Setting $\varphi = \psi$ in (4.5) shows that

$$\|\mathcal{F}\varphi\|_{L^2} = (2\pi)^{n/2}\|\varphi\|_{L^2}\,. \tag{4.7}$$

In particular this proves, given the known density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, that $\mathcal{F}$ is an isomorphism, with inverse $\mathcal{G}$, as in (4.6).

□

For any $m \in \mathbb{R}$

$$\langle x\rangle^m L^2(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n)\,;\, \langle x\rangle^{-m}\hat{u} \in L^2(\mathbb{R}^n)\right\}$$

is a well-defined subspace. We define the *Sobolev spaces* on $\mathbb{R}^n$ by, for $m \geq 0$

$$H^m(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n)\,;\, \hat{u} = \mathcal{F}u \in \langle\xi\rangle^{-m}L^2(\mathbb{R}^n)\right\}\,. \tag{4.8}$$

$$\text{Thus } H^m(\mathbb{R}^n) \subset H^{m'}(\mathbb{R}^n) \text{ if } m \geq m'\,,\ H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)\,.$$

LEMMA 4.4. *If $m \in \mathbb{N}$ is an integer, then*

$$u \in H^m(\mathbb{R}^n) \Leftrightarrow D^\alpha u \in L^2(\mathbb{R}^n) \ \forall \ |\alpha| \leq m\,. \tag{4.9}$$

PROOF. By definition, $u \in H^m(\mathbb{R}^n)$ implies that $\langle\xi\rangle^{-m}\hat{u} \in L^2(\mathbb{R}^n)$. Since $\widehat{D^\alpha u} = \xi^\alpha \hat{u}$ this certainly implies that $D^\alpha u \in L^2(\mathbb{R}^n)$ for $|\alpha| \leq m$. Conversely if $D^\alpha u \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$ then $\xi^\alpha \hat{u} \in L^2(\mathbb{R}^n)$ for all $|\alpha| \leq m$ and since

$$\langle\xi\rangle^m \leq C_m \sum_{|\alpha|\leq m} |\xi^\alpha| \,.$$

this in turn implies that $\langle\xi\rangle^m \hat{u} \in L^2(\mathbb{R}^n)$.

□

Now that we have considered the Fourier transform of Schwartz test functions we can use the usual method, of duality, to extend it to tempered distributions. If we set $\eta = \overline{\hat{\psi}}$ then $\hat{\psi} = \overline{\eta}$ and $\psi = \mathcal{G}\hat{\psi} = \mathcal{G}\overline{\eta}$ so

$$\begin{aligned}\overline{\psi}(x) = (2\pi)^{-n} \int e^{-ix\cdot\xi}\overline{\hat{\psi}}(\xi)\,d\xi \\ = (2\pi)^{-n} \int e^{-ix\cdot\xi}\eta(\xi)\,d\xi = (2\pi)^{-n}\hat{\eta}(x).\end{aligned}$$

Substituting in (4.5) we find that

$$\int \varphi\hat{\eta}\,dx = \int \hat{\varphi}\eta\,d\xi\,.$$

Now, recalling how we embed $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ we see that

$$u_{\hat{\varphi}}(\eta) = u_\varphi(\hat{\eta}) \ \forall \ \eta \in \mathcal{S}(\mathbb{R}^n)\,. \tag{4.10}$$

DEFINITION 4.5. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ we* define *its Fourier transform by*

$$\hat{u}(\varphi) = u(\hat{\varphi}) \ \forall \ \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{4.11}$$

As a composite map, $\hat{u} = u \cdot \mathcal{F}$, with each term continuous, $\hat{u}$ is continuous, i.e., $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$.

PROPOSITION 4.6. *The definition* (4.7) *gives an isomorphism*

$$\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,,\ \mathcal{F}u = \hat{u}$$

*satisfying the identities*

$$\widehat{D^\alpha u} = \xi^\alpha u\,,\ \widehat{x^\alpha u} = (-1)^{|\alpha|}D^\alpha\hat{u}\,. \tag{4.12}$$

Proof. Since $\hat{u} = u \circ \mathcal{F}$ and $\mathcal{G}$ is the 2-sided inverse of $\mathcal{F}$,

$$u = \hat{u} \circ \mathcal{G} \tag{4.13}$$

gives the inverse to $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, showing it to be an isomorphism. The identities (4.12) follow from their counterparts on $\mathcal{S}(\mathbb{R}^n)$:

$$\begin{aligned}\widehat{D^\alpha u}(\varphi) &= D^\alpha u(\hat{\varphi}) = u((-1)^{|\alpha|} D^\alpha \hat{\varphi}) \\ &= u(\widehat{\xi^\alpha \varphi}) = \hat{u}(\xi^\alpha \varphi) = \xi^\alpha \hat{u}(\varphi)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)\,.\end{aligned}$$

□

We can also define Sobolev spaces of *negative* order:

$$H^m(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n)\,;\, \hat{u} \in \langle \xi \rangle^{-m} L^2(\mathbb{R}^n) \right\}\,. \tag{4.14}$$

Proposition 4.7. *If $m \le 0$ is an integer then $u \in H^m(\mathbb{R}^n)$ if and only if it can be written in the form*

$$u = \sum_{|\alpha| \le -m} D^\alpha v_\alpha\,,\ v_\alpha \in L^2(\mathbb{R}^n)\,. \tag{4.15}$$

Proof. If $u \in \mathcal{S}'(\mathbb{R}^n)$ is of the form (4.15) then

$$\hat{u} = \sum_{|\alpha| \le -m} \xi^\alpha \hat{v}_\alpha \text{ with } \hat{v}\alpha \in L^2(\mathbb{R}^n)\,. \tag{4.16}$$

Thus $\langle \xi \rangle^m \hat{u} = \sum_{|\alpha| \le -m} \xi^\alpha \langle \xi \rangle^m \hat{v}_\alpha$. Since all the factors $\xi^\alpha \langle \xi \rangle^m$ are bounded, each term here is in $L^2(\mathbb{R}^n)$, so $\langle \xi \rangle^m \hat{u} \in L^2(\mathbb{R}^n)$ which is the definition, $u \in \langle \xi \rangle^{-m} L^2(\mathbb{R}^n)$.

Conversely, suppose $u \in H^m(\mathbb{R}^n)$, i.e., $\langle \xi \rangle^m \hat{u} \in L^2(\mathbb{R}^n)$. The function

$$\left( \sum_{|\alpha| \le -m} |\xi^\alpha| \right) \cdot \langle \xi \rangle^m \in L^2(\mathbb{R}^n)\ (m < 0)$$

is bounded below by a positive constant. Thus

$$v = \left( \sum_{|\alpha| \le -m} |\xi^\alpha| \right)^{-1} \hat{u} \in L^2(\mathbb{R}^n)\,.$$

Each of the functions $\hat{v}_\alpha = \operatorname{sgn}(\xi^\alpha) \hat{v} \in L^2(\mathbb{R}^n)$ so the identity (4.16), and hence (4.15), follows with these choices.

□

PROPOSITION 4.8. *Each of the Sobolev spaces $H^m(\mathbb{R}^n)$ is a Hilbert space with the norm and inner product*

$$\|u\|_{H^m} = \left(\int_{\mathbb{R}^n} |\hat{u}(\xi)|^2 \langle\xi\rangle^{2m}\, d\xi\right)^{1/2}, \tag{4.17}$$

$$\langle u, v\rangle = \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}\langle\xi\rangle^{2m}\, d\xi\,.$$

*The Schwartz space $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^m(\mathbb{R}^n)$ is dense for each $m$ and the pairing*

$$H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n) \ni (u, u') \longmapsto \tag{4.18}$$

$$((u, u')) = \int_{\mathbb{R}^n} \hat{u'}(\xi)\hat{u'}(\cdot\xi)\, d\xi \in \mathbb{C}$$

*gives an identification $(H^m(\mathbb{R}^n))' = H^{-m}(\mathbb{R}^n)$.*

PROOF. The Hilbert space property follows essentially directly from the definition (4.14) since $\langle\xi\rangle^{-m}L^2(\mathbb{R}^n)$ is a Hilbert space with the norm (4.17). Similarly the density of $\mathcal{S}$ in $H^m(\mathbb{R}^n)$ follows, since $\mathcal{S}(\mathbb{R}^n)$ dense in $L^2(\mathbb{R}^n)$ (Problem L11.P3) implies $\langle\xi\rangle^{-m}\mathcal{S}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)$ is dense in $\langle\xi\rangle^{-m}L^2(\mathbb{R}^n)$ and so, since $\mathcal{F}$ is an isomorphism in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$.

Finally observe that the pairing in (4.18) makes sense, since $\langle\xi\rangle^{-m}\hat{u}(\xi)$, $\langle\xi\rangle^m\hat{u'}(\xi) \in L^2(\mathbb{R}^n)$ implies

$$\hat{u}(\xi))\hat{u'}(-\xi) \in L^1(\mathbb{R}^n)\,.$$

Furthermore, by the self-duality of $L^2(\mathbb{R}^n)$ each continuous linear functional

$$U : H^m(\mathbb{R}^n) \to \mathbb{C}\,,\; U(u) \le C\|u\|_{H^m}$$

can be written uniquely in the form

$$U(u) = ((u, u')) \text{ for some } u' \in H^{-m}(\mathbb{R}^n)\,.$$

□

Notice that if $u$, $u' \in \mathcal{S}(\mathbb{R}^n)$ then

$$((u, u')) = \int_{\mathbb{R}^n} u(x)u'(x)\, dx\,.$$

This is always how we "pair" functions — it is the natural pairing on $L^2(\mathbb{R}^n)$. Thus in (4.18) what we have shown is that this pairing on test function

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (u, u') \longmapsto ((u, u')) = \int_{\mathbb{R}^n} u(x)u'(x)\, dx$$

extends by *continuity* to $H^m(\mathbb{R}^n) \times H^{-m}(\mathbb{R}^n)$ (for each fixed $m$) when it identifies $H^{-m}(\mathbb{R}^n)$ as the dual of $H^m(\mathbb{R}^n)$. This was our 'picture' at the beginning.

For $m > 0$ the spaces $H^m(\mathbb{R}^n)$ represents elements of $L^2(\mathbb{R}^n)$ that have "$m$" derivatives in $L^2(\mathbb{R}^n)$. For $m < 0$ the elements are ?? of "up to $-m$" derivatives of $L^2$ functions. For integers this is precisely ??.

## 5. Sobolev embedding

The properties of Sobolev spaces are briefly discussed above. If $m$ is a positive integer then $u \in H^m(\mathbb{R}^n)$ 'means' that $u$ has up to $m$ derivatives in $L^2(\mathbb{R}^n)$. The question naturally arises as to the sense in which these 'weak' derivatives correspond to old-fashioned 'strong' derivatives. Of course when $m$ is not an integer it is a little harder to imagine what these 'fractional derivatives' are. However the main result is:

THEOREM 5.1 (Sobolev embedding). *If $u \in H^m(\mathbb{R}^n)$ where $m > n/2$ then $u \in \mathcal{C}_0^0(\mathbb{R}^n)$, i.e.,*

$$H^m(\mathbb{R}^n) \subset \mathcal{C}_0^0(\mathbb{R}^n)\,,\ m > n/2\,. \tag{5.1}$$

PROOF. By definition, $u \in H^m(\mathbb{R}^n)$ means $v \in \mathcal{S}'(\mathbb{R}^n)$ and $\langle\xi\rangle^m \hat{u}(\xi) \in L^2(\mathbb{R}^n)$. Suppose first that $u \in \mathcal{S}(\mathbb{R}^n)$. The Fourier inversion formula shows that

$$\begin{aligned}(2\pi)^n\,|u(x)| &= \left|\int e^{ix\cdot\xi}\hat{u}(\xi)\,d\xi\right| \\ &\leq \left(\int_{\mathbb{R}^n}\langle\xi\rangle^{2m}\,|\hat{u}(\xi)|^2\,d\xi\right)^{1/2}\cdot\left(\sum_{\mathbb{R}^n}\langle\xi\rangle^{-2m}\,d\xi\right)^{1/2}.\end{aligned}$$

Now, if $m > n/2$ then the second integral is finite. Since the first integral is the norm on $H^m(\mathbb{R}^n)$ we see that

$$\sup_{\mathbb{R}^n}|u(x)| = \|u\|_{L^\infty} \leq (2\pi)^{-n}\|u\|_{H^m}\,,\ m > n/2\,. \tag{5.2}$$

This is all for $u \in \mathcal{S}(\mathbb{R}^n)$, but $\mathcal{S}(\mathbb{R}^n) \hookrightarrow H^m(\mathbb{R}^n)$ is dense. The estimate (5.2) shows that if $u_j \to u$ in $H^m(\mathbb{R}^n)$, with $u_j \in \mathcal{S}(\mathbb{R}^n)$, then $u_j \to u'$ in $\mathcal{C}_0^0(\mathbb{R}^n)$. In fact $u' = u$ in $\mathcal{S}'(\mathbb{R}^n)$ since $u_j \to u$ in $L^2(\mathbb{R}^n)$ and $u_j \to u'$ in $\mathcal{C}_0^0(\mathbb{R}^n)$ both imply that $\int u_j\varphi$ converges, so

$$\int_{\mathbb{R}^n} u_j\varphi \to \int_{\mathbb{R}^n} u\varphi = \int_{\mathbb{R}^n} u'\varphi \ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

$\square$

Notice here the precise meaning of $u = u'$, $u \in H^m(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, $u' \in \mathcal{C}_0^0(\mathbb{R}^n)$. When identifying $u \in L^2(\mathbb{R}^n)$ with the corresponding tempered distribution, the values on any set of measure zero 'are lost'. Thus as *functions* (5.1) means that each $u \in H^m(\mathbb{R}^n)$ has a representative $u' \in \mathcal{C}_0^0(\mathbb{R}^n)$.

We can extend this to higher derivatives by noting that

PROPOSITION 5.2. *If $u \in H^m(\mathbb{R}^n)$, $m \in \mathbb{R}$, then $D^\alpha u \in H^{m-|\alpha|}(\mathbb{R}^n)$ and*

$$D^\alpha : H^m(\mathbb{R}^n) \to H^{m-|\alpha|}(\mathbb{R}^n) \tag{5.3}$$

*is continuous.*

PROOF. First it is enough to show that each $D_j$ defines a continuous linear map

$$D_j : H^m(\mathbb{R}^n) \to H^{m-1}(\mathbb{R}^n) \ \forall \ j \tag{5.4}$$

since then (5.3) follows by composition.

If $m \in \mathbb{R}$ then $u \in H^m(\mathbb{R}^n)$ means $\hat{u} \in \langle\xi\rangle^{-m} L^2(\mathbb{R}^n)$. Since $\widehat{D_j u} = \xi_j \cdot \hat{u}$, and

$$|\xi_j| \, \langle\xi\rangle^{-m} \le C_m \langle\xi\rangle^{-m+1} \ \forall \ m$$

we conclude that $D_j u \in H^{m-1}(\mathbb{R}^n)$ and

$$\|D_j u\|_{H^{m-1}} \le C_m \|u\|_{H^m} \, .$$

□

Applying this result we see

COROLLARY 5.3. *If $k \in \mathbb{N}_0$ and $m > \frac{n}{2} + k$ then*

$$H^m(\mathbb{R}^n) \subset \mathcal{C}_0^k(\mathbb{R}^n) \, . \tag{5.5}$$

PROOF. If $|\alpha| \le k$, then $D^\alpha u \in H^{m-k}(\mathbb{R}^n) \subset \mathcal{C}_0^0(\mathbb{R}^n)$. Thus the 'weak derivatives' $D^\alpha u$ are continuous. Still we have to check that this means that $u$ is itself $k$ times continuously differentiable. In fact this again follows from the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$. The continuity in (5.3) implies that if $u_j \to u$ in $H^m(\mathbb{R}^n)$, $m > \frac{n}{2} + k$, then $u_j \to u'$ in $\mathcal{C}_0^k(\mathbb{R}^n)$ (using its completeness). However $u = u'$ as before, so $u \in \mathcal{C}_0^k(\mathbb{R}^n)$.

□

In particular we see that

$$H^\infty(\mathbb{R}^n) = \bigcap_m H^m(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n) \, . \tag{5.6}$$

These functions are not in general Schwartz test functions.

Proposition 5.4. *Schwartz space can be written in terms of weighted Sobolev spaces*

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_k \langle x\rangle^{-k} H^k(\mathbb{R}^n)\,. \tag{5.7}$$

Proof. This follows directly from (5.5) since the left side is contained in

$$\bigcap_k \langle x\rangle^{-k} \mathcal{C}_0^{k-n}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n).$$

□

Theorem 5.5 (Schwartz representation). *Any tempered distribution can be written in the form of a finite sum*

$$u = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} x^\alpha D_x^\beta u_{\alpha\beta}\,,\ u_{\alpha\beta} \in \mathcal{C}_0^0(\mathbb{R}^n). \tag{5.8}$$

or *in the form*

$$u = \sum_{\substack{|\alpha|\le m\\ |\beta|\le m}} D_x^\beta (x^\alpha v_{\alpha\beta}),\ \ v_{\alpha\beta} \in \mathcal{C}_0^0(\mathbb{R}^n). \tag{5.9}$$

Thus every tempered distribution is a finite sum of derivatives of continuous functions of poynomial growth.

Proof. Essentially by definition any $u \in \mathcal{S}'(\mathbb{R}^n)$ is continuous with respect to *one* of the norms $\|\langle x\rangle^k \varphi\|_{\mathcal{C}^k}$. From the Sobolev embedding theorem we deduce that, with $m > k + n/2$,

$$|u(\varphi)| \le C\|\langle x\rangle^k\varphi\|_{H^m}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

This is the same as

$$\left|\langle x\rangle^{-k} u(\varphi)\right| \le C\|\varphi\|_{H^m}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n).$$

which shows that $\langle x\rangle^{-k}u \in H^{-m}(\mathbb{R}^n)$, i.e., from Proposition 4.8,

$$\langle x\rangle^{-k} u = \sum_{|\alpha|\le m} D^\alpha u_\alpha\,,\ u_\alpha \in L^2(\mathbb{R}^n)\,.$$

In fact, choose $j > n/2$ and consider $v_\alpha \in H^j(\mathbb{R}^n)$ defined by $\hat{v}_\alpha = \langle\xi\rangle^{-j}\hat{u}_\alpha$. As in the proof of Proposition 4.14 we conclude that

$$u_\alpha = \sum_{|\beta|\le j} D^\beta u'_{\alpha,\beta}\,,\ u'_{\alpha,\beta} \in H^j(\mathbb{R}^n) \subset \mathcal{C}_0^0(\mathbb{R}^n)\,.$$

Thus,[8]

$$u = \langle x\rangle^k \sum_{|\gamma|\le M} D^\gamma_\alpha v_\gamma\,,\ \ v_\gamma \in \mathcal{C}^0_0(\mathbb{R}^n)\,. \tag{5.10}$$

To get (5.9) we 'commute' the factor $\langle x\rangle^k$ to the inside; since I have not done such an argument carefully so far, let me do it as a lemma.

LEMMA 5.6. *For any $\gamma \in \mathbb{N}_0^n$ there are polynomials $p_{\alpha,\gamma}(x)$ of degrees at most $|\gamma-\alpha|$ such that*

$$\langle x\rangle^k D^\gamma v = \sum_{\alpha\le\gamma} D^{\gamma-\alpha}\left(p_{\alpha,\gamma}\langle x\rangle^{k-2|\gamma-\alpha|}v\right)\,.$$

PROOF. In fact it is convenient to prove a more general result. Suppose $p$ is a polynomial of a degree at most $j$ then there exist polynomials of degrees at most $j+|\gamma-\alpha|$ such that

$$p\langle x\rangle^k D^\gamma v = \sum_{\alpha\le\gamma} D^{\gamma-\alpha}(p_{\alpha,\gamma}\langle x\rangle^{k-2|\gamma-\alpha|}v)\,. \tag{5.11}$$

The lemma follows from this by taking $p=1$.

Furthermore, the identity (5.11) is trivial when $\gamma=0$, and proceeding by induction we can suppose it is known whenever $|\gamma|\le L$. Taking $|\gamma| = L+1$,

$$D^\gamma = D_j D^{\gamma'}\ \ |\gamma'| = L.$$

Writing the identity for $\gamma'$ as

$$p\langle x\rangle^k D^{\gamma'} = \sum_{\alpha'\le\gamma'} D^{\gamma'-\alpha'}(p_{\alpha',\gamma'}\langle x\rangle^{k-2|\gamma'-\alpha'|}v)$$

we may differentiate with respect to $x_j$. This gives

$$p\langle x\rangle^k D^\gamma = -D_j(p\langle x\rangle^k)\cdot D^{\gamma'}v + \sum_{|\alpha'|\le\gamma} D^{\gamma-\alpha'}(p'_{\alpha',\gamma'}\langle x\rangle^{k-2|\gamma-\alpha|+2}v)\,.$$

The first term on the right expands to

$$(-(D_jp)\cdot\langle x\rangle^k D^{\gamma'}v - \frac{1}{i}kpx_j\langle x\rangle^{k-2}D^{\gamma'}v)\,.$$

We may apply the inductive hypothesis to each of these terms and rewrite the result in the form (5.11); it is only necessary to check the order of the polynomials, and recall that $\langle x\rangle^2$ is a polynomial of degree 2. □

[8] This is probably the most useful form of the representation theorem!

Applying Lemma 5.6 to (5.10) gives (5.9), once negative powers of $\langle x\rangle$ are absorbed into the continuous functions. Then (5.8) follows from (5.9) and Leibniz's formula. □

## 6. Differential operators.

In the last third of the course we will apply what we have learned about distributions, and a little more, to understand properties of differential operators with constant coefficients. Before I start talking about these, I want to prove another density result.

So far we have *not* defined a topology on $\mathcal{S}'(\mathbb{R}^n)$ – I will leave this as an optional exercise.[9] However we shall consider a notion of convergence. Suppose $u_j \in \mathcal{S}'(\mathbb{R}^n)$ is a sequence in $\mathcal{S}'(\mathbb{R}^n)$. It is said to *converge weakly* to $u \in \mathcal{S}'(\mathbb{R}^n)$ if

$$u_j(\varphi) \to u(\varphi) \;\forall\; \varphi \in \mathcal{S}(\mathbb{R}^n)\,. \tag{6.1}$$

There is no 'uniformity' assumed here, it is rather like pointwise convergence (except the linearity of the functions makes it seem stronger).

PROPOSITION 6.1. *The subspace $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is weakly dense, i.e., each $u \in \mathcal{S}'(\mathbb{R}^n)$ is the weak limit of a subspace $u_j \in \mathcal{S}(\mathbb{R}^n)$.*

PROOF. We can use Schwartz representation theorem to write, for some $m$ depending on $u$,

$$u = \langle x\rangle^m \sum_{|\alpha|\le m} D^\alpha u_\alpha\,,\ u_\alpha \in L^2(\mathbb{R}^n)\,.$$

We know that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, in the sense of metric spaces so we can find $u_{\alpha,j} \in \mathcal{S}(\mathbb{R}^n)$, $u_{\alpha,j} \to u_\alpha$ in $L^2(\mathbb{R}^n)$. The density result then follows from the basic properties of weak convergence. □

PROPOSITION 6.2. *If $u_j \to u$ and $u'_j \to u'$ weakly in $\mathcal{S}'(\mathbb{R}^n)$ then $cu_j \to cu$, $u_j + u'_j \to u + u'$, $D^\alpha u_j \to D^\alpha u$ and $\langle x\rangle^m u_j \to \langle x\rangle^m u$ weakly in $\mathcal{S}'(\mathbb{R}^n)$.*

PROOF. This follows by writing everyting in terms of pairings, for example if $\varphi \in \mathcal{S}(\mathbb{R}^n)$

$$D^\alpha u_j(\varphi) = u_j((-1)^{(\alpha)} D^\alpha\varphi) \to u((-1)^{(\alpha)} D^\alpha\varphi) = D^\alpha u(\varphi)\,.$$

□

This weak density shows that our definition of $D_j$, and $x_j\times$ are unique if we require Proposition 6.2 to hold.

[9]Problem 34.

We have discussed differentiation as an operator (meaning just a linear map between spaces of function-like objects)

$$D_j : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,.$$

Any polynomial on $\mathbb{R}^n$

$$p(\xi) = \sum_{|\alpha|\le m} p_\alpha \xi^\alpha\,,\ p_\alpha \in \mathbb{C}$$

defines a differential operator[10]

$$p(D)u = \sum_{|\alpha|\le m} p_\alpha D^\alpha u\,. \tag{6.2}$$

Before discussing any general theorems let me consider some examples.

$$\text{On } \mathbb{R}^2,\ \overline{\partial} = \partial_x + i\partial_y \text{“d-bar operator”} \tag{6.3}$$

$$\text{on } \mathbb{R}^n,\ \Delta = \sum_{j=1}^n D_j^2 \text{“Laplacian”} \tag{6.4}$$

$$\text{on } \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1},\ D_t^2 - \Delta \text{“Wave operator”} \tag{6.5}$$

$$\text{on} \mathbb{R}\times\mathrm{R}^{\mathrm{n}} = \mathbb{R}^{\mathrm{n}+1},\ \partial_{\mathrm{t}} + \Delta \text{“Heat operator”} \tag{6.6}$$

$$\text{on } \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1},\ D_t + \Delta \text{“Schrödinger operator”} \tag{6.7}$$

Functions, or distributions, satisfying $\overline{\partial}u = 0$ are said to be *holomorphic*, those satisfying $\Delta u = 0$ are said to be *harmonic*.

Definition 6.3. *An element $E \in \mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$P(D)E = \delta \tag{6.8}$$

is said *to be a (tempered) fundamental solution of $P(D)$.*

Theorem 6.4 (without proof). *Every non-zero constant coefficient differential operator has a tempered fundamental solution.*

This is quite hard to prove and not as interetsing as it might seem. We will however give lots of examples, starting with $\overline{\partial}$. Consider the function

$$E(x,y) = \frac{1}{2\pi}(x+iy)^{-1}\,,\ \ (x,y) \neq 0\,. \tag{6.9}$$

[10]More correctly a partial differential operator with constant coefficients.

LEMMA 6.5. *$E(x, y)$ is locally integrable and so defines $E \in \mathcal{S}'(\mathbb{R}^2)$ by*

$$E(\varphi) = \frac{1}{2\pi}\int_{\mathbb{R}^2}(x+iy)^{-1}\varphi(x,y)\,dx\,dy\,, \tag{6.10}$$

*and $E$ so defined is a tempered fundamental solution of $\overline{\partial}$.*

PROOF. Since $(x + iy)^{-1}$ is smooth and bounded away from the origin the local integrability follows from the estimate, using polar coordinates,

$$\int_{|(x,y)|\le 1}\frac{dx\,dy}{|x+iy|} = \int_0^{2\pi}\int_0^1 \frac{r\,dr\,d\theta}{r} = 2\pi\,. \tag{6.11}$$

Differentiating directly in the region where it is smooth,

$$\partial_x(x+iy)^{-1} = -(x+iy)^{-2}\,,\ \partial_y(x+iy)^{-1} = -i(x \in iy)^{-2}$$

so indeed, $\overline{\partial}E = 0$ in $(x, y) \neq 0$.[11]

The derivative is *really* defined by

$$\begin{aligned}(\overline{\partial}E)(\varphi) &= E(-\overline{\partial}\varphi)\\ &= \lim_{\epsilon\downarrow 0} -\frac{1}{2\pi}\int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}}(x+iy)^{-1}\ \overline{\partial}\varphi\,dx\,dy\,.\end{aligned} \tag{6.12}$$

Here I have cut the space $\{|x| \le \epsilon\,,\ |y| \le \epsilon\}$ out of the integral and used the local integrability in taking the limit as $\epsilon \downarrow 0$. Integrating by parts in $x$ we find

$$\begin{aligned}-\int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}}(x+iy)^{-1}\partial_x\varphi\,dx\,dy &= \int_{\substack{|x|\ge\epsilon\\|y|\ge\epsilon}}(\partial_x(x+iy)^{-1})\varphi\,dx\,dy\\ &+\int_{\substack{|y|\le\epsilon\\x=\epsilon}}(x+iy)^{-1}\varphi(x,y)\,dy - \int_{\substack{|y|\le\epsilon\\x=-\epsilon}}(x+iy)^{-1}\varphi(x,y)\,dy\,.\end{aligned}$$

There is a corrsponding formula for integration by parts in $y$ so, recalling that $\overline{\partial}E = 0$ away from $(0, 0)$,

$$\begin{aligned}&2\pi\overline{\partial}E(\varphi) =\\ &\lim_{\epsilon\downarrow 0}\int_{|y|\le\epsilon}[(\epsilon+iy)^{-1}\varphi(\epsilon,y) - (-\epsilon+iy)^{-1}\varphi(-\epsilon,y)]\,dy\\ &\qquad + i\lim_{\epsilon\downarrow 0}\int_{|x|\le\epsilon}[(x+i\epsilon)^{-1}\varphi(x,\epsilon) - (x-i\epsilon)^{-1}\varphi(x,\epsilon)]\,dx\,,\end{aligned} \tag{6.13}$$

[11] Thus at this stage we know $\overline{\partial}E$ must be a sum of derivatives of $\delta$.

assuming that both limits exist. Now, we can write

$$\varphi(x,y) = \varphi(0,0) + x\psi_1(x_1 y) + y\psi_2(x,y)\,.$$

Replacing $\varphi$ by either $x\psi_1$ or $y\psi_2$ in (6.13) both limits are zero. For example

$$\left|\int_{|y|\le\epsilon} (\epsilon+iy)^{-1}\epsilon\psi_1(\epsilon,y)\,dy\right| \le \int_{|y|\le\epsilon} |\psi_1| \to 0\,.$$

Thus we get the same result in (6.13) by replacing $\varphi(x,y)$ by $\varphi(0,0)$. Then $2\pi\overline{\partial}E(\varphi) = c\varphi(0)$,

$$c = \lim_{\epsilon\downarrow 0} 2\epsilon\int_{|y|\le\epsilon} \frac{dy}{\epsilon^2+y^2} = \lim_{\epsilon\downarrow 0} < \int_{|y|\le 1}\frac{dy}{1+y^2} = 2\pi\,.$$

$\square$

Let me remind you that we have already discussed the convolution of functions

$$u * v(x) = \int u(x-y)v(y)\,dy = v * u(x)\,.$$

This makes sense provided $u$ is of slow growth and $s \in \mathcal{S}(\mathbb{R}^n)$. In fact we can rewrite the definition in terms of pairing

$$(u*\varphi)(x) = \langle u, \varphi(x-\cdot)\rangle \tag{6.14}$$

where the $\cdot$ indicates the variable in the pairing.

Theorem 6.6 (Hörmander, Theorem 4.1.1). *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $u*\varphi \in \mathcal{S}'(\mathbb{R}^n)\cap\mathcal{C}^\infty(\mathbb{R}^n)$ and if* $\operatorname{supp}(\varphi) \Subset \mathbb{R}^n$

$$\operatorname{supp}(u*\varphi) \subset \operatorname{supp}(u) + \operatorname{supp}(\varphi)\,.$$

*For any multi-index $\alpha$*

$$D^\alpha(u*\varphi) = D^\alpha u * \varphi = u * D^\alpha\varphi\,.$$

Proof. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then for any fixed $x \in \mathbb{R}^n$,

$$\varphi(x-\cdot) \in \mathcal{S}(\mathbb{R}^n)\,.$$

Indeed the seminorm estimates required are

$$\sup_y (1+|y|^2)^{k/2}\,|D^\alpha{}_y\varphi(x-y)| < \infty \;\forall\; \alpha, k > 0\,.$$

Since $D^\alpha{}_y\varphi(x-y) = (-1)^{|\alpha|}(D^\alpha\varphi)(x-y)$ and

$$(1+|y|^2) \le (1+|x-y|^2)(1+|x|^2)$$

we conclude that

$$\|(1+|y|^2)^{k/2}D^\alpha{}_y(x-y)\|_{L^\infty} \le (1+|x|^2)^{k/2}\|\langle y\rangle^k D^\alpha{}_y\varphi(y)\|_{L^\infty}\,.$$

The continuity of $u \in \mathcal{S}'(\mathbb{R}^n)$ means that for some $k$

$$|u(\varphi)| \le C \sup_{|\alpha|\le k} \|(y)^k D^\alpha \varphi\|_{L^\infty}$$

so it follows that

$$|u * \varphi(x)| = |\langle u, \varphi(x - \cdot)\rangle| \le C(1 + |x|^2)^{k/2}\,. \tag{6.15}$$

The argument above shows that $x \mapsto \varphi(x - \cdot)$ is a continuous function of $x \in \mathbb{R}^n$ with values in $\mathcal{S}(\mathbb{R}^n)$, so $u * \varphi$ is continuous and satisfies (6.15). It is therefore an element of $\mathcal{S}'(\mathbb{R}^n)$.

Differentiability follows in the same way since for each $j$, with $e_j$ the $j$th unit vector

$$\frac{\varphi(x + se_j - y) - \varphi(x - y)}{s} \in \mathcal{S}(\mathbb{R}^n)$$

is continuous in $x \in \mathbb{R}^n$, $s \in \mathbb{R}$. Thus, $u * \varphi$ has continuous partial derivatives and

$$D_j u * \varphi = u * D_j \varphi\,.$$

The same argument then shows that $u*\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$. That $D_j(u*\varphi) = D_j u * \varphi$ follows from the definition of derivative of distributions

$$\begin{aligned} D_j(u * \varphi(x)) &= (u * D_j\varphi)(x) \\ &= \langle u, D_{x_j}\varphi(x - y)\rangle = -\langle u(y), D_{y_j}\varphi(x - y)\rangle_y \\ &= (D_j u) * \varphi\,. \end{aligned}$$

Finally consider the support property. Here we are assuming that $\operatorname{supp}(\varphi)$ is compact; we also know that $\operatorname{supp}(u)$ is a closed set. We have to show that

$$\overline{x} \notin \operatorname{supp}(u) + \operatorname{supp}(\varphi) \tag{6.16}$$

implies $u * \varphi(x') = 0$ for $x'$ near $\overline{x}$. Now (6.16) just means that

$$\operatorname{supp}\varphi(\overline{x} - \cdot) \cap \operatorname{supp}(u) = \phi\,, \tag{6.17}$$

Since $\operatorname{supp}\varphi(x - \cdot) = \{y \in \mathbb{R}^n; x - y \in \operatorname{supp}(\varphi)\}$, so both statements mean that there is *no* $y \in \operatorname{supp}(\varphi)$ with $\overline{x} - y \in \operatorname{supp}(u)$. This can also be written

$$\operatorname{supp}(\varphi) \cap \operatorname{supp} u(x - \cdot) = \phi$$

and as we showed when discussing supports implies

$$u * \varphi(x') = \langle u(x' - \cdot), \varphi\rangle = 0\,.$$

From (6.17) this is an *open* condition on $x'$, so the support property follows.

□

Now suppose $\varphi$, $\psi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Then

$$(u * \varphi) * \psi = u * (\varphi * \psi)\,. \tag{6.18}$$

This is really Hörmander's Lemma 4.1.3 and Theorem 4.1.2; I ask you to prove it as Problem 35.

We have shown that $u * \varphi$ is $\mathcal{C}^\infty$ if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, i.e., the regularity of $u * \varphi$ follows from the regularity of *one* of the factors. This makes it reasonable to expect that $u * v$ can be defined when $u \in \mathcal{S}'(\mathbb{R}^n)$, $v \in \mathcal{S}'(\mathbb{R}^n)$ and one of them has compact support. If $v \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then

$$u * v(\varphi) = \int \langle u(\cdot), v(x - \cdot)\rangle \varphi(x)\, dx = \int \langle u(\cdot), v(x - \cdot)\rangle \check{v}\varphi(-x)\, dx$$

where $\check{\varphi}(z) = \varphi(-z)$. In fact using Problem 35,

$$u * v(\varphi) = ((u * v) * \check{\varphi})(0) = (u * (v * \check{\varphi}))(0)\,. \tag{6.19}$$

Here, $v, \varphi$ are both smooth, but notice

Lemma 6.7. *If $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ then $v * \varphi \in \mathcal{S}(\mathbb{R}^n)$.*

Proof. Since $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support there exists $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\chi v = v$. Then

$$\begin{aligned} v * \varphi(x) &= (\chi v) * \varphi(x) = \langle \chi v(y), \varphi(x - y)\rangle_y \\ &= \langle u(y), \chi(y)\varphi(x - y)\rangle_y\,. \end{aligned}$$

Thus, for some $k$,

$$|v * \varphi(x)| \le C\|\chi(y)\varphi(x - y)\|_{(k)}$$

where $\|\ \|_{(k)}$ is one of our norms on $\mathcal{S}(\mathbb{R}^n)$. Since $\chi$ is supported in some large ball,

$$\begin{aligned} &\|\chi(y)\varphi(x - y)\|_{(k)} \\ &\quad \le \sup_{|\alpha| \le k} \left|\langle y\rangle^k D^\alpha{}_y(\chi(y)\varphi(x - y))\right| \\ &\quad \le C \sup_{|y| \le R} \sup_{|\alpha| \le k} |(D^\alpha\varphi)(x - y)| \\ &\quad \le C_N \sup_{|y| \le R} (1 + |x - y|^2)^{-N/2} \\ &\quad \le C_N(1 + |x|^2)^{-N/2}\,. \end{aligned}$$

Thus $(1 + |x|^2)^{N/2}\,|v * \varphi|$ is bounded for each $N$. The same argument applies to the derivative using Theorem 6.6, so

$$v * \varphi \in \mathcal{S}(\mathbb{R}^n)\,.$$

□

In fact we get a little more, since we see that for each $k$ there exists $k'$ and $C$ (depending on $k$ and $v$) such that

$$\|v * \varphi\|_{(k)} \le C\|\varphi\|_{(k')}\,.$$

This means that

$$v* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear map.

Now (6.19) allows us to define $u*v$ when $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ has compact support by

$$u * v(\varphi) = u * (v * \check{\varphi})(0)\,.$$

Using the continuity above, I ask you to check that $u * v \in \mathcal{S}'(\mathbb{R}^n)$ in Problem 36. For the moment let me assume that this convolution has the same properties as before – I ask you to check the main parts of this in Problem 37.

Recall that $E \in \mathcal{S}'(\mathbb{R}^n)$ is a fundamental situation for $P(D)$, a constant coefficient differential operator, if $P(D)E = \delta$. We also use a weaker notion.

Definition 6.8. *A* parametrix *for a constant coefficient differential operator $P(D)$ is a distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$P(D)F = \delta + \psi\,,\ \ \psi \in \mathcal{C}^\infty(\mathbb{R}^n)\,. \tag{6.20}$$

*An operator $P(D)$ is said to be* hypoelliptic *if it has a parametrix satisfying*

$$\operatorname{sing\,supp}(F) \subset \{0\}\ , \tag{6.21}$$

*where for any $u \in \mathcal{S}'(\mathbb{R}^n)$*

$$\begin{aligned}(\operatorname{sing\,supp}(u))^{\complement} = \{\overline{x} \in \mathbb{R}^n;\ \exists\, \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)\,,\\ \varphi(\overline{x}) \neq 0, \varphi u \in \mathcal{C}^\infty_c(\mathbb{R}^n)\}\ .\end{aligned} \tag{6.22}$$

Since the same $\varphi$ must work for nearby points in (6.22), the set $\operatorname{sing\,supp}(u)$ is *closed*. Furthermore

$$\operatorname{sing\,supp}(u) \subset \operatorname{supp}(u)\,. \tag{6.23}$$

As Problem 37 I ask you to show that if $K \Subset \mathbb{R}^n$ and $K \cap \operatorname{sing\,supp}(u) = \phi$ the $\exists\ \varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\varphi(x) = 1$ in a neighbourhood of $K$ such that $\varphi u \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. In particular

$$\operatorname{sing\,supp}(u) = \phi \Rightarrow u \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)\,. \tag{6.24}$$

THEOREM 6.9. *If $P(D)$ is hypoelliptic then*

$$\text{sing supp}(u) = \text{sing supp}(P(D)u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{6.25}$$

PROOF. One half of this is true for *any* differential operator:

LEMMA 6.10. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ then for any polynomial*

$$\text{sing supp}(P(D)u) \subset \text{sing supp}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,. \tag{6.26}$$

$\square$

PROOF. We must show that $\overline{x} \notin \text{sing supp}(u) \Rightarrow \overline{x} \notin \text{sing supp}(P(D)u)$. Now, if $\overline{x} \notin \text{sing supp}(u)$ we can find $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi \equiv 1$ near $\overline{x}$, such that $\varphi u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then

$$\begin{aligned} P(D)u &= P(D)(\varphi u + (1-\varphi)u) \\ &= P(D)(\varphi u) + P(D)((1-\varphi)u)\,. \end{aligned}$$

The first term is $\mathcal{C}^\infty$ and $\overline{x} \notin \text{supp}(P(D)((1-\varphi)u))$, so $\overline{x} \notin \text{sing supp}(P(D)u)$. $\square$

It remains to show the converse of (6.26) where $P(D)$ is assumed to be hypoelliptic. Take $F$, a parametrix for $P(D)$ with $\text{sing supp}\, u \subset \{0\}$ and assume, or rather arrange, that $F$ have compact support. In fact if $\overline{x} \notin \text{sing supp}(P(D)u)$ we can arrange that

$$(\text{supp}(F) + \overline{x}) \cap \text{sing supp}(P(D)u) = \phi\,.$$

Now $P(D)F = \delta\psi$ with $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so

$$u = \delta * u = (P(D)F) * u - \psi * u.$$

Since $\psi * u \in \mathcal{C}^\infty$ it suffices to show that $\bar{x} \notin \text{sing supp}\,((P(D)u) * f)\,.$

Take $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\varphi f \in \mathcal{C}^\infty$, $f = P(D)u$ but

$$(\text{supp}\, F + \overline{x}) \cap \text{supp}(\varphi) = 0\,.$$

Then $f = f_1 + f_2$, $f_1 = \varphi f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so

$$f * F = f_1 * F + f_2 * F$$

where $f_1 * F \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\overline{x} \notin \text{supp}(f_2 * F)$. It follows that $\overline{x} \notin \text{sing supp}(u)$.

EXAMPLE 6.1. If $u$ is holomorphic on $\mathbb{R}^n$, $\overline{\partial} u = 0$, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$.

Recall from last time that a differential operator $P(D)$ is said to be hypoelliptic if there exists $F \in \mathcal{S}'(\mathbb{R}^n)$ with

$$P(D)F - \delta \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ and } \text{sing supp}(F) \subset \{0\}\,. \tag{6.27}$$

The second condition here means that if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\varphi(x) = 1$ in $|x| < \epsilon$ for some $\epsilon > 0$ then $(1-\varphi)F \in \mathcal{C}^\infty(\mathbb{R}^n)$. Since $P(D)((1-\varphi)F) \in \mathcal{C}^\infty(\mathbb{R}^n)$ we conclude that

$$P(D)(\varphi F) - \delta \in \mathcal{C}_c^\infty(\mathbb{R}^n)$$

and we may well suppose that $F$, replaced now by $\varphi F$, has compact support. Last time I showed that

$$\text{If } P(D) \text{ is hypoelliptic and } u \in \mathcal{S}'(\mathbb{R}^n) \text{ then}$$
$$\operatorname{sing\,supp}(u) = \operatorname{sing\,supp}(P(D)u)\,.$$

I will remind you of the proof later.

First however I want to discuss the important notion of *ellipticity*. Remember that $P(D)$ is 'really' just a polynomial, called the *characteristic polynomial*

$$P(\xi) = \sum_{|\alpha| \le m} C_\alpha \xi^\alpha\,.$$

It has the property

$$\widehat{P(D)u}(\xi) = P(\xi)\hat{u}(\xi)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n)\,.$$

This shows (if it isn't already obvious) that we can remove $P(\xi)$ from $P(D)$ thought of as an operator on $\mathcal{S}'(\mathbb{R}^n)$.

We can think of *inverting* $P(D)$ by dividing by $P(\xi)$. This works well provided $P(\xi) \neq 0$, for all $\xi \in \mathbb{R}^n$. An example of this is

$$P(\xi) = |\xi|^2 + 1 = \sum_{j=1}^n +1\,.$$

However even the Laplacian, $\Delta = \sum_{j=1}^n D_j^2$, does not satisfy this rather stringent condition.

It is reasonable to expect the top order derivatives to be the most important. We therefore consider

$$P_m(\xi) = \sum_{|\alpha| = m} C_\alpha \xi^\alpha$$

the leading part, or *principal symbol*, of $P(D)$.

Definition 6.11. *A polynomial $P(\xi)$, or $P(D)$, is said to be* elliptic *of order $m$ provided $P_m(\xi) \neq 0$ for all $0 \neq \xi \in \mathbb{R}^n$.*

So what I want to show today is

Theorem 6.12. *Every elliptic differential operator $P(D)$ is hypoelliptic.*

We want to find a *parametrix* for $P(D)$; we already know that we might as well suppose that $F$ has compact support. Taking the Fourier transform of (6.27) we see that $\widehat{F}$ should satisfy

$$P(\xi)\widehat{F}(\xi) = 1 + \widehat{\psi},\ \widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)\,. \tag{6.28}$$

Here we use the fact that $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, so $\widehat{\psi} \in \mathcal{S}(\mathbb{R}^n)$ too.

First suppose that $P(\xi) = P_m(\xi)$ is actually homogeneous of degree $m$. Thus

$$P_m(\xi) = |\xi|^m\, P_m(\widehat{\xi}),\ \widehat{\xi} = \xi/\,|\xi|\ ,\ \xi \neq 0\,.$$

The assumption at ellipticity means that

$$P_m(\widehat{\xi}) \neq 0\ \forall\ \widehat{\xi} \in \mathcal{S}^{n-1} = \{\xi \in \mathbb{R}^n; |\xi| = 1\}\ . \tag{6.29}$$

Since $\mathcal{S}^{n-1}$ is *compact* and $P_m$ is continuous

$$\left|P_m(\widehat{\xi})\right| \geq C > 0\ \forall\ \widehat{\xi} \in \mathcal{S}^{n-1}\,, \tag{6.30}$$

for some constant $C$. Using homogeneity

$$\left|P_m(\widehat{\xi})\right| \geq C\,|\xi|^m\ ,\ C > 0\ \forall\ \xi \in \mathbb{R}^n\,. \tag{6.31}$$

Now, to get $\widehat{F}$ from (6.28) we want to divide by $P_m(\xi)$ or multiply by $1/P_m(\xi)$. The only problem with defining $1/P_m(\xi)$ is at $\xi = 0$. We shall simply avoid this unfortunate point by choosing $P \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ as before, with $\varphi(\xi) = 1$ in $|\xi| \leq 1$.

Lemma 6.13. *If $P_m(\xi)$ is homogeneous of degree $m$ and elliptic then*

$$Q(\xi) = \frac{(1 - \varphi(\xi))}{P_m(\xi)} \in \mathcal{S}'(\mathbb{R}^n) \tag{6.32}$$

*is the Fourier transform of a parametrix for $P_m(D)$, satisfying* (6.27).

Proof. Clearly $Q(\xi)$ is a continuous function and $|Q(\xi)| \leq C(1 + |\xi|)^{-m}\ \forall\ \xi \in \mathbb{R}^n$, so $Q \in \mathcal{S}'(\mathbb{R}^n)$. It therefore *is* the Fourier transform of some $F \in \mathcal{S}'(\mathbb{R}^n)$. Furthermore

$$\begin{aligned}\widehat{P_m(D)F}(\xi) &= P_m(\xi)\widehat{F} = P_m(\xi)Q(\xi)\\ &= 1 - \varphi(\xi)\,,\\ \Rightarrow P_m(D)F &= \delta + \psi\,,\ \widehat{\psi}(\xi) = -\varphi(\xi)\,.\end{aligned}$$

Since $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$. Thus $F$ *is* a parametrix for $P_m(D)$. We still need to show the 'hard part' that

$$\operatorname{sing\,supp}(F) \subset \{0\}\ . \tag{6.33}$$

$\square$

We can show (6.33) by considering the distributions $x^\alpha F$. The idea is that for $|\alpha|$ large, $x^\alpha$ vanishes rather rapidly at the origin and this should 'weaken' the singularity of $F$ there. In fact we shall show that

$$x^\alpha F \in H^{|\alpha|+m-n-1}(\mathbb{R}^n)\,,\ |\alpha| > n+1-m\,. \tag{6.34}$$

If you recall, these Sobolev spaces are defined in terms of the Fourier transform, namely we must show that

$$\widehat{x^\alpha F} \in \langle\xi\rangle^{-|\alpha|-m+n+1} L^2(\mathbb{R}^n)\,.$$

Now $\widehat{x^\alpha F} = (-1)^{|\alpha|} D^\alpha{}_\xi \widehat{F}$, so what we need to cinsider is the behaviour of the derivatives of $\widehat{F}$, which is just $Q(\xi)$ in (6.32).

LEMMA 6.14. *Let $P(\xi)$ be a polynomial of degree $m$ satisfying*

$$|P(\xi)| \ge C\,|\xi|^m \ \text{ in } \ |\xi| > 1/C \text{ for some } C > 0\,, \tag{6.35}$$

*then for some constants $C_\alpha$*

$$\left| D^\alpha \frac{1}{P(\xi)} \right| \le C_\alpha\, |\xi|^{-m-|\alpha|} \ \text{ in } \ |\xi| > 1/C\,. \tag{6.36}$$

PROOF. The estimate in (6.36) for $\alpha = 0$ is just (6.35). To prove the higher estimates that for each $\alpha$ there is a polynomial of degree at most $(m-1)\,|\alpha|$ such that

$$D^\alpha \frac{1}{P(\xi)} = \frac{L_\alpha(\xi)}{(P(\xi))^{1+|\alpha|}}\,. \tag{6.37}$$

Once we know (6.37) we get (6.36) straight away since

$$\left| D^\alpha \frac{1}{P(\xi)} \right| \le \frac{C'_\alpha\, |\xi|^{(m-1)|\alpha|}}{C^{1+|\alpha|}\, |\xi|^{m(1+|\alpha|)}} \le C_\alpha\, |\xi|^{-m-|\alpha|}\ .$$

We can prove (6.37) by induction, since it is certainly true for $\alpha = 0$. Suppose it is true for $|\alpha| \le k$. To get the same identity for each $\beta$ with $|\beta| = k+1$ it is enough to differentiate one of the identities with $|\alpha| = k$ once. Thus

$$D^\beta \frac{1}{P(\xi)} = D_j D^\alpha \frac{1}{P(\xi)} = \frac{D_j L_\alpha(\xi)}{P(\xi)^{1+|\alpha|}} - \frac{(1+|\alpha|) L_\alpha D_j P(\xi)}{(P(\xi))^{2+|\alpha|}}\,.$$

Since $L_\beta(\xi) = P(\xi) D_j L_\alpha(\xi) - (1+|\alpha|) L_\alpha(\xi) D_j P(\xi)$ is a polynomial of degree at most $(m-1)\,|\alpha| + m - 1 = (m-1)\,|\beta|$ this proves the lemma. $\square$

Going backwards, observe that $Q(\xi) = \frac{1-\varphi}{P_m(\xi)}$ is smooth in $|\xi| \le 1/C$, so (6.36) implies that

$$|D^\alpha Q(\xi)| \le C_\alpha (1 + |\xi|)^{-m-|\alpha|} \tag{6.38}$$
$$\Rightarrow \langle \xi \rangle^\ell D^\alpha Q \in L^2(\mathbb{R}^n) \text{ if } \ell - m - |\alpha| < -\frac{n}{2}\,,$$

which certainly holds if $\ell = |\alpha| + m - n - 1$, giving (6.34). Now, by Sobolev's embedding theorem

$$x^\alpha F \in \mathcal{C}^k \text{ if } |\alpha| > n + 1 - m + k + \frac{n}{2}\,.$$

In particular this means that if we choose $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}(\mu)$ then for every $k$, $\mu / |x|^{2k}$ is smooth and

$$\mu F = \frac{\mu}{|x|^{2k}} |x|^{2k} F \in \mathcal{C}^{2\ell - 2n}\,,\ \ell > n\,.$$

Thus $\mu F \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and this is what we wanted to show, $\operatorname{sing\,supp}(F) \subset \{0\}$.

So now we have actually proved that $P_m(D)$ is hypoelliptic if it is elliptic. Rather than go through the proof again to make sure, let me go on to the general case and in doing so review it.

Proof. *Proof of theorem.* We need to show that if $P(\xi)$ is elliptic then $P(D)$ has a parametrix $F$ as in (6.27). From the discussion above the ellipticity of $P(\xi)$ implies (and is equivalent to)

$$|P_m(\xi)| \ge c\,|\xi|^m\,,\ c > 0\,.$$

On the other hand

$$P(\xi) - P_m(\xi) = \sum_{|\alpha| < m} C_\alpha \xi^\alpha$$

is a polynomial of degree at most $m - 1$, so

$$|P(\xi) - P_m(\xi)|\,2 \le C'(1 + |\xi|)^{m-1}\,.$$

This means that id $C > 0$ is large enough then in $|\xi| > C$, $C'(1 + |\xi|)^{m-1} < \frac{c}{2}\,|\xi|^m$, so

$$|P(\xi)| \ge |P_m(\xi)| - |P(\xi) - P_m(\xi)|$$
$$\ge c\,|\xi|^m - C'(1 + |\xi|)^{m-1} \ge \frac{c}{2}\,|\xi|^m\,.$$

This means that $P(\xi)$ itself satisfies the conditions of Lemma 6.14. Thus if $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 in a large enough ball then $Q(xi) = (1 - \varphi(\xi))/P(\xi)$ in $\mathcal{C}^\infty$ and satisfies (6.36) which can be written

$$|D^\alpha Q(\xi)| \le C_\alpha (1 + |\xi|)^{m-|\alpha|}\,.$$

The discussion above now shows that defining $F \in \mathcal{S}'(\mathbb{R}^n)$ by $\widehat{F}(\xi) = Q(\xi)$ gives a solution to (6.27).

$\square$

The last step in the proof is to show that if $F \in \mathcal{S}'(\mathbb{R}^n)$ has compact support, and satisfies (6.27), then

$$\begin{gathered} u \in \mathcal{S}(\mathbb{R}^n)\,,\ P(D)u \in \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n) \\ \Rightarrow u = F * (P(D)u) - \psi * u \in \mathcal{C}^\infty(\mathbb{R}^n)\,. \end{gathered}$$

Let me refine this result a little bit.

Proposition 6.15. *If $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\mu \in \mathcal{S}'(\mathbb{R}^n)$ has compact support then*

$$\operatorname{sing\,supp}(u * f) \subset \operatorname{sing\,supp}(u) + \operatorname{sing\,supp}(f).$$

Proof. We need to show that $p \notin \operatorname{sing\,supp}(u) \in \operatorname{sing\,supp}(f)$ then $p \notin \operatorname{sing\,supp}(u * f)$. Once we can fix $p$, we might as well suppose that $f$ has compact support too. Indeed, choose a large ball $B(R, 0)$ so that

$$z \notin B(0, R) \Rightarrow p \notin \operatorname{supp}(u) + B(0, R)\,.$$

This is possible by the assumed boundedness of $\operatorname{supp}(u)$. Then choose $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\varphi = 1$ on $B(0, R)$; it follows from Theorem L16.2, or rather its extension to distributions, that $\phi \notin \operatorname{supp}(u(1-\varphi)f)$, so we can replace $f$ by $\varphi f$, noting that $\operatorname{sing\,supp}(\varphi f) \subset \operatorname{sing\,supp}(f)$. Now if $f$ has compact support we can choose compact neighbourhoods $K_1$, $K_2$ of $\operatorname{sing\,supp}(u)$ and $\operatorname{sing\,supp}(f)$ such that $p \notin K_1 + K_2$. Furthermore we an decompose $u = u_1 + u_2$, $f = f_1 + f_2$ so that $\operatorname{supp}(u_1) \subset K_1$, $\operatorname{supp}(f_2) \subset K_2$ and $u_2, f_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$. It follows that

$$u * f = u_1 * f_1 + u_2 * f_2 + u_1 * f_2 + u_2 * f_2\,.$$

Now, $p \notin \operatorname{supp}(u_1 * f_1)$, by the support property of convolution and the three other terms are $\mathcal{C}^\infty$, since at least one of the factors is $\mathcal{C}^\infty$. Thus $p \notin \operatorname{sing\,supp}(u * f)$. $\square$

The most important example of a differential operator which is hypoelliptic, but not elliptic, is the heat operator

$$\partial_t + \Delta = \partial_t - \sum_{j=1}^n \partial_{x_j}^2\,. \tag{6.39}$$

In fact the distribution

$$E(t,x) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & t \geq 0 \\ 0 & t \leq 0 \end{cases} \tag{6.40}$$

is a fundamental solution. First we need to check that $E$ *is* a distribution. Certainly $E$ is $\mathcal{C}^\infty$ in $t > 0$. Moreover as $t \downarrow 0$ in $x \neq 0$ it vanishes with all derivatives, so it is $\mathcal{C}^\infty$ except at $t = 0$, $x = 0$. Since it is clearly measurable we will check that it is locally integrable near the origin, i.e.,

$$\int_{\substack{0 \le t \le 1 \\ |x| \le 1}} E(t,x)\, dx\, dt < \infty\,, \tag{6.41}$$

since $E \ge 0$. We can change variables, setting $X = x/t^{1/2}$, so $dx = t^{n/2}\, dX$ and the integral becomes

$$\frac{1}{(4\pi)^{n/2}} \int_0^{\prime} \int_{|X| \le t^{-1/2}} \exp(-\frac{|X|^2}{4})\, dx\, dt \ < \infty\,.$$

Since $E$ is actually bounded near infinity, it follows that $E \in \mathcal{S}'\mathbb{R}^n$,

$$E(\varphi) = \int_{t \ge 0} E(t,x)\varphi(t,x)\, dx\, dt \ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^{n+1})\,.$$

As before we want to compute

$$\begin{aligned} (\partial_t + \Delta)E(\varphi) &= E(-\partial_t\varphi + \Delta\varphi) \\ &= \lim_{\mathcal{E}\downarrow 0} \int_{\mathcal{E}}^{\infty} \int_{\mathbb{R}^n} E(t,x)(-\partial_t\varphi + \Delta\varphi)\, dx\, dt\,. \end{aligned} \tag{6.42}$$

First we check that $(\partial_t + \Delta)E = 0$ in $t > 0$, where it is a $\mathcal{C}^\infty$ function. This is a straightforward computation:

$$\begin{aligned} \partial_t E &= -\frac{n}{2t}E + \frac{|x|^2}{4t^2}E \\ \partial_{x_j} E &= -\frac{x_j}{2t}E\,,\ \partial^2_{x_j} E = -\frac{1}{2t}E + \frac{x_j^2}{4t^2}E \\ \Rightarrow \Delta E &= \frac{n}{2t}E + \frac{|x|^2}{4t^2}E\,. \end{aligned}$$

Now we can integrate by parts in (6.42) to get

$$(\partial_t + \Delta)E(\varphi) = \lim_{\mathcal{E}\downarrow 0} \int_{\mathbb{R}^n} \varphi(\mathcal{E},x)\frac{e^{-|x|^2/4\mathcal{E}}}{(4\pi\mathcal{E})^{n/2}}\, dx\,.$$

Making the same change of variables as before, $X = x/2\mathcal{E}^{1/2}$,

$$(\partial_t + \Delta)E(\varphi) = \lim_{\mathcal{E}\downarrow 0} \int_{\mathbb{R}^n} \varphi(\mathcal{E},\mathcal{E}^{1/2}X)\frac{e^{-|x|^2}}{\pi^{n/2}}\, dX\,.$$

As $\mathcal{E} \downarrow 0$ the integral here is bounded by the integrable function $C \exp(-|X|^2)$, for some $C > 0$, so by Lebesgue's theorem of dominated convergence, conveys to the integral of the limit. This is

$$\varphi(0,0) \cdot \int_{\mathbb{R}^n} e^{-|x|^2} \frac{dx}{\pi^{n/2}} = \varphi(0,0)\,.$$

Thus

$$(\partial_t + \Delta)E(\varphi) = \varphi(0,0) \Rightarrow (\partial_t + \Delta)E = \delta_t \delta_x\,,$$

so $E$ is indeed a fundamental solution. Since it vanishes in $t < 0$ it is called a *forward fundamrntal* solution.

Let's see what we can use it for.

PROPOSITION 6.16. *If $f \in \mathcal{S}'\mathbb{R}^n$ has compact support $\exists\,!u \in \mathcal{S}'\mathbb{R}^n$ with* $\operatorname{supp}(m) \subset \{t \geq -T\}$ *for some $T$ and*

$$(\partial_t + \Delta)u = f \ \text{ in } \mathbb{R}^{n+1}\,. \tag{6.43}$$

PROOF. Naturally we try $u = E * f$. That it satisfies (6.43)follows from the properties of convolution. Similarly if $T$ is such that $\operatorname{supp}(f) \subset \{t \geq T\}$ then

$$\operatorname{supp}(u) \subset \operatorname{supp}(f) + \operatorname{supp}(E) \subset \{t \geq T]\,.$$

So we need to show *uniqueness*. If $u_1, u_2 \in \mathcal{S}'\mathbb{R}^n$ in two solutions of (6.43) then their difference $v = u_1 - u_2$ satisfies the 'homogeneous' equation $(\partial_t + \Delta)v = 0$. Furthermore, $v = 0$ in $t < T'$ for some $T'$. Given any $E \in \mathbb{R}$ choose $\varphi(t) \in \mathcal{C}^\infty(\mathbb{R})$ with $\varphi(t) = 0$ in $t > \bar{t} + 1$, $\varphi(t) = 1$ in $t < \bar{t}$ and consider

$$E_{\bar{t}} = \varphi(t)E = F_1 + F_2\,,$$

where $F_1 = \psi E_{\bar{t}}$ for some $\psi \in \mathcal{C}_c^\infty \mathbb{R}^{n+1})$, $\psi = 1$ near 0. Thus $F_1$ has comapct support and in fact $F_2 \in \mathcal{S}\mathbb{R}^n$. I ask you to check this last statement as Problem L18.P1.

Anyway,

$$(\partial_t + \Delta)(F_1 + F_2) = \delta + \psi \in \mathcal{S}\mathbb{R}^n\,,\ \psi_{\bar{t}} = 0\ t \leq \bar{t}\,.$$

Now,

$$(\partial_t + \Delta)(E_t * u) = 0 = u + \psi_{\bar{t}} * u\,.$$

Since $\operatorname{supp}(\psi_{\bar{t}}) \subset \big\{t \geq \bar{t}]$, the second tier here is supported in $t \geq \bar{t} \geq T'$. Thus $u = 0$ in $t < \bar{t} + T'$, but $\bar{t}$ is arbitrary, so $u = 0$. $\square$

Notice that the assumption that $u \in \mathcal{S}'\mathbb{R}^n$ is not redundant in the statement of the Proposition, if we allow "large" solutions they become non-unique. Problem L18.P2 asks you to apply the fundamental solution to solve the initial value problem for the heat operator.

Next we make similar use of the fundamental solution for Laplace's operator. If $n \geq 3$ the

$$E = C_n |x|^{-n+2} \tag{6.44}$$

is a fundamental solution. You should check that $\Delta E_n = 0$ in $x \neq 0$ directly, I will show later that $\Delta E_n = \delta$, for the appropriate choice of $C_n$, but you can do it directly, as in the case $n = 3$.

THEOREM 6.17. *If $f \in \mathcal{S}\mathbb{R}^n$ $\exists\, !u \in \mathcal{C}_0^\infty \mathbb{R}^n$ such that $\Delta u = f$.*

PROOF. Since convolution $u = E * f \in \mathcal{S}'\mathbb{R}^n \cap \mathcal{C}^\infty \mathbb{R}^n$ is defined we certainly get a solution to $\Delta u = f$ this way. We need to check that $u \in \mathcal{C}_0^\infty \mathbb{R}^n$. First we know that $\Delta$ is hypoelliptic so we can decompose

$$E = F_1 + F_2\,,\ F_1 \in \mathcal{S}'\mathbb{R}^n\,, \operatorname{supp} F, \Subset \mathbb{R}^n$$

and then $F_2 \in \mathcal{C}^\infty \mathbb{R}^n$. In fact we can see from (6.44) that

$$|D^\alpha F_2(x)| \leq C_\alpha (1 + |x|)^{-n+2-|\alpha|}\,.$$

Now, $F_1 * f \in \mathcal{S}\mathbb{R}^n$, as we showed before, and continuing the integral we see that

$$\begin{aligned} |D^\alpha u| &\leq |D^\alpha F_2 * f| + C_N (1 + |x|)^{-N} \ \forall\ N \\ &\leq C'_\alpha (1 + |x|)^{-n+2-|\alpha|}\,. \end{aligned}$$

Since $n > 2$ it follows that $u \in \mathcal{C}_0^\infty \mathbb{R}^n$.

So only the uniqueness remains. If there are two solutions, $u_1, u_2$ for a given $f$ then $v = u_1 - u_2 \in \mathcal{C}_0^\infty \mathbb{R}^n$ satisfies $\Delta v = 0$. Since $v \in \mathcal{S}'\mathbb{R}^n$ we can take the Fourier transform and see that

$$|\chi|^2 \,\widehat{v}(\chi) = 0 \Rightarrow \operatorname{supp}(\widehat{v}) \subset \{0\}\ .$$

an earlier problem was to conclude from this that $\widehat{v} = \sum_{|\alpha| \leq m} C_\alpha D^\alpha \delta$ for some constants $C_\alpha$. This in turn implies that $v$ is a polynomial. However the only polynomials in $\mathcal{C}_0^0 \mathbb{R}^n$ are identically 0. Thus $v = 0$ and uniqueness follows. $\square$

## 7. Cone support and wavefront set

In discussing the singular support of a tempered distibution above, notice that

$$\operatorname{singsupp}(u) = \emptyset$$

only implies that $u \in \mathcal{C}^\infty(\mathbb{R}^n)$, not as one might want, that $u \in \mathcal{S}(\mathbb{R}^n)$. We can however 'refine' the concept of singular support a little to get this.

Let us think of the sphere $\mathbb{S}^{n-1}$ as the set of 'asymptotic directions' in $\mathbb{R}^n$. That is, we identify a point in $\mathbb{S}^{n-1}$ with a half-line $\{a\bar{x}; a \in$

$(0,\infty)\}$ for $0 \neq \bar{x} \in \mathbb{R}^n$. Since two points give the same half-line if and only if they are positive multiples of each other, this means we think of the sphere as the quotient

$$\mathbb{S}^{n-1} = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}^+. \tag{7.1}$$

Of course if we have a metric on $\mathbb{R}^n$, for instance the usual Euclidean metric, then we can identify $\mathbb{S}^{n-1}$ with the unit sphere. However (7.1) does not require a choice of metric.

Now, suppose we consider functions on $\mathbb{R}^n \setminus \{0\}$ which are (positively) homogeneous of degree 0. That is $f(a\bar{x}) = f(\bar{x})$, for all $a > 0$, and they are just functions on $\mathbb{S}^{n-1}$. Smooth functions on $\mathbb{S}^{n-1}$ correspond (if you like by definition) with smooth functions on $\mathbb{R}^n \setminus \{0\}$ which are homogeneous of degree 0. Let us take such a function $\psi \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$, $\psi(ax) = \psi(x)$ for all $a > 0$. Now, to make this smooth on $\mathbb{R}^n$ we need to cut it off near 0. So choose a cutoff function $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, with $\chi(x) = 1$ in $|x| < 1$. Then

$$\psi_R(x) = \psi(x)(1 - \chi(x/R)) \in \mathcal{C}^\infty(\mathbb{R}^n), \tag{7.2}$$

for any $R > 0$. This function is supported in $|x| \geq R$. Now, if $\psi$ has support near some point $\omega \in \mathbb{S}^{n-1}$ then for $R$ large the corresponding function $\psi_R$ will 'localize near $\omega$ as a point at infinity of $\mathbb{R}^n$.' Rather than try to understand this directly, let us consider a corresponding analytic construction.

First of all, a function of the form $\psi_R$ is a multiplier on $\mathcal{S}(\mathbb{R}^n)$. That is,

$$\psi_R \cdot : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{7.3}$$

To see this, the main problem is to estimate the derivatives at infinity, since the product of smooth functions is smooth. This in turn amounts to estimating the deriviatives of $\psi$ in $|x| \geq 1$. This we can do using the homogeneity.

Lemma 7.1. *If $\psi \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\})$ is homogeneous of degree 0 then*

$$|D^\alpha \psi| \leq C_\alpha |x|^{-|\alpha|}. \tag{7.4}$$

Proof. I should not have even called this a lemma. By the chain rule, the derivative of order $\alpha$ is a homogeneous function of degree $-|\alpha|$ from which (7.4) follows. □

For the smoothed versio, $\psi_R$, of $\psi$ this gives the estimates

$$|D^\alpha \psi_R(x)| \leq C_\alpha \langle x \rangle^{-|\alpha|}. \tag{7.5}$$

This allows us to estimate the derivatives of the product of a Schwartz function and $\psi_R$ :

$$(7.6)\quad x^\beta D^\alpha(\psi_R f) = \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma} D^{\alpha-\gamma}\psi_R x^\beta D^\gamma f \Longrightarrow \sup_{|x|\ge1}|x^\beta D^\alpha(\psi_R f)| \le C\sup\|f\|_k$$

for some seminorm on $\mathcal{S}(\mathbb{R}^n)$. Thus the map (7.3) is actually continuous. This continuity means that $\psi_R$ is a multiplier on $\mathcal{S}'(\mathbb{R}^n)$, defined as usual by duality:

$$(7.7)\qquad \psi_R u(f) = u(\psi_R f)\ \forall\ f\in\mathcal{S}(\mathbb{R}^n).$$

Definition 7.2. *The cone-support and cone-singular-support of a tempered distribution are the subsets* $\operatorname{Csp}(u)\subset\mathbb{R}^n\cup\mathbb{S}^{n-1}$ *and* $\operatorname{Css}(u)\subset\mathbb{R}^n\cup\mathbb{S}^{n-1}$ *defined by the conditions*

(7.8)

$$\begin{aligned}
\operatorname{Csp}(u)\cap\mathbb{R}^n &= \operatorname{supp}(u)\\
(\operatorname{Csp}(u))^\complement\cap\mathbb{S}^{n-1} &= \{\omega\in\mathbb{S}^{n-1};\\
&\qquad \exists\ R>0,\ \psi\in\mathcal{C}^\infty(\mathbb{S}^{n-1}),\ \psi(\omega)\ne0,\ \psi_R u=0\},\\
\operatorname{Css}(u)\cap\mathbb{R}^n &= \operatorname{singsupp}(u)\\
(\operatorname{Css}(u))^\complement\cap\mathbb{S}^{n-1} &= \{\omega\in\mathbb{S}^{n-1};\\
&\qquad \exists\ R>0,\ \psi\in\mathcal{C}^\infty(\mathbb{S}^{n-1}),\ \psi(\omega)\ne0,\ \psi_R u\in\mathcal{S}(\mathbb{R}^n)\}.
\end{aligned}$$

That is, on the $\mathbb{R}^n$ part these are the same sets as before but 'at infinity' they are defined by conic localization on $\mathbb{S}^{n-1}$.

In considering $\operatorname{Csp}(u)$ and $\operatorname{Css}(u)$ it is convenient to combine $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ into a compactification of $\mathbb{R}^n$. To do so (topologically) let us identify $\mathbb{R}^n$ with the interior of the unit ball with respect to the Euclidean metric using the map

$$(7.9)\qquad \mathbb{R}^n\ni x\longmapsto\frac{x}{\langle x\rangle}\in\{y\in\mathbb{R}^n;|y|\le1\}=\mathbb{B}^n.$$

Clearly $|x|<\langle x\rangle$ and for $0\le a<1$, $|x|=a\langle x\rangle$ has only the solution $|x|=a/(1-a^2)^{\frac12}$. Thus if we combine (7.9) with the identification of $\mathbb{S}^n$ with the unit sphere we get an identification

$$(7.10)\qquad \mathbb{R}^n\cup\mathbb{S}^{n-1}\simeq\mathbb{B}^n.$$

Using this identification we can, and will, regard $\operatorname{Csp}(u)$ and $\operatorname{Css}(u)$ as subsets of $\mathbb{B}^n$.[12]

[12]In fact while the topology here is correct the smooth structure on $\mathbb{B}^n$ is *not the right one*™– see Problem?? For our purposes here this issue is irrelevant.

LEMMA 7.3. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$, $\operatorname{Csp}(u)$ *and* $\operatorname{Css}(u)$ *are closed subsets of* $\mathbb{B}^n$ *and if* $\tilde{\psi} \in \mathcal{C}^\infty(\mathbb{S}^n)$ *has* $\operatorname{supp}(\tilde{\psi}) \cap \operatorname{Css}(u) = \emptyset$ *then for* $R$ *sufficiently large* $\tilde{\psi}_R u \in \mathcal{S}(\mathbb{R}^n)$.

PROOF. Directly from the definition we know that $\operatorname{Csp}(u) \cap \mathbb{R}^n$ is closed, as is $\operatorname{Css}(u) \cap \mathbb{R}^n$. Thus, in each case, we need to show that if $\omega \in \mathbb{S}^{n-1}$ and $\omega \notin \operatorname{Csp}(u)$ then $\operatorname{Csp}(u)$ is disjoint from some neighbourhood of $\omega$ in $\mathbb{B}^n$. However, by definition,

$$U = \{x \in \mathbb{R}^n; \psi_R(x) \neq 0\} \cup \{\omega' \in \mathbb{S}^{n-1}; \psi(\omega') \neq 0\}$$

is such a neighbourhood. Thus the fact that $\operatorname{Csp}(u)$ is closed follows directly from the definition. The argument for $\operatorname{Css}(u)$ is essentially the same.

The second result follows by the use of a partition of unity on $\mathbb{S}^{n-1}$. Thus, for each point in $\operatorname{supp}(\psi) \subset \mathbb{S}^{n-1}$ there exists a conic localizer for which $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$. By compactness we may choose a finite number of these functions $\psi_j$ such that the open sets $\{\psi_j(\omega) > 0\}$ cover $\operatorname{supp}(\tilde{\psi})$. By assumption $(\psi_j)_{R_j} u \in \mathcal{S}(\mathbb{R}^n)$ for some $R_j > 0$. However this will remain true if $R_j$ is increased, so we may suppose that $R_j = R$ is independent of $j$. Then for function

$$\mu = \sum_j |\psi_j|^2 \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$$

we have $\mu_R u \in \mathcal{S}(\mathbb{R}^n)$. Since $\tilde{\psi} = \psi' \mu$ for some $\mu \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ it follows that $\tilde{\psi}_{R+1} u \in \mathcal{S}(\mathbb{R}^n)$ as claimed. □

COROLLARY 7.4. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *then* $\operatorname{Css}(u) = \emptyset$ *if and only if* $u \in \mathcal{S}(\mathbb{R}^n)$.

PROOF. Certainly $\operatorname{Css}(u) = \emptyset$ if $u \in \mathcal{S}(\mathbb{R}^n)$. If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{Css}(u) = \emptyset$ then from Lemma 7.3, $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$ where $\psi = 1$. Thus $v = (1 - \psi_R)u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ has $\operatorname{singsupp}(v) = \emptyset$ so $v \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and hence $u \in \mathcal{S}(\mathbb{R}^n)$. □

Of course the analogous result for $\operatorname{Csp}(u)$, that $\operatorname{Csp}(u) = \emptyset$ if and only if $u = 0$ follows from the fact that this is true if $\operatorname{supp}(u) = \emptyset$. I will treat a few other properties as self-evident. For instance

(7.11)
$$\operatorname{Csp}(\phi u) \subset \operatorname{Csp}(u),\ \operatorname{Css}(\phi u) \subset \operatorname{Css}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n),\ \phi \in \mathcal{S}(\mathbb{R}^n)$$

and

$$\begin{aligned}(7.12)\quad \mathrm{Csp}(c_1u_1+c_2u_2)&\subset \mathrm{Csp}(u_1)\cup\mathrm{Csp}(u_2),\\ \mathrm{Css}(c_1u_1+c_2u_2)&\subset \mathrm{Css}(u_1)\cup\mathrm{Css}(u_2)\\ &\forall\ u_1,u_2\in\mathcal{S}'(\mathbb{R}^n),\ c_1,c_2\in\mathbb{C}.\end{aligned}$$

One useful consequence of having the cone support at our disposal is that we can discuss sufficient conditions to allow us to multiply distributions; we will get better conditions below using the same idea but applied to the wavefront set but this preliminary discussion is used there. In general the product of two distributions is not defined, and indeed not definable, as a distribution. However, we can always multiply an element of $\mathcal{S}'(\mathbb{R}^n)$ and an element of $\mathcal{S}(\mathbb{R}^n)$.

To try to understand multiplication look at the question of *pairing* between two distributions.

Lemma 7.5. *If $K_i\subset\mathbb{B}^n$, $i=1,2$, are two disjoint closed (hence compact) subsets then we can define an unambiguous pairing*

(7.13)
$$\{u\in\mathcal{S}'(\mathbb{R}^n);\mathrm{Css}(u)\subset K_1\}\times\{u\in\mathcal{S}'(\mathbb{R}^n);\mathrm{Css}(u)\subset K_2\}\ni(u_1,u_2)\longrightarrow u_1(u_2)\in\mathbb{C}.$$

Proof. To define the pairing, choose a function $\psi\in\mathcal{C}^\infty(\mathbb{S}^{n-1})$ which is identically equal to 1 in a neighbourhood of $K_1\cap\mathbb{S}^{n-1}$ and with support disjoint from $K_2\cap\mathbb{S}^{n-1}$. Then extend it to be homogeneous, as above, and cut off to get $\psi_R$. If $R$ is large enough $\mathrm{Csp}(\psi_R)$ is disjoint from $K_2$. Then $\psi_R+(1-\psi)_R=1+\nu$ where $\nu\in\mathcal{C}^\infty_c(\mathbb{R}^n)$. We can find another function $\mu\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\psi_1=\psi_R+\mu=1$ in a neighbourhood of $K_1$ and with $\mathrm{Csp}(\psi_1)$ disjoint from $K_2$. Once we have this, for $u_1$ and $u_2$ as in (7.13),

$$\psi_1u_2\in\mathcal{S}(\mathbb{R}^n)\text{ and }(1-\psi_1)u_1\in\mathcal{S}(\mathbb{R}^n)\tag{7.14}$$

since in both cases Css is empty from the definition. Thus we can define the desired pairing between $u_1$ and $u_2$ by

$$u_1(u_2)=u_1(\psi_1u_2)+u_2((1-\psi_1)u_1).\tag{7.15}$$

Of course we should check that this definition is independent of the cut-off function used in it. However, if we go through the definition and choose a different function $\psi'$ to start with, extend it homogeneoulsy and cut off (probably at a different $R$) and then find a correction term $\mu'$ then the 1-parameter linear homotopy between them

$$\psi_1(t)=t\psi_1+(1-t)\psi'_1,\ t\in[0,1]\tag{7.16}$$

satisfies all the conditions required of $\psi_1$ in formula (7.14). Thus in fact we get a smooth family of pairings, which we can write for the moment as

$$(u_1,u_2)_t = u_1(\psi_1(t)u_2) + u_2((1-\psi_1(t))u_1). \tag{7.17}$$

By inspection, this is an affine-linear function of $t$ with derivative

$$u_1((\psi_1-\psi_1')u_2) + u_2((\psi_1'-\psi_1))u_1). \tag{7.18}$$

Now, we just have to justify moving the smooth function in (7.18) to see that this gives zero. This should be possible since $\mathrm{Csp}(\psi_1'-\psi_1)$ is disjoint from *both* $K_1$ and $K_2$.

In fact, to be very careful for once, we should construct another function $\chi$ in the same way as we constructed $\psi_1$ to be homogenous near infinity and smooth and such that $\mathrm{Csp}(\chi)$ is also disjoint from both $K_1$ and $K_2$ but $\chi=1$ on $\mathrm{Csp}(\psi_1'-\psi_1)$. Then $\chi(\psi_1'-\psi_1)=\psi_1'-\psi_1$ so we can insert it in (7.18) and justify

$$\begin{aligned} u_1((\psi_1-\psi_1')u_2) &= u_1(\chi^2(\psi_1-\psi_1')u_2) = (\chi u_1)((\psi_1-\psi_1')\chi u_2) \\ &= (\chi u_2)(\psi_1-\psi_1')\chi u_1) = u_2(\psi_1-\psi_1')\chi u_1). \end{aligned} \tag{7.19}$$

Here the second equality is just the identity for $\chi$ as a (multiplicative) linear map on $\mathcal{S}(\mathbb{R}^n)$ and hence $\mathcal{S}'(\mathbb{R}^n)$ and the operation to give the crucial, third, equality is permissible because both elements are in $\mathcal{S}(\mathbb{R}^n)$. $\square$

Once we have defined the pairing between tempered distibutions with disjoint conic singular supports, in the sense of (7.14), (7.15), we can define the product under the same conditions. Namely to define the product of say $u_1$ and $u_2$ we simply set

$$\begin{aligned} &u_1u_2(\phi) = u_1(\phi u_2) = u_2(\phi u_1)\ \forall\ \phi\in\mathcal{S}(\mathbb{R}^n), \\ &\qquad\qquad\text{provided }\ \mathrm{Css}(u_1)\cap\mathrm{Css}(u_2)=\emptyset. \end{aligned} \tag{7.20}$$

Indeed, this would be true if one of $u_1$ or $u_2$ was itself in $\mathcal{S}(\mathbb{R}^n)$ and makes sense in general. I leave it to you to check the continuity statement required to prove that the product is actually a tempered distibution (Problem 78).

One can also give a similar discussion of the convolution of two tempered distributions. Once again we do not have a definition of $u*v$ as a tempered distribution for all $u,\ v\in\mathcal{S}'(\mathbb{R}^n)$. We do know how to define the convolution if either $u$ or $v$ is compactly supported, or if either is in $\mathcal{S}(\mathbb{R}^n)$. This leads directly to

LEMMA 7.6. *If* $\mathrm{Css}(u) \cap \mathbb{S}^{n-1} = \emptyset$ *then* $u * v$ *is defined unambiguously by*

$$u * v = u_1 * v + u_2 * v,\ u_1 = (1 - \chi(\frac{x}{r}))u,\ u_2 = u - u_1 \tag{7.21}$$

*where* $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ *has* $\chi(x) = 1$ *in* $|x| \le 1$ *and* $R$ *is sufficiently large; there is a similar definition if* $\mathrm{Css}(v) \cap \mathbb{S}^{n-1} = \emptyset$.

PROOF. Since $\mathrm{Css}(u) \cap \mathbb{S}^{n-1} = \emptyset$, we know that $\mathrm{Css}(u_1) = \emptyset$ if $R$ is large enough, so then both terms on the right in (7.21) are well-defined. To see that the result is independent of $R$ just observe that the difference of the right-hand side for two values of $R$ is of the form $w * v - w * v$ with $w$ compactly supported. □

Now, we can go even further using a slightly more sophisticated decomposition based on

LEMMA 7.7. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\mathrm{Css}(u) \cap \Gamma = \emptyset$ *where* $\Gamma \subset \mathbb{S}^{n-1}$ *is a closed set, then* $u = u_1 + u_2$ *where* $\mathrm{Csp}(u_1) \cap \Gamma = \emptyset$ *and* $u_2 \in \mathcal{S}(\mathbb{R}^n)$; *in fact*

$$\begin{gathered} u = u_1' + u_1'' + u_2 \text{ where } u_1' \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \text{ and} \\ 0 \notin \mathrm{supp}(u_1''),\ x \in \mathbb{R}^n \setminus \{0\},\ x/|x| \in \Gamma \Longrightarrow x \notin \mathrm{supp}(u_1''). \end{gathered} \tag{7.22}$$

PROOF. A covering argument which you should provide. □

Let $\Gamma_i \subset \mathbb{R}^n$, $i = 1, 2$, be closed cones. That is they are closed sets such that if $x \in \Gamma_i$ and $a > 0$ then $ax \in \Gamma_i$. Suppose in addition that

$$\Gamma_1 \cap (-\Gamma_2) = \{0\}. \tag{7.23}$$

That is, if $x \in \Gamma_1$ and $-x \in \Gamma_2$ then $x = 0$. Then it follows that for some $c > 0$,

$$x \in \Gamma_1,\ y \in \Gamma_2 \Longrightarrow |x + y| \ge c(|x| + |y|). \tag{7.24}$$

To see this consider $x + y$ where $x \in \Gamma_1$, $y \in \Gamma_2$ and $|y| \le |x|$. We can assume that $x \ne 0$, otherwise the estimate is trivially true with $c = 1$, and then $Y = y/|x| \in \Gamma_1$ and $X = x/|x| \in \Gamma_2$ have $|Y| \le 1$ and $|X| = 1$. However $X + Y \ne 0$, since $|X| = 1$, so by the continuity of the sum, $|X + Y| \ge 2c > 0$ for some $c > 0$. Thus $|X + Y| \ge c(|X| + |Y|)$ and the result follows by scaling back. The other case, of $|x| \le |y|$ follows by the same argument with $x$ and $y$ interchanged, so (7.24) is a consequence of (7.23).

LEMMA 7.8. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\mathrm{Css}(\phi * u) \subset \mathrm{Css}(u) \cap \mathbb{S}^{n-1}. \tag{7.25}$$

Proof. We already know that $\phi * u$ is smooth, so $\mathrm{Css}(\phi * u) \subset \mathbb{S}^{n-1}$. Thus, we need to show that if $\omega \in \mathbb{S}^{n-1}$ and $\omega \notin \mathrm{Css}(u)$ then $\omega \notin \mathrm{Css}(\phi * u)$.

Fix such a point $\omega \in \mathbb{S}^{n-1} \setminus \mathrm{Css}(u)$ and take a closed set $\Gamma \subset \mathbb{S}^{n-1}$ which is a neighbourhood of $\omega$ but which is still disjoint from $\mathrm{Css}(u)$ and then apply Lemma 7.7. The two terms $\phi * u_2$, where $u_2 \in \mathcal{S}(\mathbb{R}^n)$ and $\phi * u_1'$ where $u_1' \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ are both in $\mathcal{S}(\mathbb{R}^n)$ so we can assume that $u$ has the support properties of $u_1''$. In particular there is a smaller closed subset $\Gamma_1 \subset \mathbb{S}^{n-1}$ which is still a neighbourhood of $\omega$ but which does not meet $\Gamma_2$, which is the closure of the complement of $\Gamma$. If we replace these $\Gamma_i$ by the closed cones of which they are the 'cross-sections' then we are in the situation of (7.23) and (7.23), except for the signs. That is, there is a constant $c > 0$ such that

$$|x - y| \geq c(|x| + |y|). \tag{7.26}$$

Now, we can assume that there is a cutoff function $\psi_R$ which has support in $\Gamma_2$ and is such that $u = \psi_R u$. For any conic cutoff, $\psi_R'$, with support in $\Gamma_1$

$$\psi_R'(\phi * u) = \langle \psi_R u, \phi(x - \cdot) \rangle = \langle u(y), \psi_R(y)\psi_R'(x)\phi(x - y) \rangle. \tag{7.27}$$

The continuity of $u$ means that this is estimated by some Schwartz seminorm

$$\begin{aligned} &\sup_{y, |\alpha| \leq k} |D_y^\alpha(\psi_R(y)\psi_R'(x)\phi(x - y))|(1 + |y|)^k \\ &\quad \leq C_N \|\phi\| \sup_y (1 + |x| + |y|)^{-N}(1 + |y|)^k \leq C_N \|\phi\|(1 + |x|)^{-N+k} \end{aligned} \tag{7.28}$$

for some Schwartz seminorm on $\phi$. Here we have used the estimate (7.24), in the form (7.26), using the properties of the supports of $\psi_R'$ and $\psi_R$. Since this is true for any $N$ and similar estimates hold for the derivatives, it follows that $\psi_R'(u * \phi) \in \mathcal{S}(\mathbb{R}^n)$ and hence that $\omega \notin \mathrm{Css}(u * \phi)$. □

Corollary 7.9. *Under the conditions of Lemma 7.6*

$$\mathrm{Css}(u * v) \subset (\mathrm{singsupp}(u) + \mathrm{singsupp}(v)) \cup (\mathrm{Css}(v) \cap \mathbb{S}^{n-1}). \tag{7.29}$$

Proof. We can apply Lemma 7.8 to the first term in (7.21) to conclude that it has conic singular support contained in the second term in (7.29). Thus it is enough to show that (7.29) holds when $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$. In that case we know that the singular support of the convolution is contained in the first term in (7.29), so it is enough to consider the conic singular support in the sphere at infinity. Thus, if $\omega \notin \mathrm{Css}(v)$ we need to show that $\omega \notin \mathrm{Css}(u * v)$. Using Lemma 7.7

we can decompose $v = v_1 + v_2 + v_3$ as a sum of a Schwartz term, a compact supported term and a term which does not have $\omega$ in its conic support. Then $u * v_1$ is Schwartz, $u * v_2$ has compact support and satisfies (7.29) and $\omega$ is not in the cone support of $u * v_3$. Thus (7.29) holds in general. $\square$

Lemma 7.10. *If $u,\ v \in \mathcal{S}'(\mathbb{R}^n)$ and $\omega \in \operatorname{Css}(u) \cap \mathbb{S}^{n-1} \Longrightarrow -\omega \notin \operatorname{Css}(v)$ then their convolution is defined unambiguously, using the pairing in Lemma 7.5, by*

$$u * v(\phi) = u(\check{v} * \phi)\ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n). \tag{7.30}$$

Proof. Since $\check{v}(x) = v(-x)$, $\operatorname{Css}(\check{v}) = -\operatorname{Css}(v)$ so applying Lemma 7.8 we know that

$$\operatorname{Css}(\check{v} * \phi) \subset -\operatorname{Css}(v) \cap \mathbb{S}^{n-1}. \tag{7.31}$$

Thus, $\operatorname{Css}(v) \cap \operatorname{Css}(\check{v} * \phi) = \emptyset$ and the pairing on the right in (7.30) is well-defined by Lemma 7.5. Continuity follows from your work in Problem 78. $\square$

In Problem 79 I ask you to get a bound on $\operatorname{Css}(u * v) \cap \mathbb{S}^{n-1}$ under the conditions in Lemma 7.10.

Let me do what is actually a fundamental computation.

Lemma 7.11. *For a conic cutoff, $\psi_R$, where $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$,*

$$\operatorname{Css}(\widehat{\psi_R}) \subset \{0\}. \tag{7.32}$$

Proof. This is actually much easier than it seems. Namely we already know that $D^\alpha(\psi_R)$ is smooth and homogeneous of degree $-|\alpha|$ near infinity. From the same argument it follows that

$$D^\alpha(x^\beta \psi_R) \in L^2(\mathbb{R}^n) \text{ if } |\alpha| > |\beta| + n/2 \tag{7.33}$$

since this is a smooth function homogeneous of degree less than $-n/2$ near infinity, hence square-integrable. Now, taking the Fourier transform gives

$$\xi^\alpha D^\beta(\widehat{\psi_R}) \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| > |\beta| + n/2. \tag{7.34}$$

If we localize in a cone near infinity, using a (completely unrelated) cutoff $\psi'_{R'}(\xi)$ then we must get a Schwartz function since

$$|\xi|^{|\alpha|} \psi'_{R'}(\xi) D^\beta(\widehat{\psi_R}) \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| > |\beta| + n/2 \Longrightarrow \psi'_{R'}(\xi)\widehat{\psi_R} \in \mathcal{S}(\mathbb{R}^n). \tag{7.35}$$

Indeed this argument applies anywhere that $\xi \neq 0$ and so shows that (7.32) holds. $\square$

Now, we have obtained some reasonable looking conditions under which the product $uv$ or the convolution $u*v$ of two elements of $\mathcal{S}'(\mathbb{R}^n)$ is defined. However, reasonable as they might be there is clearly a flaw, or at least a deficiency, in the discussion. We know that in the simplest of cases,

$$\widehat{u*v} = \widehat{u}\widehat{v}. \tag{7.36}$$

Thus, it is very natural to expect a relationship between the conditions under which the product of the Fourier transforms is defined and the conditions under which the convolution is defined. Is there? Well, not much it would seem, since on the one hand we are considering the relationship between $\operatorname{Css}(\widehat{u})$ and $\operatorname{Css}(\widehat{v})$ and on the other the relationship between $\operatorname{Css}(u) \cap \mathbb{S}^{n-1}$ and $\operatorname{Css}(v) \cap \mathbb{S}^{n-1}$. If these are to be related, we would have to find a relationship of some sort between $\operatorname{Css}(u)$ and $\operatorname{Css}(\widehat{u})$. As we shall see, there is one but it is not very strong as can be guessed from Lemma 7.11. This is not so much a bad thing as a sign that we should look for another notion which combines aspects of both $\operatorname{Css}(u)$ and $\operatorname{Css}(\widehat{u})$. This we will do through the notion of *wavefront set*. In fact we define two related objects. The first is the more conventional, the second is more natural in our present discussion.

Definition 7.12. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ we define the* wavefront set *of $u$ to be*

$$\begin{aligned}\operatorname{WF}(u) = \{(x,\omega) \in \mathbb{R}^n \times \mathbb{S}^{n-1};\\ \exists\ \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n),\ \phi(x) \neq 0,\ \omega \notin \operatorname{Css}(\widehat{\phi u})\}^{\complement}\end{aligned} \tag{7.37}$$

*and more generally the* scattering wavefront set *by*

$$\begin{aligned}\operatorname{WF}_{\mathrm{sc}}(u) = \operatorname{WF}(u) \cup \{(\omega,p) \in \mathbb{S}^{n-1} \times \mathbb{B}^n;\\ \exists\ \psi \in \mathcal{C}^\infty(\mathbb{S}^n),\ \psi(\omega) \neq 0,\ R>0 \text{ such that } p \notin \operatorname{Css}(\widehat{\psi_R u})\}^{\complement}.\end{aligned} \tag{7.38}$$

So, the definition is really always the same. To show that $(p,q) \notin \operatorname{WF}_{\mathrm{sc}}(u)$ we need to find 'a cutoff $\Phi$ near $p$' – depending on whether $p \in \mathbb{R}^n$ or $p \in \mathbb{S}^{n-1}$ this is either $\Phi = \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $F = \phi(p) \neq 0$ or a $\psi_R$ where $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ has $\psi(p) \neq 0$ – such that $q \notin \operatorname{Css}(\widehat{\Phi u})$. One crucial property is

Lemma 7.13. *If $(p,q) \notin \operatorname{WF}_{\mathrm{sc}}(u)$ then if $p \in \mathbb{R}^n$ there exists a neighbourhood $U \subset \mathbb{R}^n$ of $p$ and a neighbourhood $U \subset \mathbb{B}^n$ of $q$ such that for all $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with support in $U$, $U' \cap \operatorname{Css}(\widehat{\phi u}) = \emptyset$; similarly if $p \in \mathbb{S}^{n-1}$ then there exists a neigbourhood $\tilde{U} \subset \mathbb{B}^n$ of $p$ such that $U' \cap \operatorname{Css}(\widehat{\psi_R u}) = \emptyset$ if $\operatorname{Csp}(\omega_R) \subset \tilde{U}$.*

PROOF. First suppose $p \in \mathbb{R}^n$. From the definition of conic singular support, (7.37) means precisely that there exists $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$, $\psi(\omega) \neq 0$ and $R$ such that

$$\psi_R(\widehat{\phi u}) \in \mathcal{S}(\mathbb{R}^n). \tag{7.39}$$

Since we know that $\widehat{\phi u} \in \mathcal{C}^\infty(\mathbb{R}^n)$, this is actually true for all $R > 0$ as soon as it is true for one value. Furthermore, if $\phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ has $\operatorname{supp}(\phi') \subset \{\phi \neq 0\}$ then $\omega \notin \operatorname{Css}(\widehat{\phi' u})$ follows from $\omega \notin \operatorname{Css}(\widehat{\phi u})$. Indeed we can then write $\phi' = \mu\phi$ where $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ so it suffices to show that if $v \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ has $\omega \notin \operatorname{Css}(\widehat{v})$ then $\omega \notin \operatorname{Css}(\widehat{\mu v})$ if $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Since $\widehat{\mu v} = (2\pi)^{-n} v * \widehat{u}$ where $\check{v} = \widehat{\mu} \in \mathcal{S}(\mathbb{R}^n)$, applying Lemma 7.8 we see that $\operatorname{Css}(v * \widehat{v}) \subset \operatorname{Css}(\widehat{v})$, so indeed $\omega \notin \operatorname{Css}(\widehat{\phi' u})$.

The case that $p \in \mathbb{S}^{n-1}$ is similar. Namely we have one cut-off $\psi_R$ with $\psi(p) \neq 0$ and $q \notin \operatorname{Css}(\widehat{\omega_R u})$. We can take $U = \{\psi_{R+10} \neq 0\}$ since if $\psi'_{R'}$ has conic support in $U$ then $\psi'_{R'} = \psi'' R' \psi_R$ for some $\psi'' \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$. Thus

$$\widehat{\psi'_{R'} u} = v * \widehat{\psi_R u},\ \check{v} = \widehat{\omega''_{R''}}. \tag{7.40}$$

From Lemma 7.11 and Corollary7.9 we deduce that

$$\operatorname{Css}(\widehat{\psi'_{R'} u}) \subset \operatorname{Css}(\widehat{\omega_R u}) \tag{7.41}$$

and hence the result follows with $U'$ a small neighourhood of $q$. $\square$

PROPOSITION 7.14. *For any $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$\begin{aligned} \mathrm{WF}_{\mathrm{sc}}(u) \subset \partial(\mathbb{B}^n \times \mathbb{B}^n) &= (\mathbb{B}^n \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{B}^n) \\ &= (\mathbb{R}^n \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \cup (\mathbb{S}^{n-1} \times \mathbb{R}^n) \end{aligned} \tag{7.42}$$

*and* $\mathrm{WF}(u) \subset \mathbb{R}^n$ *are closed sets and under projection onto the first variable*

$$\pi_1(\mathrm{WF}(u)) = \operatorname{singsupp}(u) \subset \mathbb{R}^n,\ \pi_1(\mathrm{WF}_{\mathrm{sc}}(u)) = \operatorname{Css}(u) \subset \mathbb{B}^n. \tag{7.43}$$

PROOF. To prove the first part of (7.43) we need to show that if $(\bar{x}, \omega) \notin \mathrm{WF}(u)$ for all $\omega \in \mathbb{S}^{n-1}$ with $\bar{x} \in \mathbb{R}^n$ fixed, then $\bar{x} \notin \operatorname{singsupp}(u)$. The definition (7.37) means that for each $\omega \in \mathbb{S}^{n-1}$ there exists $\phi_\omega \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi_\omega(\bar{x}) \neq 0$ such that $\omega \notin \operatorname{Css}(\widehat{\phi_\omega u})$. Since $\operatorname{Css}(\phi u)$ is closed and $\mathbb{S}^{n-1}$ is compact, a finite number of these cutoffs, $\phi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, can be chosen so that $\phi_j(\bar{x}) \neq 0$ with the $\mathbb{S}^{n-1} \setminus \operatorname{Css}(\widehat{\phi_j u})$ covering $\mathbb{S}^{n-1}$. Now applying Lemma 7.13 above, we can find one $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, with support in $\bigcap_j \{\phi_j(x) \neq 0\}$ and $\phi(\bar{x}) \neq 0$, such that $\operatorname{Css}(\widehat{\phi u}) \subset \operatorname{Css}(\widehat{\phi_j u})$ for each $j$ and hence $\phi u \in \mathcal{S}(\mathbb{R}^n)$ (since it is already smooth). Thus indeed it follows that $\bar{x} \notin \operatorname{singsupp}(u)$. The converse, that $\bar{x} \notin \operatorname{singsupp}(u)$ implies $(\bar{x}, \omega) \notin \mathrm{WF}(u)$ for all $\omega \in \mathbb{S}^{n-1}$ is immediate.

The argument to prove the second part of (7.43) is similar. Since, by definition, $\mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{R}^n \times \mathbb{B}^n) = \mathrm{WF}(u)$ and $\mathrm{Css}(u) \cap \mathbb{R}^n = \operatorname{singsupp}(u)$ we only need consider points in $\mathrm{Css}(u) \cap \mathbb{S}^{n-1}$. Now, we first check that if $\theta \notin \mathrm{Css}(u)$ then $\{\theta\} \times \mathbb{B}^n \cap \mathrm{WF}_{\mathrm{sc}}(u) = \emptyset$. By definition of $\mathrm{Css}(u)$ there is a cut-off $\psi_R$, where $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\psi(\theta) \neq 0$, such that $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$. From (7.38) this implies that $(\theta, p) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ for all $p \in \mathbb{B}^n$.

Now, Lemma 7.13 allows us to apply the same argument as used above for WF. Namely we are given that $(\theta, p) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ for all $p \in \mathbb{B}^n$. Thus, for each $p$ we may find $\psi_R$, depending on $p$, such that $\psi(\theta) \neq 0$ and $p \notin \mathrm{Css}(\widehat{\psi_R u})$. Since $\mathbb{B}^n$ is compact, we may choose a finite subset of these conic localizers, $\psi^{(j)}_{R_j}$ such that the intersection of the corresponding sets $\mathrm{Css}(\widehat{\psi^{(j)}_{R_j} u})$, is empty, i.e. their complements cover $\mathbb{B}^n$. Now, using Lemma 7.13 we may choose one $\psi$ with support in the intersection of the sets $\{\psi^{(j)} \neq 0\}$ with $\psi(\theta) \neq 0$ and one $R$ such that $\mathrm{Css}(\widehat{\psi_R u}) = \emptyset$, but this just means that $\psi_R u \in \mathcal{S}(\mathbb{R}^n)$ and so $\theta \notin \mathrm{Css}(u)$ as desired.

The fact that these sets are closed (in the appropriate sets) follows directly from Lemma7.13. □

Corollary 7.15. *For* $u \in \mathcal{S}'(\mathbb{R}^n)$,

$$\mathrm{WF}_{\mathrm{sc}}(u) = \emptyset \Longleftrightarrow u \in \mathcal{S}(\mathbb{R}^n). \tag{7.44}$$

Let me return to the definition of $\mathrm{WF}_{\mathrm{sc}}(u)$ and rewrite it, using what we have learned so far, in terms of a decomposition of $u$.

Proposition 7.16. *For any* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $(p, q) \in \partial(\mathbb{B}^n \times \mathbb{B}^n)$,

$$\begin{aligned} (p, q) \notin \mathrm{WF}_{\mathrm{sc}}(u) &\Longleftrightarrow \\ &u = u_1 + u_2,\ u_1,\ u_2 \in \mathcal{S}'(\mathbb{R}^n),\ p \notin \mathrm{Css}(u_1),\ q \notin \mathrm{Css}(\widehat{u_2}). \end{aligned} \tag{7.45}$$

Proof. For given $(p, q) \notin \mathrm{WF}_{\mathrm{sc}}(u)$, take $\Phi = \phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi \equiv 1$ near $p$, if $p \in \mathbb{R}^n$ or $\Phi = \psi_R$ with $\psi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and $\psi \equiv 1$ near $p$, if $p \in \mathbb{S}^{n-1}$. In either case $p \notin \mathrm{Css}(u_1)$ if $u_1 = (1 - \Phi)u$ directly from the definition. So $u_2 = u - u_1 = \Phi u$. If the support of $\Phi$ is small enough it follows as in the discussion in the proof of Proposition 7.14 that

$$q \notin \mathrm{Css}(\widehat{u_2}). \tag{7.46}$$

Thus we have (7.45) in the forward direction.

For reverse implication it follows directly that $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u_1)$ and that $(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u_2)$. $\square$

This restatement of the definition makes it clear that there a high degree of symmetry under the Fourier transform

Corollary 7.17. *For any $u \in \mathcal{S}'(\mathbb{R}^n)$,*

$$(p,q) \in \mathrm{WF}_{\mathrm{sc}}(u)) \Longleftrightarrow (q,-p) \in \mathrm{WF}_{\mathrm{sc}}(\hat{u}). \tag{7.47}$$

Proof. I suppose a corollary should not need a proof, but still .... The statement (7.47) is equivalent to

$$(p,q) \notin \mathrm{WF}_{\mathrm{sc}}(u)) \Longrightarrow (q,-p) \notin \mathrm{WF}_{\mathrm{sc}}(\hat{u}) \tag{7.48}$$

since the reverse is the same by Fourier inversion. By (7.45) the condition on the left is equivalent to $u = u_1 + u_2$ with $p \notin \mathrm{Css}(u_1)$, $q \notin \mathrm{Css}(\widehat{u_2})$. Hence equivalent to

$$\widehat{u} = v_1 + v_2,\ v_1 = \widehat{u_2},\ \widehat{v_2} = (2\pi)^{-n}\check{u}_1 \tag{7.49}$$

so $q \notin \mathrm{Css}(v_1)$, $-p \notin \mathrm{Css}(\widehat{v_2})$ which proves (7.47). $\square$

Now, we can exploit these notions to refine our conditions under which pairing, the product and convolution can be defined.

Theorem 7.18. *For $u$, $v \in \mathcal{S}'(\mathbb{R}^n)$*

$$uv \in \mathcal{S}'(\mathbb{R}^n) \text{ is unambiguously defined provided}$$
$$(p,\omega) \in \mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{B}^n \times \mathbb{S}^{n-1}) \Longrightarrow (p,-\omega) \notin \mathrm{WF}_{\mathrm{sc}}(v) \tag{7.50}$$

*and*

$$u * v \in \mathcal{S}'(\mathbb{R}^n) \text{ is unambiguously defined provided}$$
$$(\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(u) \cap (\mathbb{S}^{n-1} \times \mathbb{B}^n) \Longrightarrow (-\theta,q) \notin \mathrm{WF}_{\mathrm{sc}}(v). \tag{7.51}$$

Proof. Let us consider convolution first. The hypothesis, (7.51) means that for each $\theta \in \mathbb{S}^{n-1}$

$$\{q \in \mathbb{B}^{n-1}; (\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(u)\} \cap \{q \in \mathbb{B}^{n-1}; (-\theta,q) \in \mathrm{WF}_{\mathrm{sc}}(v)\} = \emptyset. \tag{7.52}$$

Now, the fact that $\mathrm{WF}_{\mathrm{sc}}$ is always a closed set means that (7.52) remains true near $\theta$ in the sense that if $U \subset \mathbb{S}^{n-1}$ is a sufficiently small neighbourhood of $\theta$ then

$$\{q \in \mathbb{B}^{n-1}; \exists\ \theta' \in U,\ (\theta',q) \in \mathrm{WF}_{\mathrm{sc}}(u)\}$$
$$\cap \{q \in \mathbb{B}^{n-1}; \exists\ \theta'' \in U,\ (-\theta'',q) \in \mathrm{WF}_{\mathrm{sc}}(v)\} = \emptyset. \tag{7.53}$$

The compactness of $\mathbb{S}^{n-1}$ means that there is a finite cover of $\mathbb{S}^{n-1}$ by such sets $U_j$. Now select a partition of unity $\psi_i$ of $\mathbb{S}^{n-1}$ which is not only subordinate to this open cover, so each $\psi_i$ is supported in one of the $U_j$ but satisfies the additional condition that

$$\text{(7.54)} \quad \operatorname{supp}(\psi_i) \cap (-\operatorname{supp}(\psi_{i'})) \neq \emptyset \Longrightarrow \\ \operatorname{supp}(\psi_i) \cup (-\operatorname{supp}(\psi_{i'})) \subset U_j \text{ for some } j.$$

Now, if we set $u_i = (\psi_i)_R u$, and $v_{i'} = (\psi_{i'})_R v$, we know that $u - \sum_i u_i$ has compact support and similarly for $v$. Since convolution is already known to be possible if (at least) one factor has compact support, it suffices to define $u_i * v_{i'}$ for every $i$, $i'$. So, first suppose that $\operatorname{supp}(\psi_i) \cap (-\operatorname{supp}(\psi_{i'})) \neq \emptyset$. In this case we conclude from (7.54) that

$$\text{(7.55)} \qquad \operatorname{Css}(\widehat{u_i}) \cap \operatorname{Css}(\widehat{v_{i'}}) = \emptyset.$$

Thus we may *define*

$$\text{(7.56)} \qquad \widehat{u_i * v_{i'}} = \widehat{u}_i \widehat{v}_{i'}$$

using (7.20). On the other hand if $\operatorname{supp} \psi_i \cap (-\operatorname{supp}(\psi_{i'})) = \emptyset$ then

$$\text{(7.57)} \qquad \operatorname{Css}(u_i) \cap (-\operatorname{Css}(v_{i'})) \cap \mathbb{S}^{n-1} = \emptyset$$

and in this case we can define $u_i * v_{i'}$ using Lemma 7.10.

Thus with such a decomposition of $u$ and $v$ all terms in the convolution are well-defined. Of course we should check that this definition is independent of choices made in the decomposition. I leave this to you.

That the product is well-defined under condition (7.50) now follows if we define it using convolution, i.e. as

$$\text{(7.58)} \qquad \widehat{uv} = f * g, \ f = \widehat{u}, \ \check{g} = \widehat{v}.$$

Indeed, using (7.47), (7.50) for $u$ and $v$ becomes (7.51) for $f$ and $g$. $\square$

## 8. Homogeneous distributions

Next time I will talk about homogeneous distributions. On $\mathbb{R}$ the functions

$$x_t^s = \begin{cases} x^s & x > 0 \\ 0 & x < 0 \end{cases}$$

where $S \in \mathbb{R}$, is locally integrable (and hence a tempered distribution) precisely when $S > -1$. As a function it is homogeneous of degree $s$. Thus if $a > 0$ then

$$(ax)_t^s = a^s x_t^s \,.$$

Thinking of $x_t^s = \mu_s$ as a distribution we can set this as

$$\begin{aligned}\mu_s(ax)(\varphi) &= \int \mu_s(ax)\varphi(x)\,dx\\ &= \int \mu_s(x)\varphi(x/a)\frac{dx}{a}\\ &= a^s\mu_s(\varphi)\,.\end{aligned}$$

Thus if we *define* $\varphi_a(x) = \frac{1}{a}\varphi(\frac{x}{a})$, for any $a > 0$, $\varphi \in \mathcal{S}(\mathbb{R})$ we can ask whether a distribution is homogeneous:

$$\mu(\varphi_a) = a^s\mu(\varphi)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}).$$

## 9. Operators and kernels

From here on a summary of parts of 18.155 used in 18.156 – to be redistributed backwards With some corrections by incorporated.

## 10. Fourier transform

The basic properties of the Fourier transform, tempered distributions and Sobolev spaces form the subject of the first half of this course. I will recall and slightly expand on such a standard treatment.

## 11. Schwartz space.

The space $\mathcal{S}(\mathbb{R}^n)$ of all complex-volumed functions with rapidly decreasing derivatives of all orders is a complete metric space with metric

$$\begin{gathered}d(u,v) = \sum_{k=0}^{\infty} 2^{-k}\frac{\|u-v\|_{(k)}}{1+\|u-v\|_{(k)}} \text{ where}\\ \|u\|_{(k)} = \sum_{|\alpha|+|\beta|\le k}\sup_{z\in\mathbb{R}^n}|z^\alpha D_z^\beta u(z)|.\end{gathered}\tag{11.1}$$

Here and below I will use the notation for derivatives

$$D_z^\alpha = D_{z_1}^{\alpha_1}\dots,D_{z_n}^{\alpha_n},\ D_{z_j} = \frac{1}{i}1\frac{\partial}{\partial z_j}.$$

These norms can be replaced by other equivalent ones, for instance by reordering the factors

$$\|u\|'_{(k)} = \sum_{|\alpha|+|\beta|\le k}\sup_{z\in\mathbb{R}^n}|D_z^\beta(z^\beta u)|.$$

In fact it is only the cumulative effect of the norms that matters, so one can use

$$\|u\|''_{(k)} = \sup_{z\in\mathbb{R}^n} |\langle z\rangle^{2k}(\Delta+1)^k u| \tag{11.2}$$

in (11.1) and the same topology results. Here

$$\langle z\rangle^2 = 1 + |z|^2,\ \Delta = \sum_{j=1}^n D_j^2$$

(so the Laplacian is formally positive, the geometers' convention). It is not quite so trivial to see that inserting (11.2) in (11.1) gives an equivalent metric.

## 12. Tempered distributions.

The space of (metrically) continuous linear maps

$$f : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C} \tag{12.1}$$

is the space of tempered distribution, denoted $\mathcal{S}'(\mathbb{R}^n)$ since it is the dual of $\mathcal{S}(\mathbb{R}^n)$. The continuity in (12.1) is equivalent to the estimates

$$\exists\ k,\ C_k > 0 \text{ s.t. } |f(\varphi)| \le C_k\|\varphi\|_{(k)}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n). \tag{12.2}$$

There are several topologies which can be considered on $\mathcal{S}'(\mathbb{R}^n)$. Unless otherwise noted we consider the *uniform topology* on $\mathcal{S}'(\mathbb{R}^n)$; a subset $U \subset \mathcal{S}'(\mathbb{R}^n)$ is open in the uniform topology if for every $u \in U$ and every $k$ sufficiently large there exists $\delta_k > 0$ (both $k$ and $\delta_k$ depending on $u$) such that

$$v \in \mathcal{S}'(\mathbb{R}^n),\ |(u-u)(\varphi) \le \delta_k\|\varphi\|_{(k)} \Rightarrow v \in U.$$

For linear maps it is straightforward to work out continuity conditions. Namely

$$\begin{aligned} P &: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)\\ Q &: \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m)\\ R &: \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^m)\\ S &: \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m) \end{aligned}$$

are, respectively, continuous for the metric and uniform topologies if

$\forall\ k\ \exists\ k',\ C$ s.t. $\|P\varphi\|_{(k)} \le C\|\varphi\|_{(k')}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n)$

$\exists\ k,\ k',\ C$ s.t. $|Q\varphi(\psi)| \le C\|\varphi\|_{(k)}\|\psi\|_{(k')}$

$\forall\ k,\ k'\ \exists\ C$ s.t. $|u(\varphi)| \le \|\varphi\|_{(k')}\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \|Ru\|_{(k)} \le C$

$\forall\ k'\ \exists\ k,\ C,\ C'$ s.t. $\|u(\varphi)\|_{(k)} \le \|\varphi\|_{(k)} \forall\ \varphi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow |Su(\psi)| \le C'\|\psi\|_{(k')}\ \forall\ \psi \in \mathcal{S}(\mathbb{R}^n).$

The particular case of $R$, for $m = 0$, where at least formally $\mathcal{S}(\mathbb{R}^0) = \mathbb{C}$, corresponds to the reflexivity of $\mathcal{S}(\mathbb{R}^n)$, that

$$R : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathbb{C} \text{ is cts. iff } \exists\ \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ s.t.}$$
$$Ru = u(\varphi) \text{ i.e. } (\mathcal{S}'(\mathbb{R}^n))' = \mathcal{S}(\mathbb{R}^n).$$

In fact another extension of the middle two of these results corresponds to the Schwartz kernel theorem:

$$\begin{aligned}
Q :&\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^m) \text{ is linear and continuous}\\
&\text{iff } \exists\ Q \in \mathcal{S}'(\mathbb{R}^m \times \mathbb{R}^n) \text{ s.t. } (Q(\varphi))(\psi) = Q(\psi \boxtimes \varphi)\ \forall\ \varphi \in \mathcal{S}(\mathbb{R}^m)\ \psi \in \mathcal{S}(\mathbb{R}^n).\\
R :&\mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ is linear and continuous}\\
&\text{iff } \exists\ R \in \mathcal{S}(\mathbb{R}^m \times \mathbb{R}^n) \text{ s.t. } (Ru)(z) = u(R(z, \cdot)).
\end{aligned}$$

Schwartz test functions are dense in tempered distributions

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$$

where the *standard inclusion* is via Lebesgue measure

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto u_\varphi \in \mathcal{S}'(\mathbb{R}^n),\ u_\varphi(\psi) = \int_{\mathbb{R}^n} \varphi(z)\psi(z)dz. \tag{12.3}$$

The basic operators of differentiation and multiplication are transferred to $\mathcal{S}'(\mathbb{R}^n)$ by duality so that they remain consistent with the (12.3):

$$\begin{aligned}
D_z u(\varphi) &= u(-D_z\varphi)\\
fu(\varphi) &= u(f\varphi)\ \forall\ f \in \mathcal{S}(\mathbb{R}^n)).
\end{aligned}$$

In fact multiplication extends to the space of function of polynomial growth:

$$\forall\ \alpha \in \mathbb{N}_0^n\ \exists\ k \text{ s.t. } |D_z^\alpha f(z)| \le C\langle z\rangle^k.$$

Thus such a function is a multiplier on $\mathcal{S}(\mathbb{R}^n)$ and hence by duality on $\mathcal{S}'(\mathbb{R}^n)$ as well.

## 13. Fourier transform

Many of the results just listed are best proved using the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$
$$\mathcal{F}\varphi(\zeta) = \hat\varphi(\zeta) = \int e^{-iz\zeta}\varphi(z)dz.$$

This map is an isomorphism that extends to an isomorphism of $\mathcal{S}'(\mathbb{R}^n)$

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$
$$\mathcal{F}\varphi(D_{z_j}u) = \zeta_j\mathcal{F}u,\ \mathcal{F}(z_ju) = -D_{\zeta_j}\mathcal{F}u$$

and also extends to an isomorphism of $L^2(\mathbb{R}^n)$ from the dense subset

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^2)\mathrm{d}ense,\ \|\mathcal{F}\varphi\|^2_{L^2} = (2\pi)^n\|\varphi\|^2_{L^2}. \tag{13.1}$$

## 14. Sobolev spaces

Plancherel's theorem, (**??**), is the basis of the definition of the (standard, later there will be others) Sobolev spaces.

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n);\ (1+|\zeta|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)\} \\ \|u\|^2_s &= \int_{\mathbb{R}^n}(1+|\zeta|^2)^s|\hat{u}(\zeta)|d\zeta, \end{aligned}$$

where we use the fact that $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ is a well-defined injection (regarded as an inclusion) by continuous extension from (12.3). Now,

$$D^\alpha : H^s(\mathbb{R}^n) \longrightarrow H^{s-|\alpha|}(\mathbb{R}^n)\ \forall\ s,\ \alpha. \tag{14.1}$$

As well as this action by constant coefficient differential operators we note here that multiplication by Schwartz functions also preserves the Sobolev spaces – this is generalized with a different proof below. I give this cruder version first partly to show a little how to estimate convolution integrals.

Proposition 14.1. *For any $s \in \mathbb{R}$ there is a continuous bilinear map extending multiplication on Schwartz space*

$$\mathcal{S}(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^n) \tag{14.2}$$

Proof. The product $\phi u$ is well-defined for any $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$. Since Schwartz functions are dense in the Sobolev spaces it suffices to assume $u \in \mathcal{S}(\mathbb{R}^n)$ and then to use continuity. The Fourier transform of the product is the convolution of the Fourier transforms

$$\widehat{\phi u} = (2\pi)^{-n}\hat{\phi} * \hat{u},\ \hat{\phi} * \hat{u}(\xi) = \int_{\mathbb{R}^n}\hat{\phi}(\xi-\eta)\hat{u}(\eta)d\eta. \tag{14.3}$$

This is proved above, but let's just note that in this case it is easy enough since all the integrals are absolutely convergent and we can compute the inverse Fourier transform of the convolution

$$\begin{aligned} &(2\pi)^{-n}\int d\xi e^{iz\cdot\xi}\int_{\mathbb{R}^n}\hat{\phi}(\xi-\eta)\hat{u}(\eta)d\eta \\ &\qquad = (2\pi)^{-n}\int d\xi e^{iz\cdot(\xi-\eta)}\int_{\mathbb{R}^n}\hat{\phi}(\xi-\eta)e^{iz\cdot\eta}\hat{u}(\eta)d\eta \\ &\qquad = (2\pi)^{-n}\int d\Xi e^{iz\cdot\Xi}\int_{\mathbb{R}^n}\hat{\phi}(\Xi)e^{iz\cdot\eta}\hat{u}(\eta)d\eta \\ &\qquad = (2\pi)^n\phi(z)u(z). \end{aligned} \tag{14.4}$$

First, take $s = 0$ and prove this way the, rather obvious, fact that $\mathcal{S}$ is a space of multipliers on $L^2$. Writing out the square of the absolute value of the integral as the product with the complex conjugate, estimating by the absolute value and then using the Cauchy-Schwarz inequality gives what we want

$$
\begin{aligned}
&|\int|\int \psi(\xi-\eta)\hat{u}(\eta)d\eta|^2 d\xi \\
&\qquad \le \int\int |\psi(\xi-\eta_1)||\hat{u}(\eta_1)||\psi(\xi-\eta_2)||\hat{u}(\eta_2)|d\eta_1 d\eta_2 d\xi \\
&\qquad \le \int\int |\psi(\xi-\eta_1)||\psi(\xi-\eta_2)||\hat{u}(\eta_2)|^2 d\eta_1 d\eta_2 \\
&\qquad \le (\int|\psi|)^2\|u\|_{L^2}^2.
\end{aligned}
\tag{14.5}
$$

Here, we have decomposed the integral as the product of $|\psi(\xi-\eta_1)|^{\frac{1}{2}}|\hat{u}(\eta_1)||\psi(\xi-\eta_2)|^{\frac{1}{2}}$ and the same term with the $\eta$ variables exchanged. The two resulting factors are then the same after changing variable so there is no square-root in the integral.

Note that what we have really shown here is the well-known result:-

Lemma 14.2. *Convolution gives is a continous bilinear map*

$$
L^1(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\ni(u,v)\longmapsto u*v\in L^2(\mathbb{R}^n),\ \|u*v\|_{L^2}\le\|u\|_{L^1}\|v\|_{L^2}.
\tag{14.6}
$$

Now, to do the general case we need to take care of the weights in the integral for the Sobolev norm

$$
\|\phi u\|_{H^s}^2=\int(1+|\xi|^2)^s|\widehat{\phi u}(\xi)|^2 d\xi.
\tag{14.7}
$$

To do so, we divide the convolution integral into two regions:-

$$
\begin{aligned}
I&=\{\eta\in\mathbb{R}^n;|\xi-\eta|\ge\frac{1}{10}(|\xi|+|\eta|)\}\\
II&=\{\eta\in\mathbb{R}^n;|\xi-\eta|\le\frac{1}{10}(|\xi|+|\eta|)\}.
\end{aligned}
\tag{14.8}
$$

In the first region $\phi(\xi-\eta)$ is rapidly decreasing in both variable, so

$$
|\psi(\xi-\eta)|\le C_N(1+|\xi|)^{-N}(1+|\eta|)^{-N}
\tag{14.9}
$$

for any $N$ and as a result this contribution to the integral is rapidly decreasing:-

$$
|\int_I \psi(\xi-\eta)\hat{u}(\eta)d\eta|\le C_N(1+|\xi|)^{-n}\|u\|_{H^s}
\tag{14.10}
$$

where the $\eta$ decay is used to squelch the weight. So this certainly constributes a term to $\psi * \hat{u}$ with the bilinear bound.

To estimate the contribution from the second region, proceed as above but the insert the weight after using the Cauchy-Schwartz intequality

$$\begin{aligned}
&\int (1+|\xi|^2)^s |\int_{II} \psi(\xi-\eta)\hat{u}(\eta)d\eta|^2 d\xi \\
&\quad \le \int (1+|\xi|^2)^s \int_{II}\int_{II} |\psi(\xi-\eta_1)||\hat{u}(\eta_1)||\psi(\xi-\eta_2)||\hat{u}(\eta_2)|d\eta_1 d\eta_2 \\
&\quad \le \int\int_{II}\int_{II} (1+|\xi|^2)^s(1+|\eta_2|^2)^{-s}|\psi(\xi-\eta_1)||\psi(\xi-\eta_2)|(1+|\eta_2|^2)^s|\hat{u}(\eta_2)|^2 d\eta_1 d\eta_2
\end{aligned} \tag{14.11}$$

Exchange the order of integration and note that in region $II$ the two variables $\eta_2$ and $\xi$ are each bounded relative to the other. Thus the quotient of the weights is bounded above so the same argument applies to estimate the integral by

$$C\left(\int d\Xi|\psi(\Xi)|\right)^2 \|u\|^2_{H^s} \tag{14.12}$$

as desired. □

The Sobolev spaces are Hilbert spaces, so their duals are (conjugate) isomorphic to themselves. However, in view of our inclusion $L^2(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, we habitually identify

$$(H^s(\mathbb{R}^n))' = H^{-s}(\mathbb{R}^n),$$

with the 'extension of the $L^2$ paring'

$$(u,v) = \text{``}\int u(z)v(z)dz\text{''} = (2\pi)^{-n}\int_{\mathbb{R}^n} \langle\zeta\rangle^s\hat{u}\cdot\langle\zeta\rangle^{-s}\hat{u}d\zeta.$$

Note that then (14) is a linear, not a conjugate-linear, isomorphism since (14) is a real pairing.

The Sobolev spaces decrease with increasing $s$,

$$H^s(\mathbb{R}^n) \subset H^{s'}(\mathbb{R}^n)\ \forall\ s \ge s'.$$

One essential property is the relationship between the '$L^2$ derivatives' involved in the definition of Sobolev spaces and standard derivatives.

Namely, the Sobolev embedding theorem:

$$\begin{aligned} s > \frac{n}{2} &\Longrightarrow H^s(\mathbb{R}^n) \subset \mathcal{C}^0_\infty(\mathbb{R}^n) \\ &\qquad = \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ its continuous and bounded}\}. \\ s > \frac{n}{2} + k,\ k \in \mathbb{N} &\Longrightarrow H^s(\mathbb{R}^n) \subset \mathcal{C}^k_\infty(\mathbb{R}^n) \\ &\qquad \overset{\text{def}}{=} \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ s.t. } D^\alpha u \in \mathcal{C}^0_\infty(\mathbb{R}^n)\ \forall\ |\alpha| \le k\}. \end{aligned}$$

For positive integral $s$ the Sobolev norms are easily written in terms of the functions, without Fourier transform:

$$u \in H^k(\mathbb{R}^n) \Leftrightarrow D^\alpha u \in L^2(\mathbb{R}^n)\ \forall\ |\alpha| \le k$$
$$\|u\|_k^2 = \sum_{|\alpha| \le k} \int_{\mathbb{R}^n} |D^\alpha u|^2 dz.$$

For negative integral orders there is a similar characterization by duality, namely

$$\begin{aligned} H^{-k}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \text{ s.t. },\ \exists\ u_\alpha \in L^2(\mathbb{R}^n),\ |\alpha| \ge k \\ u = \sum_{|\alpha| \le k} D^\alpha u_\alpha\}. \end{aligned}$$

In fact there are similar "Hölder" characterizations in general. For $0 < s < 1$, $u \in H^s(\mathbb{R}^n) \Longrightarrow u \in L^2(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^{2n}} \frac{|u(z) - u(z')|^2}{|z - z'|^{n+2s}} dz dz' < \infty. \tag{14.13}$$

Then for $k < s < k+1$, $k \in \mathbb{N}$ $u \in H^s(\mathbb{R}^2)$ is equivalent to $D^\alpha \in H^{s-k}(\mathbb{R}^n)$ for all $|\alpha| \in k$, with corresponding (Hilbert) norm. Similar realizations of the norms exist for $s < 0$.

One simple consequence of this is that

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) = \bigcap_k \mathcal{C}^k_\infty(\mathbb{R}^n) = \{u; \mathbb{R}^n \longrightarrow \mathbb{C} \text{ s.t. } |D^\alpha u| \text{ is bounded } \forall\ \alpha\}$$

is a multiplier on *all* Sobolev spaces

$$\mathcal{C}^\infty_\infty(\mathbb{R}^n) \cdot H^s(\mathbb{R}^n) = H^s(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}.$$

## 15. Weighted Sobolev spaces.

It follows from the Sobolev embedding theorem that

$$\bigcap_s H^s(\mathbb{R}^n) \subset \mathcal{C}^\infty_\infty(\mathbb{R}^n); \tag{15.1}$$

in fact the intersection here is quite a lot smaller, but nowhere near as small as $\mathcal{S}(\mathbb{R}^n)$. To discuss decay at infinity, as will definitely want to do, we may use weighted Sobolev spaces.

The ordinary Sobolev spaces do not effectively define decay (or growth) at infinity. We will therefore also set

$$\begin{aligned} H^{m,l}(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n);\ \langle z\rangle^{\ell} u \in H^m(\mathbb{R}^n)\},\ m, \ell \in \mathbb{R}, \\ &= \langle z\rangle^{-\ell} H^m(\mathbb{R}^n)\,, \end{aligned}$$

where the second notation is supported to indicate that $u \in H^{m,l}(\mathbb{R}^n)$ may be written as a product $\langle z\rangle^{-\ell} v$ with $v \in H^m(\mathbb{R}^n)$. Thus

$$H^{m,\ell}(\mathbb{R}^n) \subset H^{m',\ell'}(\mathbb{R}^n) \text{ if } m \geq m' \text{ and } \ell \geq \ell',$$

so the spaces are decreasing in each index. As consequences of the *Schwartz structure theorem*

$$\begin{aligned} \mathcal{S}'(\mathbb{R}^n) &= \bigcup_{m,\ell} H^{m,\ell}(\mathbb{R}^n) \\ \mathcal{S}(\mathbb{R}^n) &= \bigcap_{m,\ell} H^{m,\ell}(\mathbb{R}^n). \end{aligned} \tag{15.2}$$

This is also true 'topologically' meaning that the first is an 'inductive limit' and the second a 'projective limit'.

Similarly, using some commutation arguments

$$\begin{aligned} D_{z_j} : H^{m,\ell}(\mathbb{R}^n) &\longrightarrow H^{m-1,\ell}(\mathbb{R}^n),\ \forall\ m,\ elll \\ \times z_j : H^{m,\ell}(\mathbb{R}^n) &\longrightarrow H^{m,\ell-1}(\mathbb{R}^n). \end{aligned}$$

Moreover there is symmetry under the Fourier transform

$$\mathcal{F} : H^{m,\ell}(\mathbb{R}^n) \longrightarrow H^{\ell,m}(\mathbb{R}^n) \text{ is an isomorphism } \forall\ m,\ \ell.$$

As with the usual Sobolev spaces, $\mathcal{S}(\mathbb{R}^n)$ is dense in all the $H^{m,\ell}(\mathbb{R}^n)$ spaces and the continuous extension of the $L^2$ paring gives an identification

$$\begin{aligned} H^{m,\ell}(\mathbb{R}^n) &\cong (H^{-m,-\ell}(\mathbb{R}^n))' \text{ fron} \\ H^{m,\ell}(\mathbb{R}^n) \times H^{-m,-\ell}(\mathbb{R}^n) &\ni u, v \mapsto \\ (u, v) &= \text{“}\int u(z)v(z)dz''. \end{aligned}$$

Let $R_s$ be the operator defined by Fourier multiplication by $\langle\zeta\rangle^s$ :

$$R_s : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\ \widehat{R_s f}(\zeta) = \langle\zeta\rangle^s \hat{f}(\zeta). \tag{15.3}$$

LEMMA 15.1. *If $\psi \in \mathcal{S}(\mathbb{R}^n)$ then*

$$M_s = [\psi, R_s *] : H^t(\mathbb{R}^n) \longrightarrow H^{t-s+1}(\mathbb{R}^n) \tag{15.4}$$

*is bounded for each $t$.*

PROOF. Since the Sobolev spaces are defined in terms of the Fourier transform, first conjugate and observe that (15.4) is equivalent to the boundeness of the integral operator with kernel

$$K_{s,t}(\zeta,\zeta') = (1+|\zeta|^2)^{\frac{t-s+1}{2}} \hat{\psi}(\zeta-\zeta') \left((1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}}\right)(1+|\zeta'|^2)^{-\frac{t}{2}} \tag{15.5}$$

on $L^2(\mathbb{R}^n)$. If we insert the characteristic function for the region near the diagonal

$$|\zeta - \zeta'| \le \frac{1}{4}(|\zeta| + |\zeta'|) \Longrightarrow |\zeta| \le 2|\zeta'|,\ |\zeta'| \le 2|\zeta| \tag{15.6}$$

then $|\zeta|$ and $|\zeta'|$ are of comparable size. Using Taylor's formula

$$\begin{gathered}(1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}} = s(\zeta-\zeta')\cdot \int_0^1 (t\zeta+(1-t\zeta')\left(1+|t\zeta+(1-t)\zeta'|^2\right)^{\frac{s}{2}-1} dt \\ \Longrightarrow \left|(1+|\zeta'|^2)^{\frac{s}{2}} - (1+|\zeta|^2)^{\frac{s}{2}}\right| \le C_s|\zeta-\zeta'|(1+|\zeta|)^{s-1}.\end{gathered} \tag{15.7}$$

It follows that in the region (15.6) the kernel in (15.5) is bounded by

$$C|\zeta - \zeta'||\hat{\psi}(\zeta-\zeta')|. \tag{15.8}$$

In the complement to (15.6) the kernel is rapidly decreasing in $\zeta$ and $\zeta'$ in view of the rapid decrease of $\hat{\psi}$. Both terms give bounded operators on $L^2$, in the first case using the same estimates that show convolution by an element of $\mathcal{S}$ to be bounded. □

LEMMA 15.2. *If $u \in H^s(\mathbb{R}^n)$ and $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then*

$$\|\psi u\|_s \le \|\psi\|_{L^\infty}\|u\|_s + C\|u\|_{s-1} \tag{15.9}$$

*where the constant depends on $s$ and $\psi$ but not $u$.*

PROOF. This is really a standard estimate for Sobolev spaces. Recall that the Sobolev norm is related to the $L^2$ norm by

$$\|u\|_s = \|\langle D\rangle^s u\|_{L^2}. \tag{15.10}$$

Here $\langle D\rangle^s$ is the convolution operator with kernel defined by its Fourier transform

$$\langle D\rangle^s u = R_s * u,\ \widehat{R_s}(\zeta) = (1+|\zeta|^2)^{\frac{s}{2}}. \tag{15.11}$$

To get (15.9) use Lemma 15.1.

From (15.4), (writing 0 for the $L^2$ norm)

$$\begin{aligned}\|\psi u\|_s &= \|R_s*(\psi u)\|_0 \le \|\psi(R_s*u)\|_0 + \|M_s u\|_0\\ &\le \|\psi\|_{L^\infty}\|R_s u\|_0 + C\|u\|_{s-1} \le \|\psi\|_{L^\infty}\|u\|_s + C\|u\|_{s-1}.\end{aligned} \tag{15.12}$$

This completes the proof of (15.9) and so of Lemma 15.2. □

## 16. Multiplicativity

Of primary importance later in our treatment of non-linear problems is some version of the multliplicative property

$$A^s(\mathbb{R}^n) = \begin{cases} H^s(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n) & s\le \frac{n}{2}\\ H^s(\mathbb{R}^n) & s>\frac{n}{2}\end{cases} \text{ is a } \mathcal{C}^\infty \text{ algebra.} \tag{16.1}$$

Here, a $\mathcal{C}^\infty$ algebra is an algebra with an additional closure property. Namely if $F:\mathbb{R}^N \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$ function vanishing at the origin and $u_1, \ldots, u_N \in A^s$ are *real-valued* then

$$F(u_1,\ldots,u_n)\in A^s.$$

I will only consider the case of real interest here, where $s$ is an integer and $s>\frac{n}{2}$. The obvious place to start is

LEMMA 16.1. *If $s>\frac{n}{2}$ then*

$$u,\ v\in H^s(\mathbb{R}^n) \Longrightarrow uv\in H^s(\mathbb{R}^n). \tag{16.2}$$

PROOF. We will prove this directly in terms of convolution. Thus, in terms of weighted Sobolev spaces $u\in H^s(\mathbb{R}^n)=H^{s,0}(\mathbb{R}^n)$ is equivalent to $\hat u\in H^{0,s}(\mathbb{R}^n)$. So (16.2) is equivalent to

$$u,\ v\in H^{0,s}(\mathbb{R}^n) \Longrightarrow u*v\in H^{0,s}(\mathbb{R}^n). \tag{16.3}$$

Using the density of $\mathcal{S}(\mathbb{R}^n)$ it suffices to prove the estimate

$$\|u*v\|_{H^{0,s}} \le C_s\|u\|_{H^{0,s}}\|v\|_{H^{0,s}} \text{ for } s>\frac{n}{2}. \tag{16.4}$$

Now, we can write $u(\zeta)=\langle\zeta\rangle^{-s}u'$ etc and convert (16.4) to an estimate on the $L^2$ norm of

$$\langle\zeta\rangle^{-s}\int\langle\xi\rangle^{-s}u'(\xi)\langle\zeta-\xi\rangle^{-s}v'(\zeta-\xi)d\xi \tag{16.5}$$

in terms of the $L^2$ norms of $u'$ and $v'\in\mathcal{S}(\mathbb{R}^n)$.

Writing out the $L^2$ norm as in the proof of Lemma 15.1 above, we need to estimate the absolute value of

(16.6)
$$\int\int\int d\zeta d\xi d\eta\langle\zeta\rangle^{2s}\langle\xi\rangle^{-s}u_1(\xi)\langle\zeta-\xi\rangle^{-s}v_1(\zeta-\xi)\langle\eta\rangle^{-s}u_2(\eta)\langle\zeta-\eta\rangle^{-s}v_2(\zeta-\eta)$$

in terms of the $L^2$ norms of the $u_i$ and $v_i$. To do so divide the integral into the four regions,

$$\begin{aligned}
&|\zeta-\xi| \le \frac{1}{4}(|\zeta|+|\xi|),\ |\zeta-\eta| \le \frac{1}{4}(|\zeta|+|\eta|)\\
&|\zeta-\xi| \le \frac{1}{4}(|\zeta|+|\xi|),\ |\zeta-\eta| \ge \frac{1}{4}(|\zeta|+|\eta|)\\
&|\zeta-\xi| \ge \frac{1}{4}(|\zeta|+|\xi|),\ |\zeta-\eta| \le \frac{1}{4}(|\zeta|+|\eta|)\\
&|\zeta-\xi| \ge \frac{1}{4}(|\zeta|+|\xi|),\ |\zeta-\eta| \ge \frac{1}{4}(|\zeta|+|\eta|).
\end{aligned} \tag{16.7}$$

Using (15.6) the integrand in (16.6) may be correspondingly bounded by

$$\begin{gathered}
C\langle\zeta-\eta\rangle^{-s}|u_1(\xi)||v_1(\zeta-\xi)|\cdot\langle\zeta-\xi\rangle^{-s}|u_2(\eta)||v_2(\zeta-\eta)|\\
C\langle\eta\rangle^{-s}|u_1(\xi)||v_1(\zeta-\xi)|\cdot\langle\zeta-\xi\rangle^{-s}|u_2(\eta)||v_2(\zeta-\eta)|\\
C\langle\zeta-\eta\rangle^{-s}|u_1(\xi)||v_1(\zeta-\xi)|\cdot\langle\xi\rangle^{-s}|u_2(\eta)||v_2(\zeta-\eta)|\\
C\langle\eta\rangle^{-s}|u_1(\xi)|v_1(\zeta-\xi)|\cdot\langle\xi\rangle^{-s}|u_2(\eta)||v_2(\zeta-\eta)|.
\end{gathered} \tag{16.8}$$

Now applying Cauchy-Schwarz inequality, with the factors as indicated, and changing variables appropriately gives the desired estimate. □

Next, we extend this argument to (many) more factors to get the following result which is close to the Gagliardo-Nirenberg estimates (since I am concentrating here on $L^2$ methods I will not actually discuss the latter).

Lemma 16.2. *If $s > \frac{n}{2}$, $N \ge 1$ and $\alpha_i \in \mathbb{N}_0^k$ for $i = 1, \dots, N$ are such that*

$$\sum_{i=1}^{N} |\alpha_i| = T \le s$$

*then*

$$u_i \in H^s(\mathbb{R}^n) \Longrightarrow U = \prod_{i=1}^{N} D^{\alpha_i} u_i \in H^{s-T}(\mathbb{R}^n),\ \|U\|_{H^{s-T}} \le C_N \prod_{i=1}^{N} \|u_i\|_{H^s}. \tag{16.9}$$

Proof. We proceed as in the proof of Lemma 16.1 using the Fourier transform to replace the product by the convolution. Thus it suffices to show that

$$u_1 * u_2 * u_3 * \cdots * u_N \in H^{0,s-T} \text{ if } u_i \in H^{0,s-\alpha_i}. \tag{16.10}$$

Writing out the convolution symmetrically in all variables,

$$u_1 * u_2 * u_3 * \cdots * u_N(\zeta) = \int_{\zeta=\sum\limits_i \xi_i} u_1(\xi_1)\cdots u_N(\xi_N) \tag{16.11}$$

it follows that we need to estimate the $L^2$ norm in $\zeta$ of

$$\langle\zeta\rangle^{s-T}\int_{\zeta=\sum\limits_i \xi_i}\langle\xi_1\rangle^{-s+a_1}v_1(\xi_1)\cdots\langle\xi_N\rangle^{-s+a_N}v_N(\xi_N) \tag{16.12}$$

for $N$ factors $v_i$ which are in $L^2$ with the $a_i = |\alpha|_i$ non-negative integers summing to $T \le s$. Again writing the square as the product with the complex conjuage it is enough to estimate integrals of the type

$$\begin{gathered}\int_{\{(\xi,\eta)\in\mathbb{R}^{2N};\sum\limits_i \xi_i=\sum\limits_i \eta_i\}}\langle\sum_i \xi\rangle^{2s-2T}\langle\xi_1\rangle^{-s+a_1}\\ v_1(\xi_1)\cdots\langle\xi_N\rangle^{-s+a_N}v_N(\xi_N)\langle\eta_1\rangle^{-s+a_1}\bar{v}_1(\eta_1)\cdots\langle\eta_N\rangle^{-s+a_N}\bar{v}_N(\eta_N).\end{gathered} \tag{16.13}$$

This is really an integral over $\mathbb{R}^{2N-1}$ with respect to Lebesgue measure. Applying Cauchy-Schwarz inequality the absolute value is estimated by

$$\int_{\{(\xi,\eta)\in\mathbb{R}^{2N};\sum\limits_i \xi_i=\sum\limits_i \eta_i\}}\prod_{i=1}^N|v_i(\xi_i)|^2\langle\sum_l \eta_l\rangle^{2s-2T}\prod_{i=1}^N\langle\eta_i\rangle^{-2s+2a_i} \tag{16.14}$$

The domain of integration, given by $\sum\limits_i \eta_i = \sum\limits_i \xi_i$, is covered by the finite number of subsets $\Gamma_j$ on which in addition $|\eta_j| \ge |\eta_i|$, for all $i$. On this set we may take the variables of integration to be $\eta_i$ for $i \ne j$ and the $\xi_l$. Then $|\eta_i| \ge |\sum\limits_l \eta_l|/N$ so the second part of the integrand in (16.14) is estimated by

$$\langle\eta_j\rangle^{-2s+2a_j}\langle\sum_l \eta_l\rangle^{2s-2T}\prod_{i\ne j}\langle\eta_i\rangle^{-2s+2a_i} \le C_N\langle\eta_j\rangle^{-2T+2a_j}\prod_{i\ne j}\langle\eta_i\rangle^{-2s+2a_i} \le C_N'\prod_{i\ne j}\langle\eta_i\rangle^{-2s} \tag{16.15}$$

Thus the integral in (16.14) is finite and the desired estimate follows. □

Proposition 16.3. *If $F \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R})$ and $u \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2}$ an integer then*

$$F(z, u(z)) \in H^s_{\text{loc}}(\mathbb{R}^n). \tag{16.16}$$

Proof. Since the result is local on $\mathbb{R}^n$ we may multiply by a compactly supported function of $z$. In fact since $u \in H^s(\mathbb{R}^n)$ is bounded we also multiply by a compactly supported function in $\mathbb{R}$ without changing the result. Thus it suffices to show that

$$F \in \mathcal{C}_c^\infty(\mathbb{R}^n \times \mathbb{R}) \Longrightarrow F(z, u(z)) \in H^s(\mathbb{R}^n). \tag{16.17}$$

Now, Lemma 16.2 can be applied to show that $F(z, u(z)) \in H^s(\mathbb{R}^n)$. Certainly $F(z, u(z)) \in L^2(\mathbb{R}^n)$ since it is continuous and has compact support. Moreover, differentiating $s$ times and applying the chain rule gives

$$D^\alpha F(z, u(z)) = \sum F_{\alpha_1,\dots,\alpha_N}(z, u(z)) D^{\alpha_1} u \cdots D^{\alpha_N} u \tag{16.18}$$

where the sum is over all (finitely many) decomposition with $\sum_{i=1}^{N} \alpha_i \leq \alpha$ and the $F_{\cdot}(z, u)$ are smooth with compact support, being various derivitives of $F(z, u)$. Thus it follows from Lemma 16.2 that all terms on the right are in $L^2(\mathbb{R}^n)$ for $|\alpha| \leq s$. □

Note that slightly more sophisticated versions of these arguments give the full result (16.1) but Proposition 16.3 suffices for our purposes below.

## 17. Some bounded operators

Lemma 17.1. *If $J \in \mathcal{C}^k(\Omega^2)$ is properly supported then the operator with kernel $J$ (also denoted $J$) is a map*

$$J : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^k_{\mathrm{loc}}(\Omega) \ \forall\ s \geq -k. \tag{17.1}$$

CHAPTER 4

# Elliptic Regularity

Includes some corrections noted by Tim Nguyen and corrections by, and some suggestions from, Jacob Bernstein.

## 1. Constant coefficient operators

A linear, constant coefficient differential operator can be thought of as a map

(1.1)
$$P(D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ of the form } P(D)u(z) = \sum_{|\alpha|\le m} c_\alpha D^\alpha u(z),$$
$$D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n},\ D_j = \frac{1}{i}\frac{\partial}{\partial z_j},$$

but it also acts on various other spaces. So, really it is just a polynomial $P(\zeta)$ in $n$ variables. This 'characteristic polynomial' has the property that

$$\mathcal{F}(P(D)u)(\zeta) = P(\zeta)\mathcal{F}u(\zeta), \tag{1.2}$$

which you may think of as a little square

$$\begin{tikzcd} \mathcal{S}(\mathbb{R}^n) \arrow[r, "P(D)"] \arrow[d, leftrightarrow, "\mathcal{F}"'] & \mathcal{S}(\mathbb{R}^n) \arrow[d, leftrightarrow, "\mathcal{F}"] \\ \mathcal{S}(\mathbb{R}^n) \arrow[r, "P\times"'] & \mathcal{S}(\mathbb{R}^n) \end{tikzcd} \tag{1.3}$$

and this is why the Fourier tranform is especially useful. However, this still does not solve the important questions directly.

Question 1.1. $P(D)$ is always injective as a map (1.1) but is usually not surjective. When is it surjective? If $\Omega \subset \mathbb{R}^n$ is a non-empty open set then

$$P(D) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega) \tag{1.4}$$

is never injective (unless $P(\zeta)$ is constnat), for which polynomials is it surjective?

The first three points are relatively easy. As a map (1.1) $P(D)$ is injective since if $P(D)u = 0$ then by (1.2), $P(\zeta)\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$. However, a zero set, in $\mathbb{R}^n$, of a non-trivial polynomial alwasys has empty interior, i.e. the set where it is non-zero is dense, so $\mathcal{F}u(\zeta) = 0$ on $\mathbb{R}^n$ (by continuity) and hence $u = 0$ by the invertibility of the Fourier transform. So (1.1) is injective (of course excepting the case that $P$ is the zero polynomial). When is it surjective? That is, when can every $f \in \mathcal{S}(\mathbb{R}^n)$ be written as $P(D)u$ with $u \in \mathcal{S}(\mathbb{R}^n)$? Taking the Fourier transform again, this is the same as asking when every $g \in \mathcal{S}(\mathbb{R}^n)$ can be written in the form $P(\zeta)v(\zeta)$ with $v \in \mathcal{S}(\mathbb{R}^n)$. If $P(\zeta)$ has a zero in $\mathbb{R}^n$ then this is not possible, since $P(\zeta)v(\zeta)$ always vanishes at such a point. It is a little trickier to see the converse, that $P(\zeta) \neq 0$ on $\mathbb{R}^n$ implies that $P(D)$ in (1.1) is surjective. Why is this not obvious? Because we need to show that $v(\zeta) = g(\zeta)/P(\zeta) \in \mathcal{S}(\mathbb{R}^n)$ whenever $g \in \mathcal{S}(\mathbb{R}^n)$. Certainly, $v \in \mathcal{C}^\infty(\mathbb{R}^n)$ but we need to show that the derivatives decay rapidly at infinity. To do this we need to get an estimate on the rate of decay of a non-vanishing polynomial

Lemma 1.1. *If $P$ is a polynomial such that $P(\zeta) \neq 0$ for all $\zeta \in \mathbb{R}^n$ then there exists $C > 0$ and $a \in \mathbb{R}$ such that*

$$|P(\zeta)| \geq C(1 + |\zeta|)^a. \tag{1.5}$$

Proof. This is a form of the Tarski-Seidenberg Lemma. Stated loosely, a semi-algebraic function has power-law bounds. Thus

$$F(R) = \inf\{|P(\zeta)|; |\zeta| \leq R\} \tag{1.6}$$

is semi-algebraic and non-vanishing so must satisfy $F(R) \geq c(1 + R)^a$ for some $c > 0$ and $a$ (possibly negative). This gives the desired bound.

Is there an elementary proof? $\square$

Thirdly the non-injectivity in (1.4) is obvious for the opposite reason. Namely for any non-constant polynomial there exists $\zeta \in \mathbb{C}^n$ such that $P(\zeta) = 0$. Since

$$P(D)e^{i\zeta\cdot z} = P(\zeta)e^{i\zeta\cdot z} \tag{1.7}$$

such a zero gives rise to a non-trivial element of the null space of (1.4). You can find an extensive discussion of the density of these sort of ‘exponential’ solutions (with polynomial factors) in all solutions in Hörmander’s book [**4**].

What about the surjectivity of (1.4)? It is not always surjective unless $\Omega$ is *convex* but there are decent answers, to find them you should look under $P$-convexity in [**4**]. If $P(\zeta)$ is elliptic then (1.4) is surjective for every open $\Omega$; maybe I will prove this later although it is not a result of great utility.

## 2. Constant coefficient elliptic operators

To discuss elliptic regularity, let me recall that any constant coefficient differential operator of order $m$ defines a continuous linear map

$$P(D) : H^{s+m}(\mathbb{R}^n) \longmapsto H^s(\mathbb{R}^n). \tag{2.1}$$

Provided $P$ is not the zero polynomial this map is *always* injective. This follows as in the discussion above for $\mathcal{S}(\mathbb{R}^n)$. Namely, if $u \in H^{s+m}(\mathbb{R}^n)$ then, by definition, $\hat{u} \in L^2_{\text{loc}}(\mathbb{R}^n)$ and if $Pu = 0$ then $P(\zeta)\hat{u}(\zeta) = 0$ off a set of measure zero. Since $P(\zeta) \neq 0$ on an open dense set it follows that $\hat{u} = 0$ off a set of measure zero and so $u = 0$ as a distribution.

As a map (2.1), $P(D)$ is is seldom surjective. It is said to be elliptic (either as a polynomial or as a differential operator) if it is of order $m$ and there is a constant $c > 0$ such that

$$|P(\zeta)| \geq c(1 + |\zeta|)^m \text{ in } \{\zeta \in \mathbb{R}^n; |\zeta| > 1/c\}. \tag{2.2}$$

Proposition 2.1. *As a map* (2.1)*, for a given* $s$, $P(D)$ *is surjective if and only if* $P$ *is elliptic and* $P(\zeta) \neq 0$ *on* $\mathbb{R}^n$ *and then it is a topological isomorphism for every* $s$.

Proof. Since the Sobolev spaces are defined as the Fourier transforms of the weighted $L^2$ spaces, that is

$$f \in H^t(\mathbb{R}^n) \Longleftrightarrow (1 + |\zeta|^2)^{t/2}\hat{f} \in L^2(\mathbb{R}^n) \tag{2.3}$$

the sufficiency of these conditions is fairly clear. Namely the combination of ellipticity, as in (2.2), and the condition that $P(\zeta) \neq 0$ for $\zeta \in \mathbb{R}^n$ means that

$$|P(\zeta)| \geq c(1 + |\zeta|^2)^{m/2},\ c > 0,\ \zeta \in \mathbb{R}^n. \tag{2.4}$$

From this it follows that $P(\zeta)$ is bounded above and below by multiples of $(1 + |\zeta|^2)^{m/2}$ and so maps the weighted $L^2$ spaces into each other

$$\times P(\zeta) : H^{0,s+m}(\mathbb{R}^n) \longrightarrow H^{0,s}(\mathbb{R}^n),\ H^{0,s} = \{u \in L^2_{\text{loc}}(\mathbb{R}^n); \langle\zeta\rangle^s u(\zeta) \in L^2(\mathbb{R}^n)\}, \tag{2.5}$$

giving an isomorphism (2.1) after Fourier transform.

The necessity follows either by direct construction or else by use of the closed graph theorem. If $P(D)$ is surjective then multiplication by $P(\zeta)$ must be an isomorphism between the corresponding weighted space $H^{0,s}(\mathbb{R}^n)$ and $H^{0,s+m}(\mathbb{R}^n)$. By the density of functions supported off the zero set of $P$ the norm of the inverse can be seen to be the inverse of

$$\inf_{\zeta \in \mathbb{R}^n} |P(\zeta)|\langle\zeta\rangle^{-m} \tag{2.6}$$

which proves ellipticity. □

Ellipticity is reasonably common in appliactions, but the condition that the characteristic polynomial not vanish at all is frequently not satisfied. In fact one of the questions I want to get to in this course – even though we are interested in variable coefficient operators – is improving (2.1) (by changing the Sobolev spaces) to get an isomorphism at least for homogeneous elliptic operators (which do not satisfy the second condition in Proposition 2.1 because they vanish at the origin). One reason for this is that we want it for monopoles.

Note that ellipticity itself is a condition on the principal part of the polynomial.

LEMMA 2.2. *A polynomial* $P(\zeta) = \sum\limits_{\alpha|\leq m} c_\alpha \zeta^\alpha$ *of degree* $m$ *is elliptic if and only if its leading part*

$$P_m(\zeta) = \sum_{|\alpha|=m} c_\alpha \zeta^\alpha \neq 0 \text{ on } \mathbb{R}^n \setminus \{0\}. \tag{2.7}$$

PROOF. Since the principal part is homogeneous of degree $m$ the requirement (2.7) is equivalent to

$$|P_m(\zeta)| \geq c|\zeta|^m,\ c = \inf_{|\zeta|=1} |P(\zeta)| > 0. \tag{2.8}$$

Thus, (2.2) follows from this, since

$$|P(\zeta)| \geq |P_m(\zeta)| - |P'(\zeta)| \geq c|\zeta|^m - C|\zeta|^{m-1} - C \geq \frac{c}{2}|\zeta|^m \text{ if } |\zeta| > C', \tag{2.9}$$

$P' = P - M_m$ being of degree at most $m-1$. Conversely, ellipticity in the sense of (2.2) implies that

$$|P_m(\zeta)| \geq |P(\zeta)| - |P'(\zeta)| \geq c|\zeta|^m - C|\zeta|^{m-1} - C > 0 \text{ in } |\zeta| > C' \tag{2.10}$$

and so $P_m(\zeta) \neq 0$ for $\zeta \in \mathbb{R}^n \setminus \{0\}$ by homogeneity. □

Let me next recall *elliptic regularity* for constant coefficient operators. Since this is a local issue, I first want to recall the local versions of the Sobolev spaces discussed in Chapter 3

DEFINITION 2.3. *If* $\Omega \subset \mathbb{R}^n$ *is an open set then*

$$H^s_{\text{loc}}(\Omega) = \left\{u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^s(\mathbb{R}^n)\ \forall\ \phi \in \mathcal{C}^\infty_c(\Omega)\right\}. \tag{2.11}$$

Again you need to know what $\mathcal{C}^{-\infty}(\Omega)$ is (it is the dual of $\mathcal{C}^\infty_c(\Omega)$) and that multiplication by $\phi \in \mathcal{C}^\infty_c(\Omega)$ defines a linear continuous map from $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ to $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and gives a bounded operator on $H^m(\mathbb{R}^n)$ for all $m$.

PROPOSITION 2.4. *If $P(D)$ is elliptic, $u \in \mathcal{C}^{-\infty}(\Omega)$ is a distribution on an open set and $P(D)u \in H^s_{\text{loc}}(\Omega)$ then $u \in H^{s+m}_{\text{loc}}(\Omega)$. Furthermore if $\phi$, $\psi \in \mathcal{C}^\infty_c(\Omega)$ with $\phi = 1$ in a neighbourhood of* $\operatorname{supp}(\psi)$ *then*

$$\|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{s+m-1} \tag{2.12}$$

*for any $M \in \mathbb{R}$, with $C'$ depending only on $\psi$, $\phi$, $M$ and $P(D)$ and $C$ depending only on $P(D)$ (so neither depends on $u$).*

Although I will no prove it here, and it is not of any use below, it is worth noting that (2.12) *characterizes* the ellipticity of a differential operator with smooth coefficients.

PROOF. Let me discuss this in two slightly different ways. The first, older, approach is via direct regularity estimates. The second is through the use of a parametrix; they are not really very different!

First the regularity estimates. An easy case of Proposition 2.4 arises if $u \in \mathcal{C}^{-\infty}_c(\Omega)$ has compact support to start with. Then $P(D)u$ also has compact support so in this case

$$u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n) \text{ and } P(D)u \in H^s(\mathbb{R}^n). \tag{2.13}$$

Then of course the Fourier transform works like a charm. Namely $P(D)u \in H^s(\mathbb{R}^n)$ means that

$$\langle\zeta\rangle^s P(\zeta)\hat{u}(\zeta) \in L^2(\mathbb{R}^n) \Longrightarrow \langle\zeta\rangle^{s+m} F(\zeta)\hat{u}(\zeta) \in L^2(\mathbb{R}^n),\ F(\zeta) = \langle\zeta\rangle^{-m}P(\zeta). \tag{2.14}$$

Ellipticity of $P(\zeta)$ implies that $F(\zeta)$ is bounded above and below on $|\zeta| > 1/c$ and hence can be inverted there by a bounded function. It follows that, given any $M \in \mathbb{R}$ the norm of $u$ in $H^{s+m}(\mathbb{R}^n)$ is bounded

$$\|u\|_{s+m} \le C\|u\|_s + C'_M\|u\|_M,\ u \in \mathcal{C}^{-\infty}(\Omega), \tag{2.15}$$

where the second term is used to bound the $L^2$ norm of the Fourier transform in $|\zeta| \le 1/c$.

To do the general case of an open set we need to use cutoffs more seriously. We want to show that $\psi u \in H^{s+m}(\mathbb{R}^n)$ where $\psi \in \mathcal{C}^\infty_c(\Omega)$ is some fixed but arbitrary element. We can always choose some function $\phi \in \mathcal{C}^\infty_c(\Omega)$ which is equal to one in a neighbourhood of the support of $\psi$. Then $\phi u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ so, by the Schwartz structure theorem, $\phi u \in H^{m+t-1}(\mathbb{R}^n)$ for some (unknown) $t \in \mathbb{R}$. We will show that $\psi u \in H^{m+T}(\mathbb{R}^n)$ where $T$ is the smaller of $s$ and $t$. To see this, compute

$$P(D)(\psi u) = \psi P(D)u + \sum_{|\beta| \le m-1, |\gamma| \ge 1} c_{\beta,\gamma} D^\gamma \psi D^\beta \phi u. \tag{2.16}$$

With the final $\phi u$ replaced by $u$ this is just the effect of expanding out the derivatives on the product. Namely, the $\psi P(D)u$ term is when no derivative hits $\psi$ and the other terms come from at least one derivative hitting $\psi$. Since $\phi = 1$ on the support of $\psi$ we may then insert $\phi$ without changing the result. Thus the first term on the right in (2.16) is in $H^s(\mathbb{R}^n)$ and all terms in the sum are in $H^t(\mathbb{R}^n)$ (since at most $m-1$ derivatives are involved and $\phi u \in H^{m+t-1}(\mathbb{R}^n)$ be definition of $t$). Applying the simple case discussed above it follows that $\psi u \in H^{m+r}(\mathbb{R}^n)$ with $r$ the minimum of $s$ and $t$. This would result in the estimate

$$\|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{s+m-1} \tag{2.17}$$

provided we knew that $\phi u \in H^{s+m-1}$ (since then $t = s$). Thus, initially we only have this estimate with $s$ replaced by $T$ where $T = \min(s,t)$. However, the only obstruction to getting the correct estimate is knowing that $\psi u \in H^{s+m-1}(\mathbb{R}^n)$.

To see this we can use a bootstrap argument. Observe that $\psi$ can be taken to be *any* smooth function with support in the interior of the set where $\phi = 1$. We can therefore insert a chain of functions, of any finite (integer) length $k \ge s-t$, between then, with each supported in the region where the previous one is equal to 1 :

(2.18)

$$\operatorname{supp}(\psi) \subset \{\phi_k = 1\}^\circ \subset \operatorname{supp}(\phi_k) \subset \cdots \subset \operatorname{supp}(\phi_1) \subset \{\phi = 1\}^\circ \subset \operatorname{supp}(\phi)$$

where $\psi$ and $\phi$ were our initial choices above. Then we can apply the argument above with $\psi = \phi_1$, then $\psi = \phi_2$ with $\phi$ replaced by $\phi_1$ and so on. The initial regularity of $\phi u \in H^{t+m-1}(\mathbb{R}^n)$ for some $t$ therefore allows us to deduce that

$$\phi_j u \in H^{m+T_j}(\mathbb{R}^n),\ T_j = \min(s, t+j-1). \tag{2.19}$$

If $k$ is large enough then $\min(s, t+k) = s$ so we conclude that $\psi u \in H^{s+m}(\mathbb{R}^n)$ for any such $\psi$ and that (2.17) holds. □

Although this is a perfectly adequate proof, I will now discuss the second method to get elliptic regularity; the main difference is that we think more in terms of operators and avoid the explicit iteration technique, by doing it all at once – but at the expense of a little more thought. Namely, going back to the easy case of a tempered distibution on $\mathbb{R}^n$ give the map a name:-

(2.20)

$$Q(D) : f \in \mathcal{S}'(\mathbb{R}^n) \longmapsto \mathcal{F}^{-1}\left(\hat{q}(\zeta)\hat{f}(\zeta)\right) \in \mathcal{S}'(\mathbb{R}^n),\ \hat{q}(\zeta) = \frac{1-\chi(\zeta)}{P(\zeta)}.$$

Here $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ is chosen to be equal to one on the set $|\zeta| \le \frac{1}{c} + 1$ corresponding to the ellipticity estimate (2.2). Thus $\hat{q}(\zeta) \in \mathcal{C}^\infty(\mathbb{R}^n)$ is bounded and in fact

$$|D^\alpha_\zeta \hat{q}(\zeta)| \le C_\alpha (1+|\zeta|)^{-m-|\alpha|}\ \forall\ \alpha. \tag{2.21}$$

This has a straightforward proof by induction. Namely, these estimates are trivial on any compact set, where the function is smooth, so we need only consider the region where $\chi(\zeta) = 0$. The inductive statement is that for some polynomials $H_\alpha$,

$$D^\alpha_\zeta \frac{1}{P(\zeta)} = \frac{H_\alpha(\zeta)}{(P(\zeta))^{|\alpha|+1}},\ \deg(H_\alpha) \le (m-1)|\alpha|. \tag{2.22}$$

From this (2.21) follows. Prove (2.22) itself by differentiating one more time and reorganizing the result.

So, in view of the estimate with $\alpha = 0$ in (2.21),

$$Q(D) : H^s(\mathbb{R}^n) \longrightarrow H^{s+m}(\mathbb{R}^n) \tag{2.23}$$

is continuous for each $s$ and it is also an essential inverse of $P(D)$ in the sense that as operators on $\mathcal{S}'(\mathbb{R}^n)$

$$Q(D)P(D) = P(D)Q(D) = \operatorname{Id} - E,\ E : H^s(\mathbb{R}^n) \longrightarrow H^\infty(\mathbb{R}^n)\ \forall\ s \in \mathbb{R}. \tag{2.24}$$

Namely, $E$ is Fourier multiplication by a smooth function of compact support (namely $1 - \hat{q}(\zeta)P(\zeta)$. So, in the global case of $\mathbb{R}^n$, we get elliptic regularity by applying $Q(D)$ to both sides of the equation $P(D)u = f$ to find

$$f \in H^s(\mathbb{R}^n) \Longrightarrow u = Eu + Qf \in H^{s+m}(\mathbb{R}^n). \tag{2.25}$$

This also gives the esimate (2.15) where the second term comes from the continuity of $E$.

The idea then, is to do the same thing for $P(D)$ acting on functions on the open set $\Omega$; this argument will subsequently be generalized to variable coefficient operators. The problem is that $Q(D)$ does not act on functions (or chapterdistributions) defined just on $\Omega$, they need to be defined on the whole of $\mathbb{R}^n$ and to be tempered before the the Fourier transform can be applied and then multiplied by $\hat{q}(\zeta)$ to define $Q(D)f$.

Now, $Q(D)$ is a convolution operator. Namely, rewriting (2.20)

$$Q(D)f = Qf = q * f,\ q \in \mathcal{S}'(\mathbb{R}^n),\ \hat{q} = \frac{1-\chi(\zeta)}{P(\zeta)}. \tag{2.26}$$

This in fact is exactly what (2.20) means, since $\mathcal{F}(q * f) = \hat{q}\hat{f}$. We can write out convolution by a smooth function (which $q$ is not, but let's not quibble) as an integral

$$q * f(\zeta) = \int_{\mathbb{R}^n} q(z - z')f(z')dz'. \tag{2.27}$$

Restating the problem, (2.27) is an integral (really a distributional pairing) over the whole of $\mathbb{R}^n$ not the subset $\Omega$. In essence the cutoff argument above inserts a cutoff $\phi$ in front of $f$ (really of course in front of $u$ but not to worry). So, let's think about inserting a cutoff into (2.27), replacing it by

$$Q_\psi f(\zeta) = \int_{\mathbb{R}^n} q(z - z')\chi(z, z')f(z')dz'. \tag{2.28}$$

Here we will take $\chi \in \mathcal{C}^\infty(\Omega^2)$. To get the integrand to have compact support in $\Omega$ for each $z \in \Omega$ we want to arrange that the projection onto the second variable, $z'$

$$\pi_L : \Omega \times \Omega \supset \operatorname{supp}(\chi) \longrightarrow \Omega \tag{2.29}$$

should be proper, meaning that the inverse image of a compact subset $K \subset \Omega$, namely $(\Omega \times K) \cap \operatorname{supp}(\chi)$, should be compact in $\Omega$.

Let me strengthen the condition on the support of $\chi$ by making it more two-sided and demand that $\chi \in \mathcal{C}^\infty(\Omega^2)$ have proper support in the following sense:

(2.30)

If $K \subset \Omega$ then $\pi_R\left((\Omega \times K) \cap \operatorname{supp}(\chi)\right) \cup \pi_L\left((L \times \Omega) \cap \operatorname{supp}(\chi)\right) \Subset \Omega$.

Here $\pi_L$, $\pi_R : \Omega^2 \longrightarrow \Omega$ are the two projections, onto left and right factors. This condition means that if we multiply the integrand in (2.28) on the left by $\phi(z)$, $\phi \in \mathcal{C}_c^\infty(\Omega)$ then the integrand has compact support in $z'$ as well – and so should exist at least as a distributional pairing. The second property we want of $\chi$ is that it should not change the properties of $q$ as a convolution operator too much. This reduces to

$$\chi = 1 \text{ in a neighbourhood of } \operatorname{Diag} = \{(z, z); z \in \Omega\} \subset \Omega^2 \tag{2.31}$$

although we could get away with the weaker condition that

$$\chi \equiv 1 \text{ in Taylor series at } \operatorname{Diag}. \tag{2.32}$$

Before discussing why these conditions help us, let me just check that it is possible to find such a $\chi$. This follows easily from the existence of a partition of unity in $\Omega$ as follows. It is possible to find functions $\phi_i \in \mathcal{C}_c^\infty(\Omega)$, $i \in \mathbb{N}$, which have locally finite supports (i.e. any compact subset of $\Omega$ only meets the supports of a finite number of the $\phi_i$, ) such that $\sum_i \phi_i(z) = 1$ in $\Omega$ and also so there exist functions $\phi_i' \in \mathcal{C}_c^\infty(\Omega)$, also with locally finite supports in the same sense and such that $\phi_i' = 1$ on a neighborhood of the support of $\phi_i$. The existence of such functions is a standard result, or if you prefer, an exercise.

Accepting that such functions exists, consider

$$\chi(z,z')=\sum_i \phi_i(z)\phi_i'(z'). \tag{2.33}$$

Any compact subset of $\Omega^2$ is contained in a compact set of the form $K\times K$ and hence meets the supports of only a finite number of terms in (2.33). Thus the sum is locally finite and hence $\chi\in\mathcal{C}^\infty(\Omega^2)$. Moreover, its support has the property (2.30). Clearly, by the assumption that $\phi_i'=1$ on the support of $\phi_i$ and that the latter form a partition of unity, $\chi(z,z)=1$. In fact $\chi(z,z')=1$ in a neighborhood of the diagonal since each $z$ has a neighborhood $N$ such that $z'\in N$, $\phi_i(z)\neq 0$ implies $\phi_i'(z')=1$. Thus we have shown that such a cutoff function $\chi$ exists.

Now, why do we want (2.31)? This arises because of the following 'pseudolocal' property of the kernel $q$.

Lemma 2.5. *Any distribution $q$ defined as the inverse Fourier transform of a function satisfying* (2.21) *has the property*

$$\operatorname{singsupp}(q)\subset\{0\} \tag{2.34}$$

Proof. This follows directly from (2.21) and the properties of the Fourier transform. Indeed these estimates show that

$$z^\alpha q(z)\in\mathcal{C}^N(\mathbb{R}^n)\text{ if }|\alpha|>n+N \tag{2.35}$$

since this is enough to show that the Fourier transform, $(i\partial_\zeta)^\alpha\hat{q}$, is $L^1$. At every point of $\mathbb{R}^n$, other than 0, one of the $z_j$ is non-zero and so, taking $z^\alpha=z_j^k$, (2.35) shows that $q(z)$ is in $C^N$ in $\mathbb{R}^n\setminus\{0\}$ for all $N$, i.e. (2.34) holds. □

Thus the distribution $q(z-z')$ is only singular at the diagonal. It follows that different choices of $\chi$ with the properties listed above lead to kernels in (2.28) which differ by smooth functions in $\Omega^2$ with proper supports.

Lemma 2.6. *A properly supported smoothing operator, which is by defninition given by an integral operator*

$$Ef(z)=\int_\Omega E(z,z')f(z')dz' \tag{2.36}$$

*where $E\in\mathcal{C}^\infty(\Omega^2)$ has proper support (so both maps*

$$\pi_L,\ \pi_R:\operatorname{supp}(E)\longrightarrow\Omega \tag{2.37}$$

*are proper), defines continuous operators*

$$E:\mathcal{C}^{-\infty}(\Omega)\longrightarrow\mathcal{C}^\infty(\Omega),\ \mathcal{C}_c^{-\infty}(\Omega)\longrightarrow\mathcal{C}_c^\infty(\Omega) \tag{2.38}$$

*and has an adjoint of the same type.*

See the discussion in Chapter 3.

PROPOSITION 2.7. *If $P(D)$ is an elliptic operator with constant coefficients then the kernel in* (2.28) *defines an operator $Q_\Omega : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega)$ which maps $H^s_{\text{loc}}(\Omega)$ to $H^{s+m}_{\text{loc}}(\Omega)$ for each $s \in \mathbb{R}$ and gives a 2-sided parametrix for $P(D)$ in $\Omega$ :*

$$P(D)Q_\Omega = \operatorname{Id} - R,\ \ Q_\Omega P(D) = \operatorname{Id} - R' \tag{2.39}$$

*where $R$ and $R'$ are properly supported smoothing operators.*

PROOF. We have already seen that changing $\chi$ in (2.28) changes $Q_\Omega$ by a smoothing operator; such a change will just change $R$ and $R'$ in (2.39) to different properly supported smoothing operators. So, we can use the explicit choice for $\chi$ made in (2.33) in terms of a partition of unity. Thus, multiplying on the left by some $\mu \in \mathcal{C}^\infty_c(\Omega)$ the sum becomes finite and

$$\mu Q_\Omega f = \sum_j \mu\psi_j q * (\psi'_j f). \tag{2.40}$$

It follows that $Q_\Omega$ acts on $\mathcal{C}^{-\infty}(\Omega)$ and, from the properties of $q$ it maps $H^s_{\text{loc}}(\mathbb{R}^n)$ to $H^{s+m}_{\text{loc}}(\mathbb{R}^n)$ for any $s$. To check (2.39) we may apply $P(D)$ to (2.40) and consider a region where $\mu = 1$. Since $P(D)q = \delta_0 - \tilde{R}$ where $\tilde{R} \in \mathcal{S}(\mathbb{R}^n)$, $P(D)Q_\Omega f = \operatorname{Id} - R$ where additional 'error terms' arise from any differentiation of $\phi_j$. All such terms have smooth kernels (since $\phi'_j = 1$ on the support of $\phi_j$ and $q(z - z')$ is smooth outside the diagonal) and are properly supported. The second identity in (2.39) comes from the same computation for the adjoints of $P(D)$ and $Q_\Omega$. $\square$

## 3. Interior elliptic estimates

Next we proceed to prove the same type of regularity and estimates, (2.17), for elliptic differential operators with variable coefficients. Thus consider

$$P(z, D) = \sum_{|\alpha| \le m} p_\alpha(z) D^\alpha,\ \ p_\alpha \in \mathcal{C}^\infty(\Omega). \tag{3.1}$$

We now assume ellipticity, of fixed order $m$, for the polynomial $P(z, \zeta)$ for each $z \in \Omega$. This is the same thing as ellipticity for the principal part, i.e. the condition for each compact subset of $\Omega$

$$|\sum_{|\alpha|=m} p_\alpha(z)\zeta^\alpha| \ge C(K)|\zeta|^m,\ \ z \in K \Subset \Omega, C(K) > 0. \tag{3.2}$$

Since the coefficients are smooth this and $\mathcal{C}^\infty(\Omega)$ is a multiplier on $H^s_{\text{loc}}(\Omega)$ such a differential operator (elliptic or not) gives continuous linear maps

$$P(z,D):H^{s+m}_{\mathrm{loc}}(\Omega)\longrightarrow H^s_{\mathrm{loc}}(\Omega),\ \forall\, s\in\mathbb{R},\ P(z,D):\mathcal{C}^\infty(\Omega)\longrightarrow\mathcal{C}^\infty(\Omega). \tag{3.3}$$

Now, we arrived at the estimate (2.12) in the constant coefficient case by iteration from the case $M=s+m-1$ (by nesting cutoff functions). Pick a point $\bar z\in\Omega$. In a small ball around $\bar z$ the coefficients are 'almost constant'. In fact, by Taylor's theorem,

$$P(z,\zeta)=P(\bar z,\zeta)+Q(z,\zeta),\ \ Q(z,\zeta)=\sum_j(z-\bar z)_jP_j(z,\bar z,\zeta) \tag{3.4}$$

where the $P_j$ are also polynomials of degree $m$ in $\zeta$ and smooth in $z$ in the ball (and in $\bar z$.) We can apply the estimate (2.12) for $P(\bar z,D)$ and $s=0$ to find

$$\|\psi u\|_m\le C\|\psi\left(P(z,D)u-Q(z,D)\right)u\|_0+C'\|\phi u\|_{m-1}. \tag{3.5}$$

Because the coefficients are small

$$\begin{aligned}\|\psi Q(z,D)u\|_0&\le\sum_{j,|\alpha|\le m}\|(z-\bar z)_jr_{j,\alpha}D^\alpha\psi u\|_0+C'\|\phi u\|_{m-1}\\&\le\delta C\|\psi u\|_m+C'\|\phi u\|_{m-1}.\end{aligned} \tag{3.6}$$

What we would like to say next is that we can choose $\delta$ so small that $\delta C<\frac12$ and so inserting (3.6) into (3.5) we would get

$$\begin{aligned}\|\psi u\|_m&\le C\|\psi P(z,D)u\|_0+C\|\psi Q(z,D)u\|_0+C'\|\phi u\|_{m-1}\\&\le C\|\psi P(z,D)u\|_0+\frac12\|\psi u\|_m+C'\|\phi u\|_{m-1}\\&\Longrightarrow\frac12\|\psi u\|_m\le C\|\psi P(z,D)u\|_0+C'\|\phi u\|_{m-1}.\end{aligned} \tag{3.7}$$

However, there is a problem here. Namely this is an *a priori* estimate – to move the norm term from right to left we need to know that it is *finite*. Really, that is what we are trying to prove! So more work is required. Nevertheless we will eventually get essentially the same estimate as in the constant coefficient case.

THEOREM 3.1. *If $P(z,D)$ is an elliptic differential operator of order $m$ with smooth coefficients in $\Omega\subset\mathbb{R}^n$ and $u\in\mathcal{C}^{-\infty}(\Omega)$ is such that $P(z,D)u\in H^s_{\mathrm{loc}}(\Omega)$ for some $s\in\mathbb{R}$ then $u\in H^{s+m}_{\mathrm{loc}}(\Omega)$ and for any $\phi$, $\psi\in\mathcal{C}^\infty_c(\Omega)$ with $\phi=1$ in a neighbourhood of* $\operatorname{supp}(\psi)$ *and $M\in\mathbb{R}$, there exist constants $C$ (depending only on $P$ and $\psi$) and $C'$ (independent of $u$) such that*

$$\|\psi u\|_{m+s}\le C\|\psi P(z,D)u\|_s+C'\|\phi u\|_M. \tag{3.8}$$

There are three main things to do. First we need to get the *a priori* estimate first for general $s$, rather than $s = 0$, and then for general $\psi$ (since up to this point it is only for $\psi$ with sufficiently small support). One problem here is that in the estimates in (3.6) the $L^2$ norm of a product is estimated by the $L^\infty$ norm of one factor and the $L^2$ norm of the other. For general Sobolev norms such an estimate does not hold, but something similar does; see Lemma 15.2. The proof of this theorem occupies the rest of this Chapter.

PROPOSITION 3.2. *Under the hypotheses of Theorem 3.1 if in addition $u \in \mathcal{C}^\infty(\Omega)$ then* (3.8) *follows.*

PROOF OF PROPOSITION 3.2. First we can generalize (3.5), now using Lemma 15.2. Thus, if $\psi$ has support near the point $\bar{z}$

$$\begin{aligned}(3.9)\quad \|\psi u\|_{s+m} &\le C\|\psi P(\bar{z},D)u\|_s + \|\phi Q(z,D)\psi u\|_s + C'\|\phi u\|_{s+m-1}\\ &\le C\|\psi P(\bar{z},D)u\|_s + \delta C\|\psi u\|_{s+m} + C'\|\phi u\|_{s+m-1}.\end{aligned}$$

This gives the extension of (3.7) to general $s$ (where we are assuming that $u$ is indeed smooth):

$$(3.10)\qquad \|\psi u\|_{s+m} \le C_s\|\psi P(z,D)u\|_s + C'\|\phi u\|_{s+m-1}.$$

Now, given a general element $\psi \in \mathcal{C}_c^\infty(\Omega)$ and $\phi \in \mathcal{C}_c^\infty(\Omega)$ with $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\psi)$ we may choose a partition of unity $\psi_j$ with respect to $\operatorname{supp}(\psi)$ for each element of which (3.10) holds for some $\phi_j \in \mathcal{C}_c^\infty(\Omega)$ where in addition $\phi = 1$ in a neighbourhood of $\operatorname{supp}(\phi_j)$. Then, with various constants

(3.11)

$$\begin{aligned}\|\psi u\|_{s+m} \le \sum_j \|\psi_j u\|_{s+m} &\le C_s \sum_j \|\psi_j \phi P(z,D)u\|_s + C' \sum_j \|\phi_j \phi u\|_{s+m-1}\\ &\le C_s(K)\|\phi P(z,D)u\|_s + C''\|\phi u\|_{s+m-1},\end{aligned}$$

where $K$ is the support of $\psi$ and Lemma 15.2 has been used again. This removes the restriction on supports.

Now, to get the full (a priori) estimate (3.8), where the error term on the right has been replaced by one with arbitrarily negative Sobolev order, it is only necessary to iterate (3.11) on a nested sequence of cutoff functions as we did earlier in the constant coefficient case.

This completes the proof of Proposition 3.2. □

So, this proves *a priori* estimates for solutions of the elliptic operator in terms of Sobolev norms. To use these we need to show the regularity of solutions and I will do this by constructing parametrices in a manner very similar to the constant coefficient case.

Theorem 3.3. *If $P(z,D)$ is an elliptic differential operator of order $m$ with smooth coefficients in $\Omega \subset \mathbb{R}^n$ then there is a continuous linear operator*

$$Q : \mathcal{C}^{-\infty}(\Omega) \longrightarrow \mathcal{C}^{-\infty}(\Omega) \tag{3.12}$$

*such that*

$$P(z,D)Q = \operatorname{Id} - R_R, \;\; QP(z,D) = \operatorname{Id} - R_L \tag{3.13}$$

*where $R_R$, $R_L$ are properly-supported smoothing operators.*

That is, both $R_R$ and $R_L$ have kernels in $\mathcal{C}^\infty(\Omega^2)$ with proper supports. We will in fact conclude that

$$Q : H^s_{\text{loc}}(\Omega) \longrightarrow H^{s+m}_{\text{loc}}(\Omega), \;\; \forall\; s \in \mathbb{R} \tag{3.14}$$

using the *a priori* estimates.

To construct at least a first approximation to $Q$ essentially the same formula as in the constant coefficient case suffices. Thus consider

$$Q_0 f(z) = \int_\Omega q(z, z-z')\chi(z,z')f(z')dz'. \tag{3.15}$$

Here $q$ is defined as last time, except it now depends on both variables, rather than just the difference, and is defined by inverse Fourier transform

$$q_0(z,Z) = \mathcal{F}^{-1}_{\zeta\longmapsto Z}\hat{q}_0(z,\zeta), \;\; \hat{q}_0 = \frac{1-\chi(z,\zeta)}{P(z,\zeta)} \tag{3.16}$$

where $\chi \in \mathcal{C}^\infty(\Omega \times \mathbb{R})$ is chosen to have compact support in the second variable, so $\operatorname{supp}(\chi) \cap (K \times \mathbb{R}^n)$ is compact for each $K \Subset \Omega$, and to be equal to 1 on such a large set that $P(z,\zeta) \neq 0$ on the support of $1 - \chi(z,\zeta)$. Thus the right side makes sense and the inverse Fourier transform exists.

Next we extend the esimates, (2.21), on the $\zeta$ derivatives of such a quotient, using the ellipticity of $P$. The same argument works for derivatives with respect to $z$, except no decay occurs. That is, for any compact set $K \Subset \Omega$

$$|D^\beta_z D^\alpha_\zeta \hat{q}_0(z,\zeta)| \leq C_{\alpha,\beta}(K)(1+|\zeta|)^{-m-|\alpha|}, \;\; z \in K. \tag{3.17}$$

Now the argument, in Lemma 2.5, concerning the singularities of $q_0$ works with $z$ derivatives as well. It shows that

$$(z_j - z'_j)^{N+k} q_0(z, z-z') \in \mathcal{C}^N(\Omega \times \mathbb{R}^n) \text{ if } k+m > n/2. \tag{3.18}$$

Thus,

$$\operatorname{singsupp} q_0 \subset \operatorname{Diag} = \{(z,z) \in \Omega^2\}. \tag{3.19}$$

The 'pseudolocality' statement (3.19), shows that as in the earlier case, changing the cutoff function in (3.15) changes $Q_0$ by a properly supported smoothing operator and this will not affect the validity of (3.13) one way or the other! For the moment not worrying too much about how to make sense of (3.15) consider (formally)

$$P(z,D)Q_0f = \int_\Omega \left(P(z,D_Z)q_0(z,Z)\right)_{Z=z-z'} \chi(z,z')f(z')dz' + E_1f + R_1f. \tag{3.20}$$

To apply $P(z,D)$ we just need to apply $D^\alpha$ to $Q_0f$, multiply the result by $p_\alpha(z)$ and add. Applying $D_z^\alpha$ (formally) under the integral sign in (3.15) each derivative may fall on either the 'parameter' $z$ in $q_0(z, z-z')$, the variable $Z = z - z'$ or else on the cutoff $\chi(z,z')$. Now, if $\chi$ is ever differentiated the result vanishes near the diagonal and as a consequence of (3.19) this gives a smooth kernel. So any such term is included in $R_1$ in (3.20) which is a smoothing operator and we only have to consider derivatives falling on the first or second variables of $q_0$. The first term in (3.20) corresponds to *all* derivatives falling on the second variable. Thus

$$E_1f = \int_\Omega e_1(z,z-z')\chi(z,z')f(z')dz' \tag{3.21}$$

is the sum of the terms which arise from at least one derivative in the 'parameter variable' $z$ in $q_0$ (which is to say ultimately the coefficients of $P(z,\zeta)$). We need to examine this in detail. First however notice that we may rewrite (3.20) as

$$P(z,D)Q_0f = \operatorname{Id} + E_1 + R_1' \tag{3.22}$$

where $E_1$ is unchanged and $R_1'$ is a new properly supported smoothing operator which comes from the fact that

$$\begin{gathered} P(z,\zeta)q_0(z,\zeta) = 1 - \rho(z,\zeta) \Longrightarrow \\ P(z,D_Z)q_0(z,Z) = \delta(Z) + r(z,Z),\ r \in \mathcal{C}^\infty(\Omega\times\mathbb{R}^n) \end{gathered} \tag{3.23}$$

from the choice of $q_0$. This part is just as in the constant coefficient case.

So, it is the new error term, $E_1$ which we must examine more carefully. This arises, as already noted, directly from the fact that the coefficients of $P(z,D)$ are not assumed to be constant, hence $q_0(z,Z)$ depends parameterically on $z$ and this is differentiated in (3.20). So, using Leibniz' formula to get an explicit representation of $e_1$ in (3.21) we see that

$$e_1(z,Z)=\sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma}D_z^{\alpha-\gamma}D_Z^\gamma q_0(z,Z). \tag{3.24}$$

The precise form of this expansion is not really significant. What *is* important is that at most $m-1$ derivatives are acting on the second variable of $q_0(z,Z)$ since all the terms where all $m$ act here have already been treated. Taking the Fourier transform in the second variable, as before, we find that

$$\hat{e}_1(z,\zeta)=\sum_{|\alpha|\le m,\ |\gamma|<m} p_\alpha(z)\binom{\alpha}{\gamma}D_z^{\alpha-\gamma}\zeta^\gamma\hat{q}_0(z,\zeta)\in\mathcal{C}^\infty(\Omega\times\mathbb{R}^n). \tag{3.25}$$

Thus $\hat{e}_1$ is the sum of products of $z$ derivatives of $q_0(z,\zeta)$ and polynomials in $\zeta$ of degree at most $m-1$ with smooth dependence on $z$. We may therefore transfer the estimates (3.17) to $e_1$ and conclude that

$$|D_z^\beta D_\zeta^\alpha\hat{e}_1(z,\zeta)|\le C_{\alpha,\beta}(K)(1+|\zeta|)^{-1-|\alpha|}. \tag{3.26}$$

Let us denote by $S^m(\Omega\times\mathbb{R}^n)\subset\mathcal{C}^\infty(\Omega\times\mathbb{R}^n)$ the linear space of functions satisfying (3.17) when $-m$ is replaced by $m$, i.e.

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)|\le C_{\alpha,\beta}(K)(1+|\zeta|)^{m-|\alpha|}\Longleftrightarrow a\in S^m(\Omega\times\mathbb{R}^n). \tag{3.27}$$

This allows (3.26) to be written succinctly as $\hat{e}_1\in S^{-1}(\Omega\times\mathbb{R}^n)$.

To summarize so far, we have chosen $\hat{q}_0\in S^{-m}(\Omega\times\mathbb{R}^n)$ such that with $Q_0$ given by (3.15),

$$P(z,D)Q_0=\operatorname{Id}+E_1+R_1' \tag{3.28}$$

where $E_1$ is given by the same formula (3.15), as (3.21), where now $\hat{e}_1\in S^{-1}(\Omega\times\mathbb{R}^n)$. In fact we can easily generalize this discussion, to do so let me use the notation

$$\operatorname{Op}(a)f(z)=\int_\Omega A(z,z-z')\chi(z,z')f(z')dz', \tag{3.29}$$
$$\text{if } \hat{A}(z,\zeta)=a(z,\zeta)\in S^m(\Omega\times\mathbb{R}^n).$$

Proposition 3.4. *If $a\in S^{m'}(\Omega\times\mathbb{R}^n)$ then*

$$P(z,D)\operatorname{Op}(a)=\operatorname{Op}(pa)+\operatorname{Op}(b)+R \tag{3.30}$$

*where $R$ is a (properly supported) smoothing operator and $b\in S^{m'+m-1}(\Omega\times\mathbb{R}^n)$.*

Proof. Follow through the discussion above with $\hat{q}_0$ replaced by $a$. □

So, we wish to get rid of the error term $E_1$ in (3.21) to as great an extent as possible. To do so we add to $Q_0$ a second term $Q_1 = \mathrm{Op}(a_1)$ where

$$a_1 = -\frac{1-\chi}{P(z,\zeta)}\hat{e}_1(z,\zeta) \in S^{-m-1}(\Omega \times \mathbb{R}^n). \tag{3.31}$$

Indeed

$$S^{m'}(\Omega \times \mathbb{R}^n) S^{m''}(\Omega \times \mathbb{R}^n) \subset S^{m'+m''}(\Omega \times \mathbb{R}^n) \tag{3.32}$$

(pretty much as though we are multiplying polynomials) as follows from Leibniz' formula and the defining estimates (3.27). With this choice of $Q_1$ the identity (3.30) becomes

$$P(z,D)Q_1 = -E_1 + \mathrm{Op}(b_2) + R_2,\ b_2 \in S^{-2}(\Omega \times \mathbb{R}^n) \tag{3.33}$$

since $p(z,\zeta)a_1 = -\hat{e}_1 + r'(z,\zeta)$ where $\mathrm{supp}(r')$ is compact in the second variable and so contributes a smoothing operator and by definition $E_1 = \mathrm{Op}(\hat{e}_1)$.

Now we can proceed by induction, let me formalize it a little.

Lemma 3.5. *If $P(z,D)$ is elliptic with smooth coefficients on $\Omega$ then we may choose a sequence of elements $a_i \in S^{-m-i}(\Omega \times \mathbb{R}^n)$ $i = 0, 1, \ldots,$ such that if $Q_i = \mathrm{Op}(a_i)$ then*

$$P(z,D)(Q_0 + Q_1 + \cdots + Q_j) = \mathrm{Id} + E_{j+1} + R'_j,\ E_{j+1} = \mathrm{Op}(b_{j+1}) \tag{3.34}$$

*with $R_j$ a smoothing operator and $b_j \in S^{-j}(\Omega \times \mathbb{R}^n)$, $j = 1, 2, \ldots.$*

Proof. We have already taken the first two steps! Namely with $a_0 = \hat{q}_0$, given by (3.16), (3.28) is just (3.34) for $j = 0$. Then, with $a_1$ given by (3.31), adding (3.33) to (3.31) gives (3.34) for $j = 1$. Proceeding by induction we may assume that we have obtained (3.34) for some $j$. Then we simply set

$$a_{j+1} = -\frac{1-\chi(z,\zeta)}{P(z,\zeta)} b_{j+1}(z,\zeta) \in S^{-j-1-m}(\Omega \times \mathbb{R}^n)$$

where we have used (3.32). Setting $Q_{j+1} = \mathrm{Op}(a_{j+1})$ the identity (3.30) becomes

$$P(z,D)Q_{j+1} = -E_{j+1} + E_{j+2} + R''_{j+1},\ E_{j+2} = \mathrm{Op}(b_{j+2}) \tag{3.35}$$

for some $b_{j+2} \in S^{-j-2}(\Omega \times \mathbb{R}^n)$. Adding (3.35) to (3.34) gives the next step in the inductive argument. □

Consider the error term in (3.34) for large $j$. From the estimates on an element $a \in S^{-j}(\Omega \times \mathbb{R}^n)$

$$|D_z^\beta D_\zeta^\alpha a(z,\zeta)| \le C_{\alpha,\beta}(K)(1+|\zeta|)^{-j-|\alpha|} \tag{3.36}$$

it follows that if $j > n + k$ then $\zeta^\gamma a$ is integrable in $\zeta$ with all its $z$ derivatives for $|\zeta| \le k$. Thus the inverse Fourier transform has continuous derivatives in all variables up to order $k$. Applied to the error term in (3.34) we conclude that

$$E_j = \mathrm{Op}(b_j) \text{ has kernel in } \mathcal{C}^{j-n-1}(\Omega^2) \text{ for large } j. \tag{3.37}$$

Thus as $j$ increases the error terms in (3.34) have increasingly smooth kernels.

Now, standard properties of operators and kernels, see Lemma 17.1, show that operator

$$Q_{(k)} = \sum_{j=0}^{k} Q_j \tag{3.38}$$

comes increasingly close to satisfying the first identity in (3.13), except that the error term is only finitely (but arbitrarily) smoothing. Since this is enough for what we want here I will banish the actual solution of (3.13) to the addenda to this Chapter.

LEMMA 3.6. *For $k$ sufficiently large, the left parametrix $Q_{(k)}$ is a continuous operator on $\mathcal{C}^\infty(\Omega)$ and*

$$Q_{(k)} : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^{s+m}_{\mathrm{loc}}(\Omega) \ \forall\ s \in \mathbb{R}. \tag{3.39}$$

PROOF. So far I have been rather cavalier in treating $\mathrm{Op}(a)$ for $a \in S^m(\Omega \times \mathbb{R}^n)$ as an operator without showing that this is really the case, however this is a rather easy exercise in distribution theory. Namely, from the basic properties of the Fourier transform and Sobolev spaces

$$A(z, z - z') \in \mathcal{C}^k(\Omega; H^{-n-1+m-k}_{\mathrm{loc}}(\Omega)) \ \forall\ k \in \mathbb{N}. \tag{3.40}$$

It follows that $\mathrm{Op}(a) : H^{n+1-m+k}_{\mathrm{c}}(\Omega)$ into $\mathcal{C}^k(\Omega)$ and in fact into $\mathcal{C}^k_{\mathrm{c}}(\Omega)$ by the properness of the support. In particular it does define an operator on $\mathcal{C}^\infty(\Omega)$ as we have been pretending and the steps above are easily justified.

A similar argument, which I will not give here since it is better to do it by duality (see the addenda), shows that for any fixed $s$

$$A : H^s_{\mathrm{loc}}(\Omega) \longrightarrow H^S_{\mathrm{loc}}(\Omega) \tag{3.41}$$

for some $S$. Of course we want something a bit more precise than this.

If $f \in H^s_{\mathrm{loc}}(\Omega)$ then it may be approximated by a sequence $f_j \in \mathcal{C}^\infty(\Omega)$ in the topology of $H^s_{\mathrm{loc}}(\Omega)$, so $\mu f_j \to \mu f$ in $H^s(\mathbb{R}^n)$ for each $\mu \in \mathcal{C}^\infty_c(\Omega)$. Set $u_j = Q_{(k)} f_j \in \mathcal{C}^\infty(\Omega)$ as we have just seen, where $k$ is fixed but will be chosen to be large. Then from our identity $P(z,D)Q_{(k)} = \operatorname{Id} + R_{(k)}$ it follows that

$$P(z,D)u_j = f_j + g_j,\ g_j = R_{(k)}f_j \to R_{(k)}f \text{ in } H^N_{\text{loc}}(\Omega) \tag{3.42}$$

for $k$ large enough depending on $s$ and $N$. Thus, for $k$ large, the right side converges in $H^s_{\text{loc}}(\Omega)$ and by (3.41), $u_j \to u$ in some $H^S_{\text{loc}}(\Omega)$. But now we can use the *a priori* estimates (3.8) on $u_j \in \mathcal{C}^\infty(\Omega)$ to conclude that

$$\|\psi u_j\|_{s+m} \le C\|\psi(f_j + g_j)\|_s + C''\|\phi u_j\|_S \tag{3.43}$$

to see that $\psi u_j$ is bounded in $H^{s+m}(\mathbb{R}^n)$ for any $\psi \in \mathcal{C}^\infty_c(\Omega)$. In fact, applied to the difference $u_j - u_l$ it shows the sequence to be Cauchy. Hence in fact $u \in H^{s+m}_{\text{loc}}(\Omega)$ and the estimates (3.8) hold for this $u$. That is, $Q_{(k)}$ has the mapping property (3.39) for large $k$. □

In fact the continuity property (3.39) holds for all $\operatorname{Op}(a)$ where $a \in S^m(\Omega \times \mathbb{R}^n)$, not just those which are parametrices for elliptic differential operators. I will comment on this below – it is one of the basic results on pseudodifferential operators.

There is also the question of the second identity in (3.13), at least in the same finite-order-error sense. To solve this we may use the transpose identity. Thus taking formal transposes this second identity should be equivalent to

$$P^tQ^t = \operatorname{Id} - R^t_L. \tag{3.44}$$

The transpose of $P(z,D)$ is the differential operotor

$$P^t(z,D) = \sum_{|\alpha|\le m} (-D)^\alpha_z p_\alpha(z). \tag{3.45}$$

This is again of order $m$ and after a lot of differenttiation to move the coefficients back to the left we see that its leading part is just $P_m(z,-D)$ where $P_m(z,D)$ is the leading part of $P(z,D)$, so it is elliptic in $\Omega$ exactly when $P$ is elliptic. To construct a solution to (3.45), up to finite order errors, we need just apply Lemma 3.5 to the transpose differential operator. This gives $Q'_{(N)} = \operatorname{Op}(a'_{(N)}$ with the property

$$P^t(z,D)Q'_{(N)} = \operatorname{Id} - R'_{(N)} \tag{3.46}$$

where the kernel of $R'_{(N)}$ is in $C^N(\Omega^2)$. Since this property is preserved under transpose we have indeed solved the second identity in (3.13) up to an arbitrarily smooth error.

Of course the claim in Theorem 3.3 is that the one operator satisfies both identities, whereas we have constructed two operators which each satisfy one of them, up to finite smoothing error terms

$$P(z,D)Q_R = \operatorname{Id} - R_R,\ Q_L P(z,D) = \operatorname{Id} - R_L. \tag{3.47}$$

However these operators must themselves be equal up to finite smoothing error terms since composing the first identity on the left with $Q_L$ and the second on the right with $Q_R$ shows that

$$Q_L - Q_L R_R = Q_L P(z,D) Q_R = Q_R - R_L Q_R \tag{3.48}$$

where the associativity of operator composition has been used. We have already checked the mapping property(3.39) for both $Q_L$ and $Q_R$, assuming the error terms are sufficiently smoothing. It follows that the composite error terms here map $H^{-p}_{\mathrm{loc}}(\Omega)$ into $H^{p}_{\mathrm{loc}}(\Omega)$ where $p \to \infty$ with $k$ with the same also true of the transposes of these operators. Such an operator has kernel in $C^{p'}(\Omega^2)$ where again $p' \to \infty$ with $k$. Thus the difference of $Q_L$ and $Q_R$ itself becomes arbitrarily smoothing as $k \to \infty$.

Finally then we have proved most of Theorem 3.3 except with arbitrarily finitely smoothing errors. In fact we have not quite proved the regularity statement that $P(z,D)u \in H^s_{\mathrm{loc}}(\Omega)$ implies $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ although we came very close in the proof of Lemma 3.6. Now that we know that $Q_{(k)}$ is also a right parametrix, i.e. satisfies the second identity in (3.8) up to arbitrarily smoothing errors, this too follows. Namely from the discussion above $Q_{(k)}$ is an operator on $\mathcal{C}^{-\infty}(\Omega)$ and

$$Q_{(k)}P(z,D)u = u + v_k,\ \psi v_k \in H^{s+m}(\Omega)$$

for large enough $k$ so (3.39) implies $u \in H^{s+m}_{\mathrm{loc}}(\Omega)$ and the *a priori* estimates magically become real estimates on all solutions.

## Addenda to Chapter 4

Asymptotic completeness to show that we really can get smoothing errors.

Some discussion of pseudodifferential operators – adjoints, composition and boundedness, but only to make clear what is going on.

Some more reassurance as regards operators, kernels and mapping properties – since I have treated these fairly shabbily!

CHAPTER 5

# Coordinate invariance and manifolds

For the geometric applications we wish to make later (and of course many others) it is important to understand how the objects discussed above behave under coordinate transformations, so that they can be transferred to manifolds (and vector bundles). The basic principle is that the results above are independent of the choice of coordinates, which is to say diffeomorphisms of open sets.

## 1. Local diffeomorphisms

Let $\Omega_i \subset \mathbb{R}^n$ be open and $f : \Omega_1 \longrightarrow \Omega_2$ be a diffeomorphism, so it is a $\mathcal{C}^\infty$ map, which is equivalent to the condition

$$f^*u \in \mathcal{C}^\infty(\Omega_1) \ \forall \ u \in \mathcal{C}^\infty(\Omega_2),\ f^*u = u \circ f,\ f^*u(z) = u(f(z)), \tag{1.1}$$

and has a $\mathcal{C}^\infty$ inverse $f^{-1} : \Omega_2 \longrightarrow \Omega_1$. Such a map induces an isomorphisms $f^* : \mathcal{C}^\infty_c(\Omega_2) \longrightarrow \mathcal{C}^\infty_c(\Omega_1)$ and $f^* : \mathcal{C}^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_1)$ with inverse $(f^{-1})^* = (f^*)^{-1}$.

Recall also that, as a homeomorphism, $f^*$ identifies the (Borel) measurable functions on $\Omega_2$ with those on $\Omega_1$. Since it is continuously differentiable it also identifies $L^1_{\text{loc}}(\Omega_2)$ with $L^1_{\text{loc}}(\Omega_1)$ and

$$u \in L^1_{\text{c}}(\Omega_2) \Longrightarrow \int_{\Omega_1} f^*u(z)|J_f(z)|dz = \int_{\Omega_1} u(z')dz',\ J_f(z) = \det \frac{\partial f_i(z)}{\partial z_j}. \tag{1.2}$$

The absolute value appears because the definition of the Lebesgue integral is through the Lebesgue measure.

It follows that $f^* : L^2_{\text{loc}}(\Omega_2) \longrightarrow L^2_{\text{loc}}(\Omega_1)$ is also an isomorphism. If $u \in L^2(\Omega_2)$ has support in some compact subset $K \Subset \Omega_2$ then $f^*u$ has support in the compact subset $f^{-1}(K) \Subset \Omega_1$ and

$$\|f^*u\|^2_{L^2} = \int_{\Omega_1} |f^*u|^2 dz \le C(K) \int_{\Omega_1} |f^*u|^2 |J_f(z)|dz = C(K)\|u\|^2_{L^2}. \tag{1.3}$$

Distributions are defined by duality, as the continuous linear functionals:-

$$u \in \mathcal{C}^{-\infty}(\Omega) \Longrightarrow u : \mathcal{C}^\infty_c(\Omega) \longrightarrow \mathbb{C}. \tag{1.4}$$

We always embed the smooth functions in the distributions using integration. This presents a small problem here, namely it is not consistent under pull-back. Indeed if $u \in \mathcal{C}^\infty(\Omega_2)$ and $\mu \in \mathcal{C}_c^\infty(\Omega_1)$ then

$$
\begin{aligned}
(1.5)\qquad & \int_{\Omega_1} f^*u(z)\mu(z)|J_f(z)|dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')dz' \text{ or} \\
& \int_{\Omega_1} f^*u(z)\mu(z)dz = \int_{\Omega_2} u(z')(f^{-1})^*\mu(z')|J_{f^{-1}}(z')|dz', \\
& \qquad\qquad\qquad\qquad \text{since } f^*J_{f^{-1}} = (J_f)^{-1}.
\end{aligned}
$$

So, if we want distributions to be 'generalized functions', so that the identification of $u \in \mathcal{C}^\infty(\Omega_2)$ as an element of $\mathcal{C}^{-\infty}(\Omega_2)$ is consistent with the identification of $f^*u \in \mathcal{C}^\infty(\Omega_1)$ as an element of $\mathcal{C}^{-\infty}(\Omega_1)$ we need to use (1.5). Thus we *define*

$$
(1.6)\qquad f^* : \mathcal{C}^{-\infty}(\Omega_2) \longrightarrow \mathcal{C}^{-\infty}(\Omega_1) \text{ by } f^*u(\mu) = u((f^{-1})^*\mu|J_{f^{-1}}|).
$$

There are better ways to think about this, namely in terms of densities, but let me not stop to do this at the moment. Of course one should check that $f^*$ is a map as indicated and that it behaves correctly under composition, so $(f \circ g)^* = g^* \circ f^*$.

As already remarked, smooth functions pull back under a diffeomorphism (or any smooth map) to be smooth. Dually, vector fields push-forward. A vector field, in local coordinates, is just a first order differential operator without constant term

$$
(1.7)\qquad V = \sum_{j=1}^n v_j(z) D_{z_j},\ D_{z_j} = D_j = \frac{1}{i}\frac{\partial}{\partial z_j}.
$$

For a diffeomorphism, the push-forward may be defined by

$$
(1.8)\qquad f^*(f_*(V)u) = Vf^*u \ \forall\ u \in \mathcal{C}^\infty(\Omega_2)
$$

where we use the fact that $f^*$ in (1.1) is an isomorphism of $\mathcal{C}^\infty(\Omega_2)$ onto $\mathcal{C}^\infty(\Omega_1)$. The chain rule is the computation of $f_*V$, namely

$$
(1.9)\qquad f_*V(f(z)) = \sum_{j,k=1}^n v_j(z)\frac{\partial f_j(z)}{\partial z_k} D_k.
$$

As always this operation is natural under composition of diffeomorphism, and in particular $(f^{-1})_*(f_*)V = V$. Thus, under a diffeomorphism, vector fields push forward to vector fields and so, more generally, differential operators push-forward to differential operators.

Now, with these definitions we have

THEOREM 1.1. *For every $s \in \mathbb{R}$, any diffeomorphism $f : \Omega_1 \longrightarrow \Omega_2$ induces an isomorphism*

$$f^* : H^s_{\mathrm{loc}}(\Omega_2) \longrightarrow H^s_{\mathrm{loc}}(\Omega_1). \tag{1.10}$$

PROOF. We know this already for $s = 0$. To prove it for $0 < s < 1$ we use the norm on $H^s(\mathbb{R}^n)$ equivalent to the standard Fourier transform norm:-

$$\|u\|_s^2 = \|u\|_{L^2}^2 + \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(\zeta)|^2}{|z - \zeta|^{2s+n}} dz d\zeta. \tag{1.11}$$

See Sect 7.9 of [**4**]. Then if $u \in H^s_{\mathrm{c}}(\Omega_2)$ has support in $K \Subset \Omega_2$ with $0 < s < 1$, certainly $u \in L^2$ so $f^*u \in L^2$ and we can bound the second part of the norm in (1.11) on $f^*u$ :

$$\begin{aligned} \int_{\mathbb{R}^{2n}} &\frac{|u(f(z)) - u(f(\zeta))|^2}{|z - \zeta|^{2s+n}} dz d\zeta \\ &= \int_{\mathbb{R}^{2n}} \frac{|u(z') - u(\zeta')|^2}{|g(z') - g(\zeta')|^{2s+n}} |J_g(z')||J_g(\zeta')| dz' d\zeta' \\ &\qquad \le C \int_{\mathbb{R}^{2n}} \frac{|u(z) - u(\zeta)|^2}{|z - \zeta|^{2s+n}} dz d\zeta \end{aligned} \tag{1.12}$$

since $C|g(z') - g(\zeta')| \ge |z' - \zeta'|$ where $g = f^{-1}$.

For the spaces of order $m + s$, $0 \le s < 1$ and $m \in \mathbb{N}$ we know that

$$u \in H^{m+s}_{\mathrm{loc}}(\Omega_2) \Longleftrightarrow Pu \in H^s_{\mathrm{loc}}(\Omega_2) \ \forall \ P \in \mathrm{Diff}^m(\Omega_2) \tag{1.13}$$

where $\mathrm{Diff}^m(\Omega)$ is the space of differential operators of order at most $m$ with smooth coefficients in $\Omega$. As noted above, differential operators map to differential operators under a diffeomorphism, so from (1.13) it follows that $H^{m+s}_{\mathrm{loc}}(\Omega_2)$ is mapped into $H^{m+s}_{\mathrm{loc}}(\Omega_1)$ by $f^*$.

For negative orders we may proceed in the same way. That is if $m \in \mathbb{N}$ and $0 \le s < 1$ then

$$u \in H^{s-m}_{\mathrm{loc}}(\Omega_2) \Longleftrightarrow u = \sum_J P_J u_J, \ P_J \in \mathrm{Diff}^m(\Omega_2), \ u_J \in H^s(\Omega_2) \tag{1.14}$$

where the sum over $J$ is finite. A similar argument then applies to prove (1.10) for all real orders. $\square$

Consider the issue of differential operators more carefully. If $P : \mathcal{C}^\infty(\Omega_1) \longrightarrow \mathcal{C}^\infty(\Omega_1)$ is a differential operator of order $m$ with smooth coefficients then, as already noted, so is

$$P_f : \mathcal{C}^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_2), \ P_f v = (f^{-1})^*(P f^* v). \tag{1.15}$$

However, the formula for the coefficients, i.e. the explicit formula for $P_f$, is rather complicated:-

$$P=\sum_{|\alpha|\le m}\Longrightarrow P_f=\sum_{|\alpha|\le m}p_\alpha(g(z'))(J_f(z')D_{z'})^\alpha \tag{1.16}$$

since we have to do some serious differentiation to move all the Jacobian terms to the left.

Even though the formula (1.16) is complicated, the leading part of it is rather simple. Observe that we can compute the leading part of a differential operator by 'oscillatory testing'. Thus, on an open set $\Omega$ consider

(1.17)

$$P(z,D)(e^{it\psi}u)=e^{it\psi}\sum_{k=0}^{m}t^kP_k(z,D)u,\ u\in\mathcal{C}^\infty(\Omega),\ \psi\in\mathcal{C}^\infty(\Omega),\ t\in\mathbb{R}.$$

Here the $P_k(z,D)$ are differential operators of order $m-k$ acting on $u$ (they involve derivatives of $\psi$ of course). Note that the only way a factor of $t$ can occur is from a derivative acting on $e^{it\psi}$ through

$$D_{z_j}e^{it\psi}=e^{it\psi}t\frac{\partial\psi}{\partial z_j}. \tag{1.18}$$

Thus, the coefficient of $t^m$ involves no differentiation of $u$ at all and is therefore multiplication by a smooth function which takes the simple form

$$\sigma_m(P)(\psi,z)=\sum_{|\alpha|=m}p_\alpha(z)(D\psi)^\alpha\in\mathcal{C}^\infty(\Omega). \tag{1.19}$$

In particular, the value of this function at any point $z\in\Omega$ is determined once we know $d\psi$, the differential of $\psi$ at that point. Using this observation, we can easily compute the leading part of $P_f$ given that of $P$ in (1.15). Namely if $\psi\in\mathcal{C}^\infty(\Omega_2)$ and $(P_f)(z')$ is the leading part of $P_f$ for

$$\begin{aligned}\text{(1.20)}\quad \sigma_m(P_f)(\psi',z')v&=\lim_{t\to\infty}t^{-m}e^{-it\psi}P_f(z',D_{z'})(e^{it\psi'}v)\\&=\lim_{t\to\infty}t^{-m}e^{-it\psi}g^*(P(z,D_z)(e^{itf^*\psi'}f^*v)\\&=g^*(\lim_{t\to\infty}t^{-m}e^{-itf^*\psi'}g^*(P(z,D_z)(e^{itf^*\psi'}f^*v)=g^*P_m(f^*\psi,z)f^*v.\end{aligned}$$

Thus

$$\sigma_m(P_f)(\psi',\zeta'))=g^*\sigma_m(P)(f^*\psi',z). \tag{1.21}$$

This allows us to 'geometrize' the transformation law for the leading part (called the principal symbol) of the differential operator $P$. To do so we think of $T^*\Omega$, for $\Omega$ and open subset of $\mathbb{R}^n$, as the union of the $T^*Z\Omega$, $z \in \Omega$, where $T^*_z\Omega$ is the linear space

$$\begin{aligned}(1.22)\quad & T^*_z\Omega = \mathcal{C}^\infty(\Omega)/\sim_z,\ \psi \sim_z \psi' \Longleftrightarrow \\ & \psi(Z) - \psi'(Z) - \psi(z) + \psi'(z) \text{ vanishes to second order at } Z = z.\end{aligned}$$

Essentially by definition of the derivative, for any $\psi \in \mathcal{C}^\infty(\Omega)$,

$$\psi \sim_z \sum_{j=1}^n \frac{\partial \psi}{\partial z}(z)(Z_j - z_j). \tag{1.23}$$

This shows that there is an isomorphism, given by the use of coordinates

$$T^*\Omega \equiv \Omega \times \mathbb{R}^n,\ [z, \psi] \longmapsto (z, d\psi(z)). \tag{1.24}$$

The point of the complicated-looking definition (1.22) is that it shows easily (and I recommend you do it explicitly) that any smooth map $h : \Omega_1 \longrightarrow \Omega_2$ induces a smooth map

$$h^*T^*\Omega_2 \longrightarrow T^*\Omega_1,\ h([h(z), \psi]) = [z, h^*\psi] \tag{1.25}$$

which for a diffeomorphism is an isomorphism.

LEMMA 1.2. *The transformation law* (1.21) *shows that for any element* $P \in \mathrm{Diff}^m(\Omega)$ *the principal symbol is well-defined as an element*

$$\sigma(P) \in \mathcal{C}^\infty(T^*\Omega) \tag{1.26}$$

*which furthermore transforms as a function under the pull-back map* (1.25) *induced by any diffeomorphism of open sets.*

PROOF. The formula (1.19) is consistent with (1.23) and hence with (1.21) in showing that $\sigma_m(P)$ is a well-defined function on $T^*\Omega$. □

## 2. Manifolds

I will only give a rather cursory discussion of manifolds here. The main cases we are interested in are practical ones, the spheres $\mathbb{S}^n$ and the balls $\mathbb{B}^n$. Still, it is obviously worth thinking about the general case, since it is the standard setting for much of modern mathematics. There are in fact several different, but equivalent, definitions of a manifold.

**2.1. Coordinate covers.** Take a Hausdorff topological (in fact metrizable) space $M$. A *coordinate patch* on $M$ is an open set and a homeomorphism

$$M \supset \Omega \xrightarrow{F} \Omega' \subset \mathbb{R}^n$$

onto an open subset of $\mathbb{R}^n$. An atlas on $M$ is a covering by such coordinate patches $(\Omega_a, F_a)$,

$$M = \bigcup_{a \in A} \Omega_a .$$

Since each $F_{ab} : \Omega'_a \to \Omega'_a$ is, by assumption, a homeomorphism, the transition maps

$$\begin{aligned} F_{ab} : \Omega'_{ab} &\to \Omega'_{ba}\,, \\ \Omega'_{ab} &= F_b(\Omega_a \cap \Omega_b)\,, \\ (\Rightarrow \Omega'_{ba} &= F_a(\Omega_a \cap \Omega_b)) \\ F_{ab} &= F_a \circ F_b^{-1} \end{aligned}$$

are also homeomorphisms of open subsets of $\mathbb{R}^n$ (in particular $n$ is constructed on components of $M$). The atlas is $\mathcal{C}^k$, $\mathcal{C}^\infty$, real analytic, etc.) if each $F_{ab}$ is $\mathcal{C}^k$, $\mathcal{C}^\infty$ or real analytic. A $\mathcal{C}^\infty$ ($\mathcal{C}^k$ or whatever) structure on $M$ is usually taken to be a *maximal* $\mathcal{C}^\infty$ atlas (meaning any coordinate patch compatible with all elements of the atlas is already in the atlas).

**2.2. Smooth functions.** A second possible definition is to take again a Hausdorff topological space and a subspace $\mathcal{F} \subset C(M)$ of the continuous real-valued function on $M$ with the following two properties.

1) For each $p \in M \exists f_1, \dots, f_n \in \mathcal{F}$ and an open set $\Omega \ni p$ such that $F = (f_1, \dots, f_n) : \Omega \to \mathbb{R}^n$ is a homeomorphism onto an open set, $\Omega' \subset \mathbb{R}^n$ and $(F^{-1})^* g \in \mathcal{C}^\infty(\Omega')\, \forall g \in \mathcal{F}$.
2) $\mathcal{F}$ is maximal with this property.

**2.3. Embedding.** Alternatively one can simply say that a ($\mathcal{C}^\infty$) manifold is a subset $M \subset \mathbb{R}^N$ such that $\forall p \in M \exists$ an open set $U \ni p$, $U \subset \mathbb{R}^N$, and $h_1, \dots, h_{N-n} \in \mathcal{C}^\infty(U)$ s.t.

$$M \cap U = \{q \in U;\ h_i(q) = 0,\ i = 1, \dots, N-n\}$$
$$dh_i(p) \text{ are linearly independent.}$$

I leave it to you to show that these definitions are equivalent in an *appropriate sense*. If we weaken the various notions of coordinates in each case, for instance in the first case, by requiring that $\Omega' \in$

$\mathbb{R}^{n-k} \times [0,\infty)^k$ for some $k$, with a corresponding version of smoothness, we arrive at the notion of a manifold with cones.[1]

So I will assume that you are reasonably familiar with the notion of a smooth ($\mathcal{C}^\infty$) manifold $M$, equipped with the space $\mathcal{C}^\infty(M)$ — this is just $\mathcal{F}$ in the second definition and in the first

$$\mathcal{C}^\infty(M) = \{u : M \to \mathbb{R};\ u \circ F^{-1} \in \mathcal{C}^\infty(\Omega') \,\forall \text{ coordinate patches}\}.$$

Typically I will not distinguish between complex and real-valued functions unless it seems necessary in this context.

Manifolds are always paracompact — so have countable covers by compact sets — and admit partitions of unity.

Proposition 2.1. *If $M = \bigcup_{a\in A} U_a$ is a cover of a manifold by open sets then there exist $\rho_a \in \mathcal{C}^\infty(M)$ s.t.* $\operatorname{supp}(\rho_a) \Subset U_a$ *(i.e., $\exists K_a \Subset U_a$ s.t. $\rho_a = 0$ on $M\backslash K_a$), these supports are locally finite, so if $K \Subset M$ then*

$$\{a \in A; \rho_a(m) \neq 0 \textit{ for some } m \in K\}$$

*is finite, and finally*

$$\sum_{a\in A} \rho_a(m) = 1,\ \forall\ m \in M.$$

It can also be arranged that

(1) $0 \le \rho_a(m) \le 1\ \forall\ a,\ \forall\ m \in M$.
(2) $\rho_a = \mu_a^2\,,\ \mu_a \in \mathcal{C}^\infty(M)$.
(3) $\exists\ \varphi_a \in \mathcal{C}^\infty(M),\ 0 \le \varphi_a \le 1,\ \varphi = 1$ in a neighborhood of $\operatorname{supp}(\rho_a)$ and the sets $\operatorname{supp}(\varphi_a)$ are locally finite.

Proof. Up to you. □

Using a partition of unity subordinate to a covering by coordinate patches we may transfer definitions from $\mathbb{R}^n$ to $M$, provided they are coordinate-invariant in the first place and preserved by multiplication by smooth functions of compact support. For instance:

Definition 2.2. *If $u : M \longrightarrow \mathbb{C}$ and $s \ge 0$ then $u \in H^s_{loc}(M)$ if for some partition of unity subordinate to a cover of $M$ by coordinate patches*

$$\begin{aligned} &(F_a^{-1})^*(\rho_a u) \in H^s(\mathbb{R}^n) \\ \textit{or } &(F_a^{-1})^*(\rho_a u) \in H^s_{loc}(\Omega'_a). \end{aligned} \tag{2.1}$$

[1] I always demand in addition that the boundary faces of a manifold with cones be a *embedded* but others differ on this. I call the more general object a *tied manifold*.

Note that there are some abuses of notation here. In the first part of (2.1) we use the fact that $(F_a^{-1})^*(\rho_a u)$, defined really on $\Omega'_a$ (the image of the coordinate patch $F_a : \Omega_a \to \Omega'_a \in \mathbb{R}^n$), vanishes outside a compact subset and so can be unambiguously extended as zero outside $\Omega'_a$ to give a function on $\mathbb{R}^n$. The second form of (2.1) is better, but there is an equivalence relation, of equality off sets of measure zero, which is being ignored. The definition doesn't work well for $s < 0$ because $u$ might then not be representable by a function so we don't know what $u'$ is to start with.

The most sysetematic approach is to define distributions on $M$ first, so we know what we are dealing with. However, there is a problem here too, because of the transformation law (1.5) that was forced on us by the local identification $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^{-\infty}(\Omega)$. Namely, we really need *densities* on $M$ before we can define distributions. I will discuss densities properly later; for the moment let me use a little ruse, sticking for simplicity to the compact case.

Definition 2.3. *If $M$ is a compact $\mathcal{C}^\infty$ manifold then $\mathcal{C}^0(M)$ is a Banach space with the supremum norm and a continuous linear functional*

$$\mu : \mathcal{C}^0(M) \longrightarrow \mathbb{R} \tag{2.2}$$

*is said to be a positive smooth measure if for every coordinate patch on $M$, $F : \Omega \longrightarrow \Omega'$ there exists $\mu_F \in \mathcal{C}^\infty(\Omega')$, $\mu_F > 0$, such that*

$$\mu(f) = \int_{\Omega'} (F^{-1})^* f \mu_F dz \ \forall\ f \in \mathcal{C}^0(M) \text{ with } \operatorname{supp}(f) \subset \Omega. \tag{2.3}$$

Now if $\mu$, $\mu' : \mathcal{C}^0(M) \longrightarrow \mathbb{R}$ is two such smooth measures then $\mu'_F = v_F \mu_F$ with $v_F \in \mathcal{C}^\infty(\Omega')$. In fact $\exists\ v \in \mathcal{C}^\infty(M)$, $v > 0$, such that $F^*_{v_F} = v$ on $\Omega$. That is, the $v$'s patch to a well-defined function globally on $M$. To see this, notice that every $g \in \mathcal{C}^0_c(\Omega')$ is of the form $(F^{-1})^* g$ for some $g \in \mathcal{C}^0(M)$ (with support in $\Omega$) so (2.3) certainly *determines* $\mu_F$ on $\Omega'$. Thus, assuming we have two smooth measures, $v_F$ is determined on $\Omega'$ for every coordinate patch. Choose a partition of unity $\rho_a$ and *define*

$$v = \sum_a \rho_a F_a^* v_{F_a} \in \mathcal{C}^\infty(M).$$

**Exercise.** Show (using the transformation of integrals under diffeomorphisms) that

$$\mu'(f) = \mu(vf)\ \forall\ f \in \mathcal{C}^\infty(M). \tag{2.4}$$

Thus we have 'proved' half of

Proposition 2.4. *Any (compact) manifold admits a positive smooth density and any two positive smooth densities are related by* (2.4) *for some (uniquely determined)* $v \in \mathcal{C}^\infty(M)$, $v > 0$.

Proof. I have already unloaded the hard part on you. The extension is similar. Namely, chose a covering of $M$ by coordinate patches and a corresponding partition of unity as above. Then simply *define*

$$\mu(f) = \sum_a \int_{\Omega'_a} (F_a^{-1})^*(\rho_a f) dz$$

using Lebesgue measure in each $\Omega'_a$. The fact that this satisfies (2.3) is similar to the exercise above. □

Now, for a compact manifold, we can define a *smooth positive density* $\mu' \in \mathcal{C}^\infty(M; \Omega)$ as a continuous linear functional of the form

$$\mu' : \mathcal{C}^0(M) \longrightarrow \mathbb{C},\ \mu'(f) = \mu(\varphi f) \text{ for some } \varphi \in \mathcal{C}^\infty(M) \tag{2.5}$$

where $\varphi$ is allowed to be complex-valued. For the moment the notation, $\mathcal{C}^\infty(M; \Omega)$, is not explained. However, the *choice* of a fixed positive $\mathcal{C}^\infty$ measure allows us to identify

$$\mathcal{C}^\infty(M; \Omega) \ni \mu' \longrightarrow \varphi \in \mathcal{C}^\infty(M),$$

meaning that this map is an isomorphism.

Lemma 2.5. *For a compact manifold, $M$, $\mathcal{C}^\infty(M; \Omega)$ is a complete metric space with the norms and distance function*

$$\|\mu'\|_{(k)} = \sup_{|\alpha| \le k} |V_1^{\alpha_1} \cdots V_p^{\alpha_1} \varphi|$$
$$d(\mu'_1, \mu'_2) = \sum_{k=0}^{\infty} 2^{-k} \frac{\|\mu'\|_{(k)}}{1 + \|\mu'\|_{(k)}}$$

*where $\{V_1, \ldots, V_p\}$ is a collection of vector fields spanning the tangent space at each point of $M$.*

This is really a result of about $\mathcal{C}^\infty(M)$ itself. I have put it this way because of the current relevance of $\mathcal{C}^\infty(M; \Omega)$.

Proof. First notice that there are indeed such vector fields on a compact manifold. Simply take a covering by coordinate patches and associated partitions of unity, $\varphi_a$, supported in the coordinate patch $\Omega_a$. Then if $\Psi_a \in \mathcal{C}^\infty(M)$ has support in $\Omega_a$ and $\Psi_a \equiv 1$ in a neighborhood of $\operatorname{supp}(\varphi_a)$ consider

$$V_{a\ell} = \Psi_a (F_a^{-1})_*(\partial_{z_\ell}),\ \ell = 1, \ldots, n,$$

just the coordinate vector fields cut off in $\Omega_a$. Clearly, taken together, these span the tangent space at each point of $M$, i.e., the local coordinate vector fields are really linear combinations of the $V_i$ given by renumbering the $V_{a\ell}$. It follows that

$$\|\mu'\|_{(k)} = \sup_{|\alpha\leq k} |V_1^{\alpha_1} \cdots V_p^{\alpha_p} \varphi| \in M$$

is a norm on $\mathcal{C}^\infty(M;\Omega)$ locally equivalent to the $\mathcal{C}^k$ norm on $\varphi_f$ on compact subsets of coordinate patches. It follows that (2.6) gives a distance function on $\mathcal{C}^\infty(M;\Omega)$ with respect to what is complete — just as for $\mathcal{S}(\mathbb{R}^n)$. $\square$

Thus we can define the space of distributions on $M$ as the space of continuous linear functionals $u \in \mathcal{C}^{-\infty}(M)$

$$u : \mathcal{C}^\infty(M;\Omega) \longrightarrow \mathbb{C},\ |u(\mu)| \leq C_k \|\mu\|_{(k)}. \tag{2.6}$$

As in the Euclidean case smooth, and even locally integrable, functions embed in $\mathcal{C}^{-\infty}(M)$ by integration

$$L^1(M) \hookrightarrow \mathcal{C}^{-\infty}(M),\ f \mapsto f(\mu) = \int_M f\mu \tag{2.7}$$

where the integral is defined unambiguously using a partition of unity subordinate to a coordinate cover:

$$\int_M f\mu = \sum_a \int_{\Omega'_a} (F_a^{-1})^*(\varphi_a f \mu_a) dz$$

since $\mu = \mu_a dz$ in local coordinates.

Definition 2.6. *The Sobolev spaces on a compact manifold are defined by reference to a coordinate case, namely if $u \in \mathcal{C}^{-\infty}(M)$ then*

$$u \in H^s(M) \Leftrightarrow u(\psi\mu) = u_a((F_a^{-1})^*\psi\mu_a),\ \forall\ \psi \in \mathcal{C}_c^\infty(\Omega_a) \text{ with } u_a \in H^s_{loc}(\Omega'_a). \tag{2.8}$$

Here the condition can be the requirement *for all* coordinate systems or for a covering by coordinate systems in view of the coordinate independence of the local Sobolev spaces on $\mathbb{R}^n$, that is the weaker condition implies the stronger.

Now we can transfer the properties of Sobolev for $\mathbb{R}^n$ to a compact manifold; in fact the compactness simplifies the properties

$$H^m(M) \subset H^{m'}(M),\ \forall\ m \geq m' \tag{2.9}$$

$$H^m(M) \hookrightarrow \mathcal{C}^k(M),\ \forall\ m > k + \frac{1}{2}\dim M \tag{2.10}$$

$$\bigcap_m H^m(M) = \mathcal{C}^\infty(M) \tag{2.11}$$

$$\bigcup_m H^m(M) = \mathcal{C}^{-\infty}(M). \tag{2.12}$$

These are indeed Hilbert(able) spaces — meaning they do not have a *natural* choice of Hilbert space structure, but they do have one. For instance

$$\langle u, v\rangle_s = \sum_a \langle (F_a^{-1})^* \varphi_a u, (F_a^{-1})^* \varphi_a v\rangle_{H^s(\mathbb{R}^n)}$$

where $\varphi_a$ is a *square* partition of unity subordinate to coordinate covers.

## 3. Vector bundles

Although it is *not* really the subject of this course, it is important to get used to the coordinate-free language of vector bundles, etc. So I will insert here at least a minimum treatment of bundles, connections and differential operators on manifolds.

CHAPTER 6

# Invertibility of elliptic operators

Next we will use the local elliptic estimates obtained earlier on open sets in $\mathbb{R}^n$ to analyse the global invertibility properties of elliptic operators on compact manifolds. This includes at least a brief discussion of spectral theory in the self-adjoint case.

## 1. Global elliptic estimates

For a single differential operator acting on functions on a compact manifold we now have a relatively simple argument to prove global elliptic estimates.

PROPOSITION 1.1. *If $M$ is a compact manifold and $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is a differential operator with $\mathcal{C}^\infty$ coefficients which is* elliptic *(in the sense that $\sigma_m(P) \neq 0$) on $T^*M\backslash 0$) then for any $s$, $M \in \mathbb{R}$ there exist constants $C_s$, $C'_M$ such that*

$$\begin{gathered} u \in H^M(M),\ Pu \in H^s(M) \Longrightarrow u \in H^{s+m}(M) \\ \|u\|_{s+m} \leq C_s\|Pu\|_s + C'_M\|u\|_M, \end{gathered} \tag{1.1}$$

*where $m$ is the order of $P$.*

PROOF. The regularity result in (1.1) follows directly from our earlier local regularity results. Namely, if $M = \bigcup_a \Omega_a$ is a (finite) covering of $M$ by coordinate patches,

$$F_a : \Omega_a \longrightarrow \Omega'_a \subset \mathbb{R}^n$$

then

$$P_a v = (F_a^{-1})^* P F_a^* v,\ v \in \mathcal{C}_c^\infty(\Omega'_a) \tag{1.2}$$

defines $P_a \in \operatorname{Diff}^m(\Omega'_a)$ which is a differential operator in local coordinates with smooth coefficients; the invariant definition of ellipticity above shows that it is elliptic for each $a$. Thus if $\varphi_a$ is a partition of unity subordinate to the open cover and $\psi_a \in \mathcal{C}_c^\infty(\Omega_a)$ are chosen with $\psi_a = 1$ in a neighbourhood of $\operatorname{supp}(\varphi_a)$ then

$$\|\varphi'_a v\|_{s+m} \leq C_{a,s}\|\psi'_a P_a v\|_s + C'_{a,m}\|\psi'_a v\|_M \tag{1.3}$$

where $\varphi'_a = (F_a^{-1})^*\varphi_a$ and similarly for $\psi'_a(F_a^{-1})^*\varphi_a \in \mathcal{C}^\infty_c(\Omega'_a)$, are the local coordinate representations. We know that (1.3) holds for every $v \in \mathcal{C}^{-\infty}(\Omega'_a)$ such that $P_a v \in H^M_{\text{loc}}(\Omega'_a)$. Applying (1.3) to $(F_a^{-1})^*u = v_a$, for $u \in H^M(M)$, it follows that $Pu \in H^s(M)$ implies $P_a v_a \in H^M_{\text{loc}}(\Omega'_a)$, by coordinate-invariance of the Sobolev spaces and then conversely

$$v_a \in H^{s+m}_{\text{loc}}(\Omega'_a)\ \forall\ a \Longrightarrow u \in H^{s+m}(M).$$

The norm on $H^s(M)$ can be taken to be

$$\|u\|_s = \left(\sum_a \|(F_a^{-1})^*(\varphi_a u)\|_s^2\right)^{1/2}$$

so the estimates in (1.1) also follow from the local estimates:

$$\begin{aligned}\|u\|^2_{s+m} &= \sum_a \|(F_a^{-1})^*(\varphi_a u)\|^2_{s+m}\\ &\le \sum_a C_{a,s}\|\psi'_a P_a (F_a^{-1})^* u\|^2_s\\ &\le C_s\|Pu\|^2_s + C'_M\|u\|^2_M.\end{aligned}$$

$\square$

Thus the elliptic regularity, and estimates, in (1.1) just follow by patching from the local estimates. The same argument applies to elliptic operators on vector bundles, once we prove the corresponding local results. This means going back to the beginning!

As discussed in Section 3, a differential operator between sections of the bundles $E_1$ and $E_2$ is represented in terms of local coordinates and local trivializations of the bundles, by a matrix of differential operators

$$P = \begin{bmatrix} P_{11}(z, D_z) & \cdots & P_{1\ell}(z, D_z)\\ \vdots & & \vdots\\ P_{n1}(z, D_z) & \cdots & P_{n\ell}(z, D_z)\end{bmatrix}.$$

The (usual) order of $P$ is the maximum of the orders of the $P_{ij}(z, D_3)$ and the symbol is just the corresponding matrix of symbols

$$\sigma_m(P)(z,\zeta) = \begin{bmatrix} \sigma_m(P_{11})(z,\zeta) & \cdots & \sigma_m(P_{1\ell})(z,\zeta)\\ \vdots & & \vdots\\ \sigma_m(P_{n1})(z,\zeta) & \cdots & \sigma_m(P_{n\ell})(z,\zeta)\end{bmatrix}. \tag{1.4}$$

Such a $P$ is said to be *elliptic* at $z$ if this matrix is invariable for all $\zeta \neq 0$, $\zeta \in \mathbb{R}^n$. Of course this implies that the matrix is square, so the two vector bundles have the same rank, $\ell$. As a differential operator, $P \in \text{Diff}^m(M, \mathbb{E})$, $\mathbb{E} = E_1, E_2$, is *elliptic* if it is elliptic at each point.

PROPOSITION 1.2. *If $P \in \mathrm{Diff}^m(M, \mathbb{E})$ is a differential operator between sections of vector bundles $(E_1, E_2) = \mathbb{E}$ which is* elliptic *of order $m$ at every point of $M$ then*

$$u \in \mathcal{C}^{-\infty}(M; E_1),\ Pu \in H^s(M, E_) \Longrightarrow u \in H^{s+m}(M; E_1) \tag{1.5}$$

*and for all $s$, $t \in \mathbb{R}$ there exist constants $C = C_s$, $C' = C'_{s,t}$ such that*

$$\|u\|_{s+m} \le C\|Pu\|_s + C'\|u\|_t. \tag{1.6}$$

*Furthermore, there is an operator*

$$Q : \mathcal{C}^\infty(M; E_2) \longrightarrow \mathcal{C}^\infty M; E_1) \tag{1.7}$$

*such that*

$$PQ - \mathrm{Id}_2 = R_2,\ QP - \mathrm{Id}_1 = R_1 \tag{1.8}$$

*are smoothing operators.*

PROOF. As already remarked, we need to go back and carry the discussion through from the beginning for systems. Fortunately this requires little more than notational change.

Starting in the constant coefficient case, we first need to observe that ellipticity of a (square) matrix system is equivalent to the ellipticity of the determinant polynomial

$$D_p(\zeta) = \det \begin{bmatrix} P_{11}(\zeta) & \cdots & P_{1k}(\zeta) \\ \vdots & & \vdots \\ P_{k1}(\zeta) & \cdots & P_{kk}(\zeta) \end{bmatrix} \tag{1.9}$$

which is a polynomial degree $km$. If the $P_i$'s are replaced by their leading parts, of homogeneity $m$, then $D_p$ is replaced by its leading part of degree $km$. From this it is clear that the ellipticity at $P$ is equivalent to the ellipticity at $D_p$. Furthermore the invertibility of matrix in (1.9), under the assumption of ellipticity, follows for $|\zeta| > C$. The inverse can be written

$$P(\zeta)^{-1} = \mathrm{cof}(P(\zeta))/D_p(\zeta).$$

Since the cofactor matrix represents the Fourier transform of a differential operator, applying the earlier discussion to $D_p$ and then composing with this differential operator gives a generalized inverse etc.

For example, if $\Omega \subset \mathbb{R}^n$ is an open set and $D_\Omega$ is the parameterix constructed above for $D_p$ on $\Omega$ then

$$Q_\Omega = \mathrm{cof}(P(D)) \circ D_\Omega$$

is a 2-sided parameterix for the matrix of operators $P$:

$$\begin{aligned} PQ_\Omega - \mathrm{Id}_{k\times k} &= R_R \\ Q_\Omega - \mathrm{Id}_{k\times k} &= R_L \end{aligned} \tag{1.10}$$

where $R_L$, $R_R$ are $k \times k$ matrices of smoothing operators. Similar considerations apply to the variable coefficient case. To construct the global parameterix for an elliptic operator $P$ we proceed as before to piece together the local parameterices $Q_a$ for $P$ with respect to a coordinate patch over which the bundles $E_1, E_2$ are trivial. Then

$$Qf = \sum_a F_a^* \psi_a' Q_a \varphi_a' (F_a)^{-1} f$$

is a global 1-sided parameterix for $P$; here $\varphi_a$ is a partition of unity and $\psi_a \in \mathcal{C}_c^\infty(\Omega_a)$ is equal to 1 in a neighborhood of its support. $\square$

(Probably should be a little more detail.)

## 2. Compact inclusion of Sobolev spaces

For any $R > 0$ consider the Sobolev spaces of elements with compact support in a ball:

$$\dot{H}^s(B) = \{u \in H^s(\mathbb{R}^n); u) = 0 \text{ in } |x| > 1\}. \tag{2.1}$$

Lemma 2.1. *Tthe inclusion map*

$$\dot{H}^s(B) \hookrightarrow \dot{H}^t(B) \text{ is compact if } s > t. \tag{2.2}$$

Proof. Recall that compactness of a linear map between (separable) Hilbert (or Banach) spaces is the condition that the image of any bounded sequence has a convergent subsequence (since we are in separable spaces this is the same as the condition that the image of the unit ball have compact closure). So, consider a bounded sequence $u_n \in \dot{H}^s(B)$. Now $u \in \dot{H}^s(B)$ implies that $u \in H^s(\mathbb{R}^n)$ and that $\phi u = u$ where $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 in a neighbourhood of the unit ball. Thus the Fourier transform satifies

$$\hat{u} = \hat{\phi} * \hat{u} \Longrightarrow \hat{u} \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{2.3}$$

In fact this is true with uniformity. That is, one can bound any derivative of $\hat{u}$ on a compact set by the norm

$$\sup_{|z| \le R} |D_j \hat{u}| + \max_j \sup_{|z| \le R} |D_j \hat{u}| \le C(R) \|u\|_{H^s} \tag{2.4}$$

where the constant does not depend on $u$. By the Ascoli-Arzela theorem, this implies that for each $R$ the sequence $\hat{u}_n$ has a convergent subsequence in $\mathcal{C}(\{|\zeta| \le R\})$. Now, by diagonalization we can extract a subsequence which converges in $\mathcal{V}_c(\{|\zeta| \le R\})$ for every $R$. This implies that the restriction to $\{|\zeta| \le R\}$ converges in the weighted $L^2$ norm corresponding to $H^t$, i.e. that $(1 + |\zeta|^2)^{t/2} \chi_R \hat{u}_{n_j} \to (1 + |\zeta|^2)^{t/2} \chi_R \hat{v}$

in $L^2$ where $\chi_R$ is the characteristic function of the ball of radius $R$. However the boundedness of $u_n$ in $H^s$ strengthens this to

$$(1+|\zeta|^2)^{t/2}\hat{u}_{n_j} \to (1+|\zeta|^2)^{t/2}\hat{v} \text{ in } L^2(\mathbb{R}^n).$$

Namely, the sequence is Cauchy in $L^(\mathbb{R}^n)$ and hence convergnet. To see this, just note that for $\epsilon > 0$ one can first choose $R$ so large that the norm outside the ball is

$$\int_{|\zeta|\ge R}(1+|\zeta|^2)^t|u_n|^2d\zeta \le (1+R^2)^{\frac{s-t}{2}}\int_{|\zeta|\ge R}(1+|\zeta|^2)^s|u_n|^2d\zeta \le C(1+R^2)^{\frac{s-t}{2}} < \epsilon/2 \tag{2.5}$$

where $C$ is the bound on the norm in $H^s$. Now, having chosen $R$, the subsequence converges in $|\zeta| \le R$. This proves the compactness. □

Once we have this local result we easily deduce the global result.

Proposition 2.2. *On a compact manifold the inclusion $H^s(M) \hookrightarrow H^t(M)$, for any $s > t$, is compact.*

Proof. If $\phi_i \in \mathcal{C}_c^\infty(U_i)$ is a partition of unity subordinate to an open cover of $M$ by coordinate patches $g_i : U_i \longrightarrow U_i' \subset \mathbb{R}^n$, then

$$u \in H^s(M) \Longrightarrow (g_i^{-1})^*\phi_i u \in H^s(\mathbb{R}^n),\ \operatorname{supp}((g_i^{-1})^*\phi_i u) \Subset U_i'. \tag{2.6}$$

Thus if $u_n$ is a bounded sequence in $H^s(M)$ then the $(g_i^{-1})^*\phi_i u_n$ form a bounded sequence in $H^s(\mathbb{R}^n)$ with fixed compact supports. It follows from Lemma 2.1 that we may choose a subsequence so that each $\phi_i u_{n_j}$ converges in $H^t(\mathbb{R}^n)$. Hence the subsequence $u_{n_j}$ converges in $H^t(M)$. □

## 3. Elliptic operators are Fredholm

If $V_1, V_2$ are two vector spaces then a linear operator $P : V_1 \to V_2$ is said to be *Fredholm* if these are finite-dimensional subspaces $N_1 \subset V_1$, $N_2 \subset V_2$ such that

$$\begin{gathered}\{v \in V_1;\ Pv = 0\} \subset N_1\\ \{w \in V_2;\ \exists\ v \in V_1,\ Pv = w\} + N_2 = V_2.\end{gathered} \tag{3.1}$$

The first condition just says that the null space is finite-dimensional and the second that the range has a finite-dimensional complement – by shrinking $N_1$ and $N_2$ if necessary we may arrange that the inclusion in (3.1) is an equality and that the sum is direct.

THEOREM 3.1. *For any elliptic operator, $P \in \mathrm{Diff}^m(M;\mathbb{E})$, acting between sections of vector bundles over a compact manifold,*

$$P : H^{s+m}(M;E_1) \longrightarrow H^s(M;E_2)$$
$$\text{and } P : \mathcal{C}^\infty(M;E_1) \longrightarrow \mathcal{C}^\infty(M;E_2)$$

*are Fredholm for all $s \in \mathbb{R}$.*

The result for the $\mathcal{C}^\infty$ spaces follows from the result for Sobolev spaces. To prove this, consider the notion of a Fredholm operator between Hilbert spaces,

$$P : H_1 \longrightarrow H_2. \tag{3.2}$$

In this case we can unwind the conditions (3.1) which are then equivalent to the three conditions

$$\begin{gathered} \mathrm{Nul}(P) \subset H_1 \text{ is finite-dimensional.} \\ \mathrm{Ran}(P) \subset H_2 \text{ is closed.} \\ \mathrm{Ran}(P))^\perp \subset H_2 \text{ is finite-dimensional.} \end{gathered} \tag{3.3}$$

Note that *any* subspace of a Hilbert space with a finite-dimensional complement is closed so (3.3) does follow from (3.1). On the other hand the ortho-complement of a subspace is the same as the ortho-complement of its closure so the first and the third conditions in (3.3) do *not* suffice to prove (3.1), in general. For instance the range of an operator can be dense but not closed.

The main lemma we need, given the global elliptic estimates, is a standard one:-

LEMMA 3.2. *If $R : H \longrightarrow H$ is a compact operator on a Hilbert space then* $\mathrm{Id} - R$ *is Fredholm.*

PROOF. A compact operator is one which maps the unit ball (and hence any bounded subset) of $H$ into a precompact set, a set with compact closure. The unit ball in the null space of $\mathrm{Id} - R$ is

$$\{u \in H;\ \|u\| = 1,\ u = Ru\} \subset R\{u \in H;\ \|u\| = 1\}$$

and is therefore precompact. Since is it closed, it is compact and any Hilbert space with a compact unit ball is finite-dimensional. Thus the null space of $\mathrm{Id} - R$ is finite-dimensional.

Consider a sequence $u_n = v_n - Rv_n$ in the range of $\mathrm{Id} - R$ and suppose $u_n \to u$ in $H$; we need to show that $u$ is in the range of $\mathrm{Id} - R$. We may assume $u \neq 0$, since 0 is in the range, and by passing to a subsequence suppose that $\|u_n\| \neq 0$; $\|u_n\| \to \|u\| \neq 0$ by assumption. Now consider $w_n = v_n/\|v_n\|$. Since $\|u_n\| \neq 0$, $\inf_n \|v_n\| \neq 0$, since other wise there is a subsequence converging to 0, and so $w_n$ is well-defined and of norm 1. Since $w_n = Rw_n + u_n/\|v_n\|$ and $\|v_n\|$ is bounded below, $w_n$ must have a convergence subsequence, by the compactness of $R$. Passing to such a subsequence, and relabelling, $w_n \to w$, $u_n \to u$, $u_n/\|v_n\| \to cu$, $c \in \mathbb{C}$. If $c = 0$ then $(\mathrm{Id} - R)w = 0$. However, we can assume in the first place that $u_n \perp \mathrm{Nul}(\mathrm{Id} - R)$ , so the same is true of $w_n$. As $\|w\| = 1$ this is a contradiction, so $\|v_n\|$ is bounded above, $c \neq 0$, and hence there is a solution to $(\mathrm{Id} - R)w = u$. Thus the range of $\mathrm{Id} - R$ is closed.

The ortho-complement of the range $\mathrm{Ran}(\mathrm{Id} - R)^\perp$ is the null space at $\mathrm{Id} - R^*$ which is also finite-dimensional since $R^*$ is compact. Thus $\mathrm{Id} - R$ is Fredholm. □

Proposition 3.3. *Any smoothing operator on a compact manifold is compact as an operator between (any) Sobolev spaces.*

Proof. By definition a smoothing operator is one with a smooth kernel. For vector bundles this can be expressed in terms of local coordinates and a partition of unity with trivialization of the bundles over the supports as follows.

$$
\begin{gathered}
Ru = \sum_{a,b} \varphi_b R \varphi_a u \\
\varphi_b R \varphi_a u = F_b^* \varphi_b' R_{ab} \varphi_a' (F_a^{-1})^* u \qquad (3.4) \\
R_{ab} v(z) = \int_{\Omega_a'} R_{ab}(z, z') v(z'),\ z \in \Omega_b',\ v \in \mathcal{C}_c^\infty(\Omega_a'; E_1)
\end{gathered}
$$

where $R_{ab}$ is a matrix of smooth sections of the localized (hence trivial by refinement) bundle on $\Omega_b' \times \Omega_a$. In fact, by inserting extra cutoffs in (3.4), we may assume that $R_{ab}$ has compact support in $\Omega_b' \times \Omega_a'$. Thus, by the compactness of sums of compact operators, it suffices to show that a single smoothing operator of compact support compact support is compact on the standard Sobolev spaces. Thus if $R \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}$

$$
H^{L'}(\mathbb{R}^n) \ni u \mapsto \int_{\mathbb{R}^n} R(z) \in H^L(\mathbb{R}^n) \qquad (3.5)
$$

is compact for any $L$, $L'$. By the continuous inclusion of Sobolev spaces it suffices to take $L' = -L$ with $L$ a large even integer. Then $(\Delta + 1)^{L/2}$ is an isomorphism from $(L^2(\mathbb{R}^n))$ to $H^{-L}(\mathbb{R}^2)$ and from $H^L(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Thus the compactness of (3.5) is equivalent to the compactness of

$$
(\Delta + 1)^{L/2} R (\Delta + 1)^{L/2} \text{ on } L^2(\mathbb{R}^n). \qquad (3.6)
$$

This is still a smoothing operator with compactly supported kernel, then we are reduced to the special case of (3.5) for $L = L' = 0$. Finally then it suffices to use Sturm's theorem, that $R$ is uniformly approximated by polynomials on a large ball. Cutting off on left and right then shows that

$$\rho(z)R_i(z,z')\rho(z') \to Rz,z') \text{ uniformly on } \mathbb{R}^{2n}$$

the $R_i$ is a polynomial (and $\rho(z)\rho(z') = 1$ on $\mathrm{supp}(R)$) with $\rho \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. The uniform convergence of the kernels implies the convergence of the operators on $L^2(\mathbb{R}^n)$ in the norm topology, so $R$ is in the norm closure of the finite rank operators on $L^2(\mathbb{R}^n)$, hence is compact. $\square$

Proof of Theorem 3.1. We know that $P$ has a 2-sided parameterix $Q : H^s(M;E_2) \longrightarrow H^{s+m}(M;E_1)$ (for any $s$) such that

$$PQ - \mathrm{Id}_2 = R_2,\ QP - \mathrm{Id}_2 = R_1,$$

are both smoothing (or at *least* $C^N$ for arbitrarily large $N$) operators. Then we can apply Proposition 3.3 and Lemma 3.2. First

$$QP = \mathrm{Id} - R_1 : H^{s+m}(M;E_1) \longrightarrow H^{s+m}(M;E_2)$$

have finite-dimensional null spaces. However, the null space of $P$ is certainly contained in the null space of $\mathrm{Id} - R$, so it too is finite-dimensional. Similarly,

$$PQ = \mathrm{Id} - R_1 : H^s(M;E_2) \longrightarrow H^s(M;E_2)$$

has closed range of finite codimension. But the range of $P$ certainly *contains* the range of $\mathrm{Id} - R$ so it too must be closed and of finite codimension. Thus $P$ is Fredholm as an operator from $H^{s+m}(M;E_2)$ to $H^s(M;E_2)$ for any $s \in \mathbb{R}$.

So consider $P$ as an operator on the $\mathcal{C}^\infty$ spaces. The null space of $P : H^m(M;E_1) \longrightarrow H^0(M;E_2)$ consists of $\mathcal{C}^\infty$ sections, by elliptic regularity, so must be equal to the null space on $\mathcal{C}^\infty(M;E_1)$ — which is therefore finite-dimensional. Similarly consider the range of $P : H^m(M;E_1) \longrightarrow H^0(M;E_2)$. We know this to have a finite-dimensional complement, with basis $v_1, \dots, v_n \in H^0(M;E_2)$. By the density of $\mathcal{C}^\infty(M;E_2)$ in $L^2(M;E_2)$ we can approximate the $v_i$'s closely by $w_i \in \mathcal{C}^\infty(M;E_2)$. On close enough approximation, the $w_i$ must span the complement. Thus $PH^m(M;E_1)$ has a complement in $L^2(M;E_2)$ which is a finite-dimensional subspace of $\mathcal{C}^\infty(M;E_2)$; call this $N_2$. If $f \in \mathcal{C}^\infty(M;E_2) \subset L^2(M;E_2)$ then there are constants $c_i$ such that

$$f - \sum_{i=1}^{N} c_i w_i = Pu,\ u \in H^m(M;E_1).$$

Again by elliptic regularity, $u \in \mathcal{C}^\infty(M; E_1)$ thus $N_2$ is a complement to $P\mathcal{C}^\infty(M; E_1)$ in $\mathcal{C}^\infty(M; E_2)$ and $P$ is Fredholm. $\square$

The point of Fredholm operators is that they are 'almost invertible' — in the sense that they are invertible up to finite-dimensional obstructions. However, a Fredholm operator may not itself be *close* to an invertible operator. This defect is measured by the index

$$\begin{gathered}\operatorname{ind}(P) = \dim \operatorname{Nul}(P) - \dim(\operatorname{Ran}(P)^\perp)\\ P : H^m(M; E_1) \longrightarrow L^2(M; E_2).\end{gathered}$$

## 4. Generalized inverses

Written, at least in part, by Chris Kottke.

As discussed above, a bounded operator between Hilbert spaces,

$$T : H_1 \longrightarrow H_2$$

is Fredholm if and only if it has a parametrix up to compact errors, that is, there exists an operator

$$S : H_2 \longrightarrow H_1$$

such that

$$TS - \mathrm{Id}_2 = R_2,\ ST - \mathrm{Id}_1 = R_1$$

are both compact on the respective Hilbert spaces $H_1$ and $H_2$. In this case of Hilbert spaces there is a "preferred" parametrix or generalized inverse.

Recall that the adjoint

$$T^* : H_2 \longrightarrow H_1$$

of any bounded operator is defined using the Riesz Representation Theorem. Thus, by the continuity of $T$, for any $u \in H_2$,

$$H_1 \ni \phi \longrightarrow \langle T\phi, u\rangle \in \mathbb{C}$$

is continuous and so there exists a unique $v \in H_1$ such that

$$\langle T\phi, u\rangle_2 = \langle \phi, v\rangle_1,\ \forall\ \phi \in H_1.$$

Thus $v$ is determined by $u$ and the resulting map

$$H_2 \ni u \mapsto v = T^*u \in H_1$$

is easily seen to be continuous giving the adjoint identity

$$\langle T\phi, u\rangle = \langle \phi, T^*u\rangle,\ \forall\ \phi \in H_1,\ u \in H_2 \tag{4.1}$$

In particular it is always the case that

$$\operatorname{Nul}(T^*) = (\operatorname{Ran}(T))^\perp \tag{4.2}$$

as follows directly from (4.1). As a useful consequence, if $\operatorname{Ran}(T)$ is closed, then $H_2 = \operatorname{Ran}(T) \oplus \operatorname{Nul}(T^*)$ is an orthogonal direct sum.

PROPOSITION 4.1. *If $T : H_1 \longrightarrow H_2$ is a Fredholm operator between Hilbert spaces then $T^*$ is also Fredholm,* $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$*, and $T$ has a unique generalized inverse $S : H_2 \longrightarrow H_1$ satisfying*

$$TS = \operatorname{Id}_2 - \Pi_{\operatorname{Nul}(P^*)},\ ST = \operatorname{Id}_1 - \Pi_{\operatorname{Nul}(P)} \tag{4.3}$$

PROOF. A straightforward exercise, but it should probably be written out! □

Notice that $\operatorname{ind}(T)$ is the difference of the two non-negative integers $\dim \operatorname{Nul}(T)$ and $\dim \operatorname{Nul}(T^*)$. Thus

$$\dim \operatorname{Nul}(T) \ \geq \ \operatorname{ind}(T) \tag{4.4}$$

$$\dim \operatorname{Nul}(T^*) \ \geq \ -\operatorname{ind}(T) \tag{4.5}$$

so if $\operatorname{ind}(T) \neq 0$ then $T$ is definitely *not* invertible. In fact it cannot then be made invertible by small bounded perturbations.

PROPOSITION 4.2. *If $H_1$ and $H_2$ are two seperable, infinite-dimensional Hilbert spaces then for all $k \in \mathbb{Z}$,*

$$\operatorname{Fr}_k = \{T : H_1 \longrightarrow H_2;\ T \text{ is Fredholm and } \operatorname{ind}(T) = k\}$$

*is a non-empty subset of $B(H_1, H_2)$, the Banach space of bounded operators from $H_1$ to $H_2$.*

PROOF. All separable Hilbert spaces of infinite dimension are isomorphic, so $\operatorname{Fr}_0$ is non-empty. More generally if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $H_1$, then the shift operator, determined by

$$S_k e_i = \begin{cases} e_{i+k}, & i \geq 1,\ k \geq 0 \\ e_{i+k}, & i \geq -k,\ k \leq 0 \\ 0, & i < -k \end{cases}$$

is easily seen to be Fredholm of index $k$ in $H_1$. Composing with an isomorphism to $H_2$ shows that $\operatorname{Fr}_k \neq \emptyset$ for all $k \in \mathbb{Z}$. □

One important property of the spaces $\operatorname{Fr}_k(H_1, H_2)$ is that they are stable under compact perturbations; that is, if $K : H_1 \longrightarrow H_2$ is a compact operator and $T \in \operatorname{Fr}_k$ then $(T + K) \in \operatorname{Fr}_k$. That $(T + K)$ is Fredholm is clear, sinces a parametrix for $T$ is a parametrix for $T + K$, but it remains to show that the index itself is stable and we do this in steps. In what follows, take $T \in \operatorname{Fr}_k(H_1, H_2)$ with kernel $N_1 \subset H_1$. Define $\tilde{T}$ by the factorization

$$T : H_1 \longrightarrow \tilde{H}_1 = H_1/N_1 \xrightarrow{\tilde{T}} \operatorname{Ran} T \hookrightarrow H_2, \tag{4.6}$$

so that $\tilde{T}$ is invertible.

LEMMA 4.3. *Suppose $T \in \mathrm{Fr}_k(H_1, H_2)$ has kernel $N_1 \subset H_1$ and $M_1 \supset N_1$ is a finite dimensional subspace of $H_1$ then defining $T' = T$ on $M_1^\perp$ and $T' = 0$ on $M_1$ gives an element $T' \in \mathrm{Fr}_k$.*

PROOF. Since $N_1 \subset M_1$, $T'$ is obtained from (4.6) by replacing $\tilde{T}$ by $\tilde{T}'$ which is defined in essentially the same way as $T'$, that is $\tilde{T}' = 0$ on $M_1/N_1$, and $\tilde{T}' = \tilde{T}$ on the orthocomplement. Thus the range of $\tilde{T}'$ in $\mathrm{Ran}(T)$ has complement $\tilde{T}(M_1/N_1)$ which has the same dimension as $M_1/N_1$. Thus $T'$ has null space $M_1$ and has range in $H_2$ with complement of dimension that of $M_1/N_1 + N_2$, and hence has index $k$. □

LEMMA 4.4. *If $A$ is a finite rank operator $A : H_1 \longrightarrow H_2$ such that $\mathrm{Ran}\, A \cap \mathrm{Ran}\, T = \{0\}$, then $T + A \in \mathrm{Fr}_k$.*

PROOF. First note that $\mathrm{Nul}(T + A) = \mathrm{Nul}\, T \cap \mathrm{Nul}\, A$ since

$$x \in \mathrm{Nul}(T+A) \Leftrightarrow Tx = -Ax \in \mathrm{Ran}\, T \cap \mathrm{Ran}\, A = \{0\} \Leftrightarrow x \in \mathrm{Nul}\, T \cap \mathrm{Nul}\, A.$$

Similarly the range of $T+A$ restricted to $\mathrm{Nul}\, T$ meets the range of $T+A$ restricted to $(\mathrm{null}\, T)^\perp$ only in 0 so the codimension of the $\mathrm{Ran}(T + A)$ is the codimension of $\mathrm{Ran}\, A_N$ where $A_N$ is $A$ as a map from $\mathrm{Nul}\, T$ to $H_2/\mathrm{Ran}\, T$. So, the equality of row and column rank for matrices,

$$\mathrm{codim}\,\mathrm{Ran}(T+A) = \mathrm{codim}\,\mathrm{Ran}\, T - \dim \mathrm{Nul}(A_N) = \dim \mathrm{Nul}(T) - k - \dim \mathrm{Nul}(A_N) = \dim \mathrm{Nul}(T+$$

Thus $T + A \in \mathrm{Fr}_k$. □

PROPOSITION 4.5. *If $A : H_1 \longrightarrow H_2$ is any finite rank operator, then $T + A \in \mathrm{Fr}_k$.*

PROOF. Let $E_2 = \mathrm{Ran}\, A \cap \mathrm{Ran}\, T$, which is finite dimensional, then $E_1 = \tilde{T}^{-1}(E_2)$ has the same dimension. Put $M_1 = E_1 \oplus N_1$ and apply Lemma 4.3 to get $T' \in \mathrm{Fr}_k$ with kernel $M_1$. Then

$$T + A = T' + A' + A$$

where $A' = T$ on $E_1$ and $A' = 0$ on $E_1^\perp$. Then $A' + A$ is a finite rank operator and $\mathrm{Ran}(A' + A) \cap \mathrm{Ran}\, T' = \{0\}$ and Lemma 4.4 applies. Thus

$$T + A = T' + (A' + A) \in \mathrm{Fr}_k(H_1, H_2).$$

□

PROPOSITION 4.6. *If $B : H_1 \longrightarrow H_2$ is compact then $T + B \in \mathrm{Fr}_k$.*

PROOF. A compact operator is the sum of a finite rank operator and an operator of arbitrarily small norm so it suffices to show that $T + C \in \mathrm{Fr}_k$ where $\|C\| < \epsilon$ for $\epsilon$ small enough and then apply Proposition 4.5. Let $P : H_1 \longrightarrow \tilde{H}_1 = H_1/N_1$ and $Q : H_2 \longrightarrow \operatorname{Ran} T$ be projection operators. Then

$$C = QCP + QC(\mathrm{Id} - P) + (\mathrm{Id} - Q)CP + (\mathrm{Id} - Q)C(\mathrm{Id} - P)$$

the last three of which are finite rank operators. Thus it suffices to show that

$$\tilde{T} + QC : \tilde{H}_1 \longrightarrow \operatorname{Ran} T$$

is invertible. The set of invertible operators is open, by the convergence of the Neumann series so the result follows. □

REMARK 1. In fact the $\mathrm{Fr}_k$ are all *connected* although I will not use this below. In fact this follows from the multiplicativity of the index:-

$$\mathrm{Fr}_k \circ \mathrm{Fr}_l = \mathrm{Fr}_{k+l} \tag{4.7}$$

and the connectedness of the group of invertible operators on a Hilbert space. The topological type of the $\mathrm{Fr}_k$ is actually a point of some importance. A fact, which you should know but I am not going to prove here is:-

THEOREM 4.7. *The open set* $\mathrm{Fr} = \bigcup_k \mathrm{Fr}_k$ *in the Banach space of bounded operators on a separable Hilbert space is a* classifying space *for even K-theory.*

That is, if $X$ is a reasonable space – for instance a compact manifold – then the space of homotopy classes of continuous maps into Fr may be canonically identified as an Abelian group with the (complex) K-theory of $X$ :

$$\mathrm{K}^0(X) = [X; \mathrm{Fr}]. \tag{4.8}$$

## 5. Self-adjoint elliptic operators

Last time I showed that elliptic differential operators, acting on functions on a compact manifold, are Fredholm on Sobolev spaces. Today I will first quickly discuss the rudiments of spectral theory for self-adjoint elliptic operators and then pass over to the general case of operators between sections of vector bundles (which is really only notationally different from the case of operators on functions).

To define self-adjointness of an operator we need to define the adjoint! To do so requires invariant integration. I have already talked about this a little, but recall from 18.155 (I hope) Riesz' theorem identifying (appropriately behaved, i.e. Borel outer continuous and inner regular) measures on a locally compact space with continuous linear functionals on $\mathcal{C}^0_0(M)$ (the space of continuous functions 'vanishing at infinity'). In the case of a manifold we define a smooth positive measure, also called a positive density, as one given in local coordinates by a smooth positive multiple of the Lebesgue measure. The existence of such a density is guaranteed by the existence of a partition of unity subordinate to a coordinate cover, since the we can take

$$\nu = \sum_j \phi_j f_j^* |dz| \tag{5.1}$$

where $|dz|$ is Lebesgue measure in the local coordinate patch corresponding to $f_j : U_j \longrightarrow U_j'$. Since we know that a smooth coordinate transforms $|dz|$ to a positive smooth multiple of the new Lebesque measure (namely the absolute value of the Jacobian) and two such positive smooth measures are related by

$$\nu' = \mu\nu,\ 0 < \mu \in \mathcal{C}^\infty(M). \tag{5.2}$$

In the case of a compact manifold this allows one to define integration of functions and hence an inner product on $L^2(M)$,

$$\langle u, v\rangle_\nu = \int_M u(z)\overline{v(z)}\nu. \tag{5.3}$$

It is with respect to such a choice of smooth density that adjoints are defined.

Lemma 5.1. *If $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is a differential operator with smooth coefficients and $\nu$ is a smooth positive measure then there exists a unque differential operator with smooth coefficients $P^* : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ such that*

$$\langle Pu, v\rangle_\nu = \langle u, P^*v\rangle_\nu\ \forall\ u,\ v \in \mathcal{C}^\infty(M). \tag{5.4}$$

Proof. First existence. If $\phi_i$ is a partition of unity subordinate to an open cover of $M$ by coordinate patches and $\phi_i' \in \mathcal{C}^\infty(M)$ have supports in the same coordinate patches, with $\phi_i' = 1$ in a neighbourhood of $\operatorname{supp}(\phi_i)$ then we know that

$$Pu = \sum_i \phi_i' P \phi_i u = \sum_i f_i^* P_i (f_i^{-1})^* u \tag{5.5}$$

where $f_i : U)i \longrightarrow U_i'$ are the coordinate charts and $P_i$ is a differential operator on $U_i'$ with smooth coefficients, all compactly supported in $U_i'$. The existence of $P^*$ follows from the existence of $(\phi_i' P \phi_i)^*$ and hence

$P_i^*$ in each coordinate patch, where the $P_i^*$ should satisfy

$$\int_{U_i'} (P_i)u'\overline{v'}\mu' dz = \int_{U_i'} u'\overline{P_i^* v'}\mu' dz, \ \forall\ u', v' \in \mathcal{C}^\infty(U_i'). \tag{5.6}$$

Here $\nu = \mu'|dz|$ with $0 < \mu' \in \mathcal{C}^\infty(U_i')$ in the local coordinates. So in fact $P_i^*$ is unique and given by

$$P_i^*(z, D)v' = \sum_{|\alpha|\le m} (\mu')^{-1} D^\alpha \overline{p_\alpha(z)}\mu' v' \text{ if } P_i = \sum_{|\alpha|\le m} p_\alpha(z) D^\alpha. \tag{5.7}$$

The uniqueness of $P^*$ follows from (5.4) since the difference of two would be an operator $Q : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ satisfying

$$\langle u, Qv\rangle_\nu = 0\ \forall\ u,\ v \in \mathcal{C}^\infty(M) \tag{5.8}$$

and this implies that $Q = 0$ as an operator. □

Proposition 5.2. *If $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is an elliptic differential operator of order $m > 0$ which is (formally) self-adjoint with respect to some smooth positive density then*

(5.9)

$$\operatorname{spec}(P) = \{\lambda \in \mathbb{C}; (P-\lambda) : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M) \text{ is not an isomorphism}\}$$

*is a discrete subset of $\mathbb{R}$, for each $\lambda \in \operatorname{spec}(P)$*

$$E(\lambda) = \{u \in \mathcal{C}^\infty(M); Pu = \lambda u\} \tag{5.10}$$

*is finite dimensional and*

$$L^2(M) = \sum_{\lambda\in\operatorname{spec}(P)} E(\lambda) \text{ is orthogonal.} \tag{5.11}$$

Formal self-adjointness just means that $P^* = P$ as differential operators acting on $\mathcal{C}^\infty(M)$. Actual self-adjointness means a little more but this follows easily from formal self-adjointness and ellipticity.

Proof. First notice that $\operatorname{spec}(P) \subset \mathbb{R}$ since if $Pu = \lambda u$ with $u \in \mathcal{C}^\infty(M)$ then

$$\lambda\|u\|\nu^2 = \langle Pu, u\rangle = \langle u, Pu\rangle = \bar\lambda\|u\|\nu^2 \tag{5.12}$$

so $\lambda \notin \mathbb{R}$ implies that the null space of $P - \lambda$ is trivial. Since we know that the range is closed and has complement the null space of $(P - \lambda)^* = P - \bar\lambda$ it follows that $P - \lambda$ is an isomorphism on $\mathcal{C}^\infty(M)$ if $\lambda \notin \mathbb{R}$.

If $\lambda \in \mathbb{R}$ then we also know that $E(\lambda)$ is finite dimensional. For any $\lambda \in \mathbb{R}$ suppose that $(P - \lambda)u = 0$ with $u \in \mathcal{C}^\infty(M)$. Then we know that $P - \lambda$ is an isomorphism from $E(\lambda)^\perp$ to itself which extends by continuity to an isomorphism from the closure of $E^\perp(\lambda)$ in $H^m(M)$ to $E^\perp(\lambda) \subset L^2(M)$. It follows that $P - \lambda'$ defines such an isomorphism for

$|\lambda = l'| < \epsilon$ for some $\epsilon > 0$. However acting on $E(\lambda)$, $P - \lambda' = (\lambda - \lambda')$ is also an isomorphism for $\lambda' \neq \lambda$ so $P - \lambda'$ is an isomorphism. This shows that $E(\lambda') = \{0\}$ for $|\lambda' - \lambda| < \epsilon$.

This leaves the completeness statement, (5.11). In fact this really amounts to the existence of a non-zero eigenvalue as we shall see. Consider the generalized inverse of $P$ acting on $L^2(M)$. It maps the orthocomplement of the null space to itself and is a compact operator, as follows from the a priori estimats for $P$ and the compactness of the embedding of $H^m(M)$ in $L^2(M)$ for $m > 0$. Futhermore it is self-adjoint. A standard result shows that a compact self-adjoint operator either has a non-zero eigenvalue or is itself zero. For the completeness it is enough to show that the generalized inverse maps the orthocomplement of the span of the $E(\lambda)$ in $L^2(M)$ into itself and is compact. It is therefore either zero or has a non-zero eigenvalue. Any corresponding eigenfunction would be an eigenfunction of $P$ and hence in one of the $E(\lambda)$ so this operator must be zero, meaning that (5.11) holds. □

For single differential operators we first considered constant coefficient operators, then extended this to variable coefficient operators by a combination of perturbation (to get the a priori estimates) and construction of parametrices (to get approximation) and finally used coordinate invariance to transfer the discussion to a (compact) manifold. If we consider matrices of operators we can follow the same path, so I shall only comment on the changes needed.

A $k \times l$ matrix of differential operators (so with $k$ rows and $l$ columns) maps $l$-vectors of smooth functions to $k$ vectors:

$$P_{ij}(D) = \sum_{|\alpha| \leq m} c_{\alpha,i,j} D^\alpha, \ (P(D)u)_i(z) = \sum_j P_{ij}(D) u_j(z). \tag{5.13}$$

The matrix $P_{ij}(\zeta)$ is invertible if and only if $k = l$ and the polynomial of order $mk$, $\det P(\zeta) \neq 0$. Such a matrix is said to be elliptic if $\det P(\zeta)$ is elliptic. The cofactor matrix defines a matrix $P'$ of differential operators of order $(k-1)m$ and we may construct a parametrix for $P$ (assuming it to be elliptic) from a parametrix for $\det P$ :

$$Q_P = Q_{\det P} P'(D). \tag{5.14}$$

It is then easy to see that it has the same mapping properties as in the case of a single operator (although notice that the product is no longer commutative because of the non-commutativity of matrix multiplication)

$$Q_P P = \operatorname{Id} - R_L, \ P Q_P = \operatorname{Id} - R_R \tag{5.15}$$

where $R_L$ and $R_R$ are given by matrices of convolution operators with all elements being Schwartz functions. For the action on vector-valued functions on an open subset of $\mathbb{R}^n$ we may proceed exactly as before, cutting off the kernel of $Q_P$ with a properly supported function which is 1 near the diagonal

$$Q_\Omega f(z) = \int_\Omega q(z-z')\chi(z,z')f(z')dz'. \tag{5.16}$$

The regularity estimates look exactly the same as before if we define the local Sobolev spaces to be simply the direct sum of $k$ copies of the usual local Sobolev spaces

(5.17)
$$Pu = f \in H^s_{\text{loc}}(\Omega) \Longrightarrow \|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{m-1} \text{ or } \|\psi u\|_{s+m} \le C\|\phi P(D)u\|_s + C''|$$

where $\psi$, $\phi \in \mathcal{C}^\infty_c(\Omega)$ and $\phi = 1$ in a neighbourhood of $\psi$ (and in the second case $C''$ depends on $M$.

Now, the variable case proceed again as before, where now we are considering a $k \times k$ matrix of differential operators of order $m$. I will not go into the details. A priori estimates in the first form in (5.17), for functions $\psi$ with small support near a point, follow by perturbation from the constant coefficient case and then in the second form by use of a partition of unity. The existence of a parametrix for the variable coefficient matrix of operators also goes through without problems – the commutativity which disappears in the matrix case was not used anyway.

As regards coordinate transformations, we get the same results as before. It is also notural to allow transformations by variable coefficient matrices. Thus if $G_i(z) \in \mathcal{C}^\infty(\Omega; \mathrm{GL}(k,\mathbb{C})$ $i = 1, 2$, are smooth family of invertible matrices we may consider the composites $PG_2$ or $G_1^{-1}P$, or more usually the 'conjugate' operator

$$G_1^{-1}P(z,D)G)2 = P'(z,D). \tag{5.18}$$

This is also a variable coefficient differential operator, elliptic if and only if $P(z,D)$ is elliptic. The Sobolev spaces $H^s_{\text{loc}}(\Omega;\mathbb{R}^k)$ are invariant under composition with such matrices, since they are the same in each variable.

Combining coordinate transformations and such matrix conjugation allows us to consider not only manifolds but also vector bundles over manifolds. Let me briefly remind you of what this is about. Over an open subset $\Omega \subset \mathbb{R}^n$ one can introduce a vector bundle as just a subbundle of some trivial $N$-dimensional bundle. That is, consider a smooth $N \times N$ matrix $\Pi \in \mathcal{C}^\infty(\Omega; M(N,\mathbb{C}))$ on $\Omega$ which is valued in the projections (i.e. idempotents) meaning that $\Pi(z)\Pi(z) = \Pi(z)$ for all $z \in \Omega$. Then the range of $\Pi(z)$ defines a linear subspace of $\mathbb{C}^N$ for each $z \in \Omega$ and together these form a vector bundle over $\Omega$. Namely these spaces fit together to define a manifold of dimension $n+k$ where $k$ is the rank of $\Pi(z)$ (constant if $\Omega$ is connected, otherwise require it be the same on all components)

$$E_\Omega = \bigcup_{z\in\Omega} E_z,\ E_z = \Pi(z)\mathbb{C}^N. \tag{5.19}$$

If $\bar{z} \in \Omega$ then we may choose a basis of $E_{\bar{z}}$ and so identify it with $\mathbb{C}^k$. By the smoothness of $\Pi(z)$ in $z$ it follows that in some small ball $B(\bar{z}, r)$, so that $\|\Pi(z)(\Pi(z) - \Pi(\bar{z}))\Pi(z)\| < \frac{1}{2}$) the map

(5.20)

$$E_{B(\bar{z},r)} = \bigcup_{z\in B(\bar{z},r)} E_z,\ E_z = \Pi(z)\mathbb{C}^N \ni (z,u) \longmapsto (z, E(\bar{z})u) \in B(\bar{z},r)\times E_{\bar{z}} \simeq B(\bar{z},r)\times\mathbb{C}^k$$

is an isomorphism. Injectivity is just injectivity of each of the maps $E_z \longrightarrow E_{\bar{z}}$ and this follows from the fact that $\Pi(z)\Pi(\bar{z})\Pi(z)$ is invertible on $E_z$; this also implies surjectivity.

## 6. Index theorem

## Addenda to Chapter 6

CHAPTER 7

# Suspended families and the resolvent

For a compact manifold, $M$, the Sobolev spaces $H^s(M;E)$ (of sections of a vector bundle $E$) are defined above by reference to local coordinates and local trivializations of $E$. If $M$ is not compact (but is paracompact, as is demanded by the definition of a manifold) the same sort of definition leads either to the spaces of sections with compact support, or the "local" spaces:

$$H^s_{\mathrm{c}}(M;E) \subset H^s_{\mathrm{loc}}(M;E),\ s \in \mathbb{R}. \tag{0.1}$$

Thus, if $F_a : \Omega_a \to \Omega'_a$ is a covering of $M$, for $a \in A$, by coordinate patches over which $E$ is trivial, $T_a : (F_a^{-1})^*E \cong \mathbb{C}^N$, and $\{\rho_a\}$ is a partition of unity subordinate to this cover then

$$\mu \in H^s_{\mathrm{loc}}(M;E) \Leftrightarrow T_a(F_a^{-1})^*(\rho_a\mu) \in H^s(\Omega'_a;\mathbb{C}^N)\ \forall\ a. \tag{0.2}$$

Practically, these spaces have serious limitations; for instance they are not Hilbert or even Banach spaaces. On the other hand they certainly have their uses and differential operators act on them in the usual way,

$$\begin{aligned} &P \in \mathrm{Diff}^m(M;\mathbb{E}) \Rightarrow \\ &\qquad P : H^{s+m}_{\mathrm{loc}}(M;E_+) \to H^s_{\mathrm{loc}}(M;E_-),\\ &\qquad P : H^{s+m}_{\mathrm{c}}(M;E_+) \to H^s_{\mathrm{c}}(M;E_-). \end{aligned} \tag{0.3}$$

However, without some limitations on the growth of elements, as is the case in $H^s_{\mathrm{loc}}(M;E)$, it is not reasonable to expect the null space of the first realization of $P$ above to be finite dimensional. Similarly in the second case it is not reasonable to expect the operator to be even close to surjective.

## 1. Product with a line

Some corrections from Fang Wang added, 25 July, 2007.

Thus, for non-compact manifolds, we need to find intermediate spaces which represent some growth constraints on functions or distributions. Of course this is precisely what we have done for $\mathbb{R}^n$ in defining the weighted Sobolev spaces,

$$H^{s,t}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n); \langle z\rangle^{-t}u \in H^s(\mathbb{R}^n)\right\}. \tag{1.1}$$

However, it turns out that even these spaces are not always what we want.

To lead up to the discussion of other spaces I will start with the simplest sort of non-compact space, the real line. To make things more interesting (and useful) I will conisider

$$X = \mathbb{R} \times M \tag{1.2}$$

where $M$ is a compact manifold. The new Sobolev spaces defined for this product will combine the features of $H^s(\mathbb{R})$ and $H^s(M)$. The Sobolev spaces on $\mathbb{R}^n$ are associated with the translation action of $\mathbb{R}^n$ on itself, in the sense that this fixes the "uniformity" at infinity through the Fourier transform. What happens on $X$ is quite similar.

First we can define "tempered distributions" on $X$. The space of Schwartz functions of rapid decay on $X$ can be fixed in terms of differential operators on $M$ and differentiation on $\mathbb{R}$.

(1.3)
$$\mathcal{S}(\mathbb{R}\times M) = \left\{u : \mathbb{R}\times M \to \mathbb{C};\ \sup_{\mathbb{R}\times M}\left|t^l D_t^k Pu(t,\cdot)\right| < \infty\ \forall\ l,\ k,\ P \in \operatorname{Diff}^*(M)\right\}.$$

Exercise 1. Define the corresponding space for sections of a vector bundle $E$ over $M$ lifted to $X$ and then put a topology on $\mathcal{S}(\mathbb{R}\times M; E)$ corresponding to these estimates and check that it is a complete metric space, just like $\mathcal{S}(\mathbb{R})$ in Chapter 3.

There are several different ways to look at

$$\mathcal{S}(\mathbb{R}\times M) \subset \mathcal{C}^\infty(\mathbb{R}\times M).$$

Namely we can think of either $\mathbb{R}$ or $M$ as "coming first" and see that

$$\mathcal{S}(\mathbb{R}\times M) = \mathcal{C}^\infty(M; \mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R}; \mathcal{C}^\infty(M)). \tag{1.4}$$

The notion of a $\mathcal{C}^\infty$ function on $M$ with values in a topological vector space is easy to define, since $\mathcal{C}^0(M; \mathcal{S}(\mathbb{R}))$ is defined using the metric space topology on $\mathcal{S}(\mathbb{R})$. In a coordinate patch on $M$ higher derivatives are defined in the usual way, using difference quotients and these definitions are coordinate-invariant. Similarly, continuity and differentiability for a map $\mathbb{R} \to \mathcal{C}^\infty(M)$ are easy to define and then

(1.5)
$$\mathcal{S}(\mathbb{R}; \mathcal{C}^\infty(M)) = \left\{u : \mathbb{R} \to \mathcal{C}^\infty(M);\ \sup_t \left\|t^k D_t^p u\right\|_{\mathcal{C}^l(M)} < \infty,\ \forall\ k,\ p,\ l\right\}.$$

Using such an interpretation of $\mathcal{S}(\mathbb{R}\times M)$, or directly, it follows easily that the 1-dimensional Fourier transform gives an isomorphism $\mathcal{F}:\mathcal{S}(\mathbb{R}\times M)\to\mathcal{S}(\mathbb{R}\times M)$ by

$$\mathcal{F}:u(t,\cdot)\longmapsto\hat{u}(\tau,\cdot)=\int_{\mathbb{R}}e^{-it\tau}u(t,\cdot)\,dt. \tag{1.6}$$

So, one might hope to use $\mathcal{F}$ to define Sobolev spaces on $\mathbb{R}\times M$ with uniform behavior as $t\to\infty$ in $\mathbb{R}$. However this is not so straightforward, although I will come back to it, since the 1-dimensional Fourier transform in (1.6) does *nothing* in the variables in $M$. Instead let us think about $L^2(\mathbb{R}\times M)$, the definition of which requires a choice of measure.

Of course there is an obvious class of product measures on $\mathbb{R}\times M$, namely $dt\cdot\nu_M$, where $\nu_M$ is a positive smooth density on $M$ and $dt$ is Lebesgue measure on $\mathbb{R}$. This corresponds to the functional

$$\int:\mathcal{C}^0_{\mathrm{c}}(\mathbb{R}\times M)\ni u\longmapsto\int u(t,\cdot)\,dt\cdot\nu\in\mathbb{C}. \tag{1.7}$$

The analogues of (1.4) correspond to Fubini's Theorem.

$$\begin{gathered}L^2_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{u:\mathbb{R}\times M\to\mathbb{C}\text{ measurable};\ \int|u(t,z)|^2\,dt\,\nu_z<\infty\right\}/\sim\ \text{a.e.}\\ L^2_{\mathrm{ti}}(\mathbb{R}\times M)=L^2(\mathbb{R};L^2(M))=L^2(M;L^2(\mathbb{R})).\end{gathered} \tag{1.8}$$

Here the subscript "ti" is supposed to denote translation-invariance (of the measure and hence the space).

We can now easily define the Sobolev spaces of positive integer order:

$$\begin{gathered}H^m_{\mathrm{ti}}(\mathbb{R}\times M)=\Big\{u\in L^2_{\mathrm{ti}}(\mathbb{R}\times M);\\ D^j_tP_ku\in L^2_{\mathrm{ti}}(\mathbb{R}\times M)\ \forall\,j\le m-k,\ 0\le k\le m,\ P_k\in\mathrm{Diff}^k(M)\Big\}.\end{gathered} \tag{1.9}$$

In fact we can write them more succinctly by defining

$$\mathrm{Diff}^k_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{Q\in\mathrm{Diff}^m(\mathbb{R}\times M);\ Q=\sum_{0\le j\le m}D^j_tP_j,\ P_j\in\mathrm{Diff}^{m-j}(M)\right\}. \tag{1.10}$$

This is the space of "$t$-translation-invariant" differential operators on $\mathbb{R}\times M$ and (1.9) reduces to

$$H^m_{\mathrm{ti}}(\mathbb{R}\times M)=\left\{u\in L^2_{\mathrm{ti}}(\mathbb{R}\times M);\ Pu\in L^2_{\mathrm{ti}}(\mathbb{R}\times M),\ \forall\ P\in\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M)\right\}. \tag{1.11}$$

I will discuss such operators in some detail below, especially the elliptic case. First, we need to consider the Sobolev spaces of non-integral order, for completeness sake if nothing else. To do this, observe that on $\mathbb{R}$ itself (so for $M = \{\text{pt}\}$), $L^2_{\text{ti}}(\mathbb{R} \times \{\text{pt}\}) = L^2(\mathbb{R})$ in the usual sense. Let us consider a special partition of unity on $\mathbb{R}$ consisting of integral translates of *one* function.

DEFINITION 1.1. *An element $\mu \in \mathcal{C}^\infty_c(\mathbb{R})$ generates a "***ti-partition of unity***" (a non-standard description) on $\mathbb{R}$ if $0 \le \mu \le 1$ and $\sum_{k\in\mathbb{Z}} \mu(t-k) = 1$.*

It is easy to construct such a $\mu$. Just take $\mu_1 \in \mathcal{C}^\infty_c(\mathbb{R})$, $\mu_1 \ge 0$ with $\mu_1(t) = 1$ in $|t| \le 1/2$. Then let

$$F(t) = \sum_{k\in\mathbb{Z}} \mu_1(t-k) \in \mathcal{C}^\infty(\mathbb{R})$$

since the sum is finite on each bounded set. Moreover $F(t) \ge 1$ and is itself invariant under translation by any integer; set $\mu(t) = \mu_1(t)/F(t)$. Then $\mu$ generates a ti-partition of unity.

Using such a function we can easily decompose $L^2(\mathbb{R})$. Thus, setting $\tau_k(t) = t - k$,

(1.12)

$$f \in L^2(\mathbb{R}) \iff (\tau_k^* f)\mu \in L^2_{\text{loc}}(\mathbb{R}) \;\forall\, k \in \mathbb{Z} \text{ and } \sum_{k\in\mathbb{Z}} \int |\tau_k^* f\mu|^2 \, dt < \infty.$$

Of course, saying $(\tau_k^* f)\mu \in L^2_{\text{loc}}(\mathbb{R})$ is the same as $(\tau_k^* f)\mu \in L^2_{\text{c}}(\mathbb{R})$. Certainly, if $f \in L^2(\mathbb{R})$ then $(\tau_k^* f)\mu \in L^2(\mathbb{R})$ and since $0 \le \mu \le 1$ and $\text{supp}(\mu) \subset [-R, R]$ for some $R$,

$$\sum_k \int |(\tau_k^* f)\mu|^2 \le C \int |f|^2 \, dt.$$

Conversely, since $\sum_{|k|\le T} \mu = 1$ on $[-1, 1]$ for some $T$, it follows that

$$\int |f|^2 \, dt \le C' \sum_k \int |(\tau_k^* f)\mu|^2 \, dt.$$

Now, $D_t \tau_k^* f = \tau_k^*(D_t f)$, so we can use (1.12) to rewrite the definition of the spaces $H^k_{\text{ti}}(\mathbb{R} \times M)$ in a form that extends to *all* orders. Namely

(1.13)

$$u \in H^s_{\text{ti}}(\mathbb{R} \times M) \iff (\tau_k^* u)\mu \in H^s_{\text{c}}(\mathbb{R} \times M) \text{ and } \sum_k \|\tau_k^* u\|_{H^s} < \infty$$

provided we choose a fixed norm on $H^s_{\text{c}}(\mathbb{R} \times M)$ giving the usual topology for functions supported in a fixed compact set, for example by embedding $[-T, T]$ in a torus $\mathbb{T}$ and then taking the norm on $H^s(\mathbb{T} \times M)$.

LEMMA 1.2. *With* $\mathrm{Diff}^m_{ti}(\mathbb{R}\times M)$ *defined by* (1.10) *and the translation-invariant Sobolev spaces by* (1.13),

$$P \in \mathrm{Diff}^m_{ti}(\mathbb{R}\times M) \Longrightarrow \\ P : H^{s+m}_{ti}(\mathbb{R}\times M) \longrightarrow H^s_{ti}(\mathbb{R}\times M)\ \forall\, s \in \mathbb{R}. \tag{1.14}$$

PROOF. This is basically an exercise. Really we also need to check a little more carefully that the two definitions of $H^(_{\mathrm{ti}}\mathbb{R}\times M)$ for $k$ a positive integer, are the same. In fact this is similar to the proof of (1.14) so is omitted. So, to prove (1.14) we will proceed by induction over $m$. For $m = 0$ there is nothing to prove. Now observe that the translation-invariant of $P$ means that $P\tau_k^* u = \tau_k^*(Pu)$ so

$$u \in H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M) \Longrightarrow \\ P(\tau_k^* u\mu) = \tau_k^*(Pu) + \sum_{m'<m} \tau_k^*(P_{m'}u)D_t^{m-m'}\mu,\ P_{m'} \in \mathrm{Diff}^{m'}_{\mathrm{ti}}(\mathbb{R}\times M). \tag{1.15}$$

The left side is in $H^s_{\mathrm{ti}}(\mathbb{R}\times M)$, with the sum over $k$ of the squares of the norms bounded, by the regularity of $u$. The same is easily seen to be true for the sum on the right by the inductive hypothesis, and hence for the first term on the right. This proves the mapping property (1.14) and continuity follows by the same argument or the closed graph theorem. □

We can, and shall, extend this in various ways. If $\mathbb{E} = (E_1, E_2)$ is a pair of vector bundles over $M$ then it lifts to a pair of vector bundles over $\mathbb{R}\times M$, which we can again denote by $\mathbb{E}$. It is then straightforward to define $\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M);\mathbb{E})$ and the Sobolev spaces $H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_i)$ and to check that (1.14) extends in the obvious way.

Then main question we want to understand is the *invertibility* of an operator such as $P$ in (1.14). However, let me look first at these Sobolev spaces a little more carefully. As already noted we really have two definitions in the case of positive integral order. Thinking about these we can also make the following provisional definitions in terms of the 1-dimensional Fourier transform discussed above – where the '$\tilde{H}$' notation is only temporary since these will turn out to be the same as the spaces just considered.

For any compact manifold define

$$
\begin{aligned}
&\tilde{H}^s_{\mathrm{ti}}(\mathbb{R}\times M) = \big\{u\in L^2(\mathbb{R}\times M);\\
&\qquad \|u\|_s^2 = \int_{\mathbb{R}}\left(\langle\tau\rangle^s|\hat{u}(\tau,\cdot)|^2_{L^2(M)} + \int_{\mathbb{R}}|\hat{u}(\tau,\cdot)|^2_{H^s(M)}\right)d\tau<\infty\big\},\ s\ge 0
\end{aligned}\tag{1.16}
$$

$$
\begin{aligned}
&\tilde{H}^s_{\mathrm{ti}}(\mathbb{R}\times M) = \big\{u\in\mathcal{S}'(\mathbb{R}\times M); u=u_1+u_2,\\
&\quad u_1\in L^2(\mathbb{R};H^s(M)),\ u_2\in L^2(M;H^s(\mathbb{R}))\big\},\ \|u\|_s^2=\inf\|u_1\|^2+\|u_2\|^2,\ s<0.
\end{aligned}\tag{1.17}
$$

The following interpolation result for Sobolev norms on $M$ should be back in Chapter 5.

Lemma 1.3. *If $M$ is a compact manifold or $\mathbb{R}^n$ then for any $m_1\ge m_2\ge m_3$ and any $R$, the Sobolev norms are related by*

$$
\|u\|_{m_2}\le C\left((1+R)^{m_2-m_1}\|u\|_{m_1}+(1+R)^{m_2-m_3}\|u\|_{m_3}\right).\tag{1.18}
$$

Proof. On $\mathbb{R}^n$ this follows directly by dividing Fourier space in two pieces

$$
\begin{aligned}
\|u\|^2_{m_2} &= \int_{|\zeta|>R}\langle\zeta\rangle^{2m_2}|\hat{u}|d\zeta+\int_{|\zeta|\le R}\langle\zeta\rangle^{2m_2}|\hat{u}|d\zeta\\
&\le\langle R\rangle^{2(m_1-m_2)}\int_{|\zeta|>R}\langle\zeta\rangle^{2m_1}|\hat{u}|d\zeta+\langle R\rangle^{2(m_2-m_3)}\int_{|\zeta|\le R}\langle\zeta\rangle^{2m_3}|\hat{u}|d\zeta\\
&\le\langle R\rangle^{2(m_1-m_2)}\|u\|^2_{m_1}+\langle R\rangle^{2(m_2-m_3)}\|u\|^2_{m_3}.
\end{aligned}\tag{1.19}
$$

On a compact manifold we have defined the norms by using a partition $\phi_i$ of unity subordinate to a covering by coordinate patches $F_i:Y_i\longrightarrow U_i'$ :

$$
\|u\|^2_m=\sum_i\|(F_i)^*(\phi_iu)\|^2_m\tag{1.20}
$$

where on the right we are using the Sobolev norms on $\mathbb{R}^n$. Thus, applying the estimates for Euclidean space to each term on the right we get the same estimate on any compact manifold. □

Corollary 1.4. *If $u\in\tilde{H}^s_{ti}(\mathbb{R}\times M)$, for $s>0$, then for any $0<t<s$*

$$
\int_{\mathbb{R}}\langle\tau\rangle^{2t}\|\hat{u}(\tau,\cdot)\|^2_{H^{s-t}(M)}d\tau<\infty\tag{1.21}
$$

*which we can interpret as meaning '$u\in H^t(\mathbb{R};H^{s-t}(M))$ or $u\in H^{s-t}(M;H^s(\mathbb{R}))$.'*

PROOF. Apply the estimate to $\hat{u}(\tau,\cdot) \in H^s(M)$, with $R = |\tau|$, $m_1 = s$ and $m_3 = 0$ and integrate over $\tau$. $\square$

LEMMA 1.5. *The Sobolev spaces* $\tilde{H}^s_{ti}(\mathbb{R} \times M)$ *and* $H^s_{ti}(\mathbb{R} \times M)$ *are the same.*

PROOF. $\square$

LEMMA 1.6. *For* $0 < s < 1$ $u \in H^s_{ti}(\mathbb{R} \times M)$ *if and only if* $u \in L^2(\mathbb{R} \times M)$ *and*

(1.22)
$$\int_{\mathbb{R}^2\times M} \frac{|u(t,z)-u(t',z)|^2}{|t-t'|^{2s+1}} dt dt'\nu + \int_{\mathbb{R}\times M^2} \frac{|u(t,z')-u(t,z)|^2}{\rho(z,z')^{s+\frac{n}{2}}} dt\nu(z)\nu(z') < \infty,$$
$$n = \dim M,$$

*where* $0 \le \rho \in \mathcal{C}^\infty(M^2)$ *vanishes exactly quadratically at* $\mathrm{Diag} \subset M^2$.

PROOF. This follows as in the cases of $\mathbb{R}^n$ and a compact manifold discussed earlier since the second term in (1.22) gives (with the $L^2$ norm) a norm on $L^2(\mathbb{R}; H^s(M))$ and the first term gives a norm on $L^2(M; H^s(\mathbb{R}))$. $\square$

Using these results we can see directly that the Sobolev spaces in (1.16) have the following 'obvious' property as in the cases of $\mathbb{R}^n$ and $M$.

LEMMA 1.7. *Schwartz space* $\mathcal{S}(\mathbb{R} \times M) = \mathcal{C}^\infty(M; \mathcal{S}(\mathbb{R}))$ *is dense in each* $H^s_{ti}(\mathbb{R} \times M)$ *and the* $L^2$ *pairing extends by continuity to a jointly continuous non-degenerate pairing*

$$H^s_{ti}(\mathbb{R} \times M) \times H^{-s}_{ti}(\mathbb{R} \times M) \longrightarrow \mathbb{C} \tag{1.23}$$

*which identifies* $H^{-s}_{ti}(\mathbb{R}\times M)$ *with the dual of* $H^s_{ti}(\mathbb{R}\times M)$ *for any* $s \in \mathbb{R}$.

PROOF. I leave the density as an exercise – use convolution in $\mathbb{R}$ and the density of $\mathcal{C}^\infty(M)$ in $H^s(M)$ (explicity, using a partition of unity on $M$ and convolution on $\mathbb{R}^n$ to get density in each coordinate patch).

Then the existence and continuity of the pairing follows from the definitions and the corresponding pairings on $\mathbb{R}$ and $M$. We can assume that $s > 0$ in (1.23) (otherwise reverse the factors). Then if $u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M)$ and $v = v_1 + v_2 \in H^{-s}_{\mathrm{ti}}(\mathbb{R} \times M)$ as in (1.17),

$$(u,v) = \int_{\mathbb{R}} (u(t,\cdot), u_1(t,\cdot))\, dt + \int_M (u(\cdot,z), v_2(\cdot,z))\, \nu_z \tag{1.24}$$

where the first pairing is the extension of the $L^2$ pairing to $H^s(M) \times H^{-s}(M)$ and in the second case to $H^s(\mathbb{R}) \times H^{-s}(\mathbb{R})$. The continuity of the pairing follows directly from (1.24).

So, it remains only to show that the pairing is non-degenerate – so that

$$(1.25) \qquad H_{\mathrm{ti}}^{-s}(\mathbb{R} \times M) \ni v \longmapsto \sup_{\|u\|_{H^s_{\mathrm{ti}}(\mathbb{R}\times M)}=1} |(u,v)|$$

is equivalent to the norm on $H_{\mathrm{ti}}^{-s}(\mathbb{R} \times M)$. We already know that this is bounded above by a multiple of the norm on $H_{\mathrm{ti}}^{-s}$ so we need the estimate the other way. To see this we just need to go back to Euclidean space. Take a partition of unity $\psi_i$ with our usual $\phi_i$ on $M$ subordinate to a coordinate cover and consider with $\phi_i = 1$ in a neighbourhood of the support of $\psi_i$. Then

$$(1.26) \qquad (u, \psi_i v) = (\phi_i u, \psi_i v)$$

allows us to extend $\psi_i v$ to a continuous linear functional on $H^s(\mathbb{R}^n)$ by reference to the local coordinates and using the fact that for $s > 0$ $(F_i^{-1})^*(\phi_i u) \in H^s(\mathbb{R}^{n+1})$. This shows that the coordinate representative of $\psi_i v$ is a sum as desired and summing over $i$ gives the desired bound. □

## 2. Translation-invariant Operators

Some corrections from Fang Wang added, 25 July, 2007.

Next I will characterize those operators $P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E})$ which give invertible maps (1.14), or rather in the case of a pair of vector bundles $\mathbb{E} = (E_1, E_2)$ over $M$ :

$$(2.1) \quad P : H^{s+m}_{\mathrm{ti}}(\mathbb{R}\times M; E_1) \longrightarrow H^s_{\mathrm{ti}}(\mathbb{R}\times M; E_2),\ P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E}).$$

This is a generalization of the 1-dimensional case, $M = \{\mathrm{pt}\}$ which we have already discussed. In fact it will become clear how to generalize some parts of the discussion below to products $\mathbb{R}^n \times M$ as well, but the case of a 1-dimensional Euclidean factor is both easier and more fundamental.

As with the constant coefficient case, there is a basic dichotomy here. A $t$-translation-invariant differential operator as in (2.1) is Fredholm if and only if it is invertible. To find necessary and sufficient conditons for invertibility we will we use the 1-dimensional Fourier transform as in (1.6).

If

$$(2.2) \quad P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M);\mathbb{E}) \Longleftrightarrow P = \sum_{i=0}^m D_t^i P_i,\ P_i \in \mathrm{Diff}^{m-i}(M;\mathbb{E})$$

then

$$P : \mathcal{S}(\mathbb{R} \times M; E_1) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_2)$$

and

$$\widehat{Pu}(\tau, \cdot) = \sum_{i=0}^{m} \tau^i P_i \hat{u}(\tau, \cdot) \tag{2.3}$$

where $\hat{u}(\tau, \cdot)$ is the 1-dimensional Fourier transform from (1.6). So we clearly need to examine the "suspended" family of operators

$$P(\tau) = \sum_{i=0}^{m} \tau^i P_i \in \mathcal{C}^\infty \left(\mathbb{C}; \operatorname{Diff}^m(M; \mathbb{E})\right). \tag{2.4}$$

I use the term "suspended" to denote the addition of a parameter to $\operatorname{Diff}^m(M; \mathbb{E})$ to get such a family—in this case polynomial. They are sometimes called "operator pencils" for reasons that escape me. Anyway, the main result we want is

Theorem 2.1. *If $P \in \operatorname{Diff}^m_{ti}(M; \mathbb{E})$ is elliptic then the suspended family $P(\tau)$ is invertible for all $\tau \in \mathbb{C} \setminus D$ with inverse*

$$P(\tau)^{-1} : H^s(M; E_2) \longrightarrow H^{s+m}(M; E_1) \tag{2.5}$$

*where*

$$D \subset \mathbb{C} \text{ is discrete and } D \subset \{\tau \in \mathbb{C};\ |\operatorname{Re}\tau| \le c|\operatorname{Im}\tau| + 1/c\} \tag{2.6}$$

*for some $c > 0$ (see Fig.* **??** *– still not quite right).*

In fact we need some more information on $P(\tau)^{-1}$ which we will pick up during the proof of this result. The translation-invariance of $P$ can be written in operator form as

$$Pu(t+s, \cdot) = (Pu)(t+s, \cdot)\ \forall\ s \in \mathbb{R} \tag{2.7}$$

Lemma 2.2. *If $P \in \operatorname{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ is elliptic then it has a parametrix*

$$Q : \mathcal{S}(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_1) \tag{2.8}$$

*which is translation-invariant in the sense of* (2.7) *and preserves the compactness of supports in* $\mathbb{R}$,

$$Q : \mathcal{C}^\infty_c(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R} \times M; E_1) \tag{2.9}$$

Proof. In the case of a compact manifold we contructed a global parametrix by patching local parametricies with a partition of unity. Here we do the same thing, treating the variable $t \in \mathbb{R}$ globally throughout. Thus if $F_a : \Omega_a \to \Omega'_a$ is a coordinate patch in $M$ over which $E_1$

and (hence) $E_2$ are trivial, $P$ becomes a square matrix of differential operators

$$P_a = \begin{bmatrix} P_{11}(z, D_t, D_z) & \cdots & P_{l1}(z, D_t, D_z) \\ \vdots & & \vdots \\ P_{1l}(z, D_t, D_z) & \cdots & P_{ll}(z, D_t, D_z) \end{bmatrix} \tag{2.10}$$

in which the coefficients do *not* depend on $t$. As discussed in Sections 2 and 3 above, we can construct a local parametrix in $\Omega'_a$ using a properly supported cutoff $\chi$. In the $t$ variable the parametrix is global anyway, so we use a fixed cutoff $\tilde{\chi} \in \mathcal{C}^\infty_c(\mathbb{R})$, $\tilde{\chi} = 1$ in $|t| < 1$, and so construct a parametrix

$$Q_a f(t, z) = \int_{\Omega'_a} q(t - t', z, z')\tilde{\chi}(t - t')\chi(z, z')f(t', z')\, dt'\, dz'. \tag{2.11}$$

This satisfies

$$P_a Q_a = \operatorname{Id} - R_a, \quad Q_a P_a = \operatorname{Id} - R'_a \tag{2.12}$$

where $R_a$ and $R'_a$ are smoothing operators on $\Omega'_a$ with kernels of the form

$$\begin{gathered} R_a f(t, z) = \int_{\Omega'_a} R_a(t - t', z, z')f(t', z')\, dt'\, dz' \\ R_a \in \mathcal{C}^\infty(\mathbb{R} \times \Omega'^2_a),\ R_a(t, z, z') = 0 \text{ if } |t| \ge 2 \end{gathered} \tag{2.13}$$

with the support proper in $\Omega'_a$.

Now, we can sum these local parametricies, which are all $t$-translation-invariant to get a global parametrix with the same properties

$$Qf = \sum_a \chi_a (F_a^{-1})^* (T_a^{-1})^* Q_a T_a^* F_a^* f \tag{2.14}$$

where $T_a$ denotes the trivialization of bundles $E_1$ and $E_2$. It follows that $Q$ satisfies (2.9) and since it is translation-invariant, also (2.8). The global version of (2.12) becomes

$$\begin{aligned} &PQ = \operatorname{Id} - R_2, \quad QP = \operatorname{Id} - R_1, \\ &R_i : \mathcal{C}^\infty_c(\mathbb{R} \times M; E_i) \longrightarrow \mathcal{C}^\infty_c(\mathbb{R} \times M; E_i), \\ &R_i f = \int_{\mathbb{R}\times M} R_i(t - t', z, z')f(t', z')\, dt'\, \nu_{z'} \end{aligned} \tag{2.15}$$

where the kernels

$$R_i \in \mathcal{C}^\infty_c \left(\mathbb{R} \times M^2; \operatorname{Hom}(E_i)\right),\ i = 1, 2. \tag{2.16}$$

$\square$

In fact we can deduce directly from (2.11) the boundedness of $Q$.

LEMMA 2.3. *The properly-supported parametrix $Q$ constructed above extends by continuity to a bounded operator*

$$\begin{aligned} Q :& H^s_{ti}(\mathbb{R} \times M; E_2) \longrightarrow H^{s+m}_{ti}(\mathbb{R} \times M; E_1) \ \forall\, s \in \mathbb{R} \\ Q :& \mathcal{S}(\mathbb{R} \times M; E_2) \longrightarrow \mathcal{S}(\mathbb{R} \times M; E_1). \end{aligned} \tag{2.17}$$

PROOF. This follows directly from the earlier discussion of elliptic regularity for each term in (2.14) to show that

$$\begin{aligned} Q : \{ f \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2;\ \mathrm{supp}(f) \subset [-2, 2] \times M \} \\ \longrightarrow \left\{ u \in H^{s+m}_{\mathrm{ti}}(\mathbb{R} \times M; E_1;\ \mathrm{supp}(u) \subset [-2 - R, 2 + R] \times M \right\} \end{aligned} \tag{2.18}$$

for some $R$ (which can in fact be taken to be small and positive). Indeed on compact sets the translation-invariant Sobolev spaces reduce to the usual ones. Then (2.17) follows from (2.18) and the translation-invariance of $Q$. Using a $\mu \in \mathcal{C}^\infty_c(\mathbb{R})$ generating a ti-paritition of unity on $\mathbb{R}$ we can decompose

$$H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2) \ni f = \sum_{k \in \mathbb{Z}} \tau^*_k (\mu \tau^*_{-k} f). \tag{2.19}$$

Then

$$Qf = \sum_{k \in \mathbb{Z}} \tau^*_k \left( Q(\mu \tau^*_{-k} f) \right). \tag{2.20}$$

The estimates corresponding to (2.18) give

$$\|Qf\|_{H^{s+m}_{\mathrm{ti}}} \le C \|f\|_{H^s_{\mathrm{ti}}}$$

if $f$ has support in $[-2, 2] \times M$. The decomposition (2.19) then gives

$$\sum \|\mu \tau^*_{-k} f\|^2_{H^s} = \|f\|^2_{H_s} < \infty \implies \|Qf\|^2 \le C' \|f\|^2_{H^s}.$$

This proves Lemma 2.3. □

Going back to the remainder term in (2.15), we can apply the 1-dimensional Fourier transform and find the following uniform results.

LEMMA 2.4. *If $R$ is a compactly supported, t-translation-invariant smoothing operator as in* (2.15) *then*

$$\widehat{Rf}(\tau, \cdot) = \widehat{R}(\tau) \widehat{f}(\tau, \cdot) \tag{2.21}$$

*where $\widehat{R}(\tau) \in \mathcal{C}^\infty(\mathbb{C} \times M^2; \mathrm{Hom}(E))$ is entire in $\tau \in \mathbb{C}$ and satisfies the estimates*

$$\forall\, k, p\ \exists\, C_{p,k} \text{ such that } \|\tau^k \widehat{R}(\tau)\|_{\mathcal{C}^p} \le C_{p,k} \exp(A|\operatorname{Im} \tau|). \tag{2.22}$$

*Here $A$ is a constant such that*

$$\operatorname{supp} R(t, \cdot) \subset [-A, A] \times M^2. \tag{2.23}$$

Proof. This is a parameter-dependent version of the usual estimates for the Fourier-Laplace transform. That is,

$$\widehat{R}(\tau,\cdot) = \int e^{-i\tau t} R(t,\cdot)\, dt \tag{2.24}$$

from which all the statements follow just as in the standard case when $R \in \mathcal{C}^\infty_c(\mathbb{R})$ has support in $[-A, A]$. □

Proposition 2.5. *If $R$ is as in Lemma 2.4 then there exists a discrete subset $D \subset \mathbb{C}$ such that $\left(\operatorname{Id} - \widehat{R}(\tau)\right)^{-1}$ exists for all $\tau \in \mathbb{C} \setminus D$ and*

$$\left(\operatorname{Id} - \widehat{R}(\tau)\right)^{-1} = \operatorname{Id} - \widehat{S}(\tau) \tag{2.25}$$

*where $\widehat{S} : \mathbb{C} \longrightarrow \mathcal{C}^\infty(M^2; \operatorname{Hom}(E))$ is a family of smoothing operators which is meromorphic in the complex plane with poles of finite order and residues of finite rank at $D$. Furthermore,*

$$D \subset \{\tau \in \mathbb{C};\ \log(|\operatorname{Re}\tau|) < c|\operatorname{Im}\tau| + 1/c\} \tag{2.26}$$

*for some $c > 0$ and for any $C > 0$, there exists $C'$ such that*

$$|\operatorname{Im}\tau| < C,\ |\operatorname{Re}\tau| > C' \implies \|\tau^k \widehat{S}(\tau)\|_{\mathcal{C}^p} \le C_{p,k}. \tag{2.27}$$

Proof. This is part of "Analytic Fredholm Theory" (although usually done with compact operators on a Hilbert space). The estimates (2.22) on $\widehat{R}(\tau)$ show that, in some region as on the right in (2.26),

$$\|\widehat{R}(\tau)\|_{L^2} \le 1/2. \tag{2.28}$$

Thus, by Neumann series,

$$\widehat{S}(\tau) = \sum_{k=1}^{\infty} \left(\widehat{R}(\tau)\right)^k \tag{2.29}$$

exists as a bounded operator on $L^2(M; E)$. In fact it follows that $\widehat{S}(\tau)$ is itself a family of smoothing operators in the region in which the Neumann series converges. Indeed, the series can be rewritten

$$\widehat{S}(\tau) = \widehat{R}(\tau) + \widehat{R}(\tau)^2 + \widehat{R}(\tau)\widehat{S}(\tau)\widehat{R}(\tau) \tag{2.30}$$

The smoothing operators form a "corner" in the bounded operators in the sense that products like the third here are smoothing if the outer two factors are. This follows from the formula for the kernel of the product

$$\int_{M\times M} \widehat{R}_1(\tau; z, z')\widehat{S}(\tau; z', z'')\widehat{R}_2(\tau; z'', \tilde{z})\, \nu_{z'}\, \nu_{z''}.$$

Thus $\widehat{S}(\tau) \in \mathcal{C}^\infty(M^2; \operatorname{Hom}(E))$ exists in a region as on the right in (2.26). To see that it extends to be meromorphic in $\mathbb{C} \setminus D$ for a discrete divisor $D$ we can use a finite-dimensional approximation to $\widehat{R}(\tau)$.

Recall — if neccessary from local coordinates — that given any $p \in \mathbb{N}, R > 0, q > 0$ there are finitely many sections $f_i^{(\tau)} \in \mathcal{C}^\infty(M; E'), g_i^{(\tau)} \in \mathcal{C}^\infty(M; E)$ and such that

$$\|\widehat{R}(\tau) - \sum_i g_i(\tau, z) \cdot f_i(\tau, z')\|_{\mathcal{C}^p} < \epsilon, \quad |\tau| < R. \tag{2.31}$$

Writing this difference as $M(\tau)$,

$$\operatorname{Id} - \widehat{R}(\tau) = \operatorname{Id} - M(\tau) + F(\tau)$$

where $F(\tau)$ is a finite rank operator. In view of (2.31), $\operatorname{Id} - M(\tau)$ is invertible and, as seen above, of the form $\operatorname{Id} - \widehat{M}(\tau)$ where $\widehat{M}(\tau)$ is holomorphic in $|\tau| < R$ as a smoothing operator.

Thus

$$\operatorname{Id} - \widehat{R}(\tau) = (\operatorname{Id} - M(\tau))(\operatorname{Id} + F(\tau) - \widehat{M}(\tau)F(\tau))$$

is invertible if and only if the finite rank perturbation of the identity by $(\operatorname{Id} - \widehat{M}(\tau))F(\tau)$ is invertible. For $R$ large, by the previous result, this finite rank perturbation must be invertible in an open set in $\{|\tau| < R\}$. Then, by standard results for finite dimensional matrices, it has a meromorphic inverse with finite rank (generalized) residues. The same is therefore true of $\operatorname{Id} - \widehat{R}(\tau)$ itself.

Since $R > 0$ is arbitrary this proves the result. $\square$

Proof. Proof of Theorem 2.1 We have proved (2.15) and the corresponding form for the Fourier transformed kernels follows:

$$\widehat{P}(\tau)\widehat{Q}'(\tau) = \operatorname{Id} - \widehat{R}_2(\tau), \; \widehat{Q}'(\tau)\widehat{P}(\tau) = \operatorname{Id} - \widehat{R}_1(\tau) \tag{2.32}$$

where $\widehat{R}_1(\tau), \widehat{R}_2(\tau)$ are families of smoothing operators as in Proposition 2.5. Applying that result to the first equation gives a new meromorphic right inverse

$$\widehat{Q}(\tau) = \widehat{Q}'(\tau)(\operatorname{Id} - \widehat{R}_2(\tau))^{-1} = \widehat{Q}'(\tau) - \widehat{Q}'(\tau)M(\tau)$$

where the first term is entire and the second is a meromorphic family of smoothing operators with finite rank residues. The same argument on the second term gives a left inverse, but his shows that $\widehat{Q}(\tau)$ must be a two-sided inverse.

This we have proved everything except the locations of the poles of $\widehat{Q}(\tau)$ — which are only constrained by (2.26) instead of (2.6). However, we can apply the same argument to $P_\theta(z, D_t, D_z) = P(z, e^{i\theta}D_t, D_z)$ for

$|\theta| < \delta$, $\delta > 0$ small, since $P_\theta$ stays elliptic. This shows that the poles of $\widehat{Q}(\tau)$ lie in a set of the form (2.6). □

## 3. Invertibility

We are now in a position to characterize those $t$-translation-invariant differential operators which give isomorphisms on the translation-invariant Sobolev spaces.

THEOREM 3.1. *An element $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; E)$ gives an isomorphism* (2.1) *(or equivalently is Fredholm) if and only if it is elliptic and $D \cap \mathbb{R} = \emptyset$, i.e. $\hat{P}(\tau)$ is invertible for all $\tau \in \mathbb{R}$.*

PROOF. We have already done most of the work for the important direction for applications, which is that the ellipticity of $P$ and the invertibility at $\hat{P}(\tau)$ for all $\tau \in \mathbb{R}$ together imply that (2.1) is an isomorphism for any $s \in \mathbb{R}$.

Recall that the ellipticity of $P$ leads to a parameterix $Q$ which is translation-invariant and has the mapping property we want, namely (2.17).

To prove the same estimate for the true inverse (and its existence) consider the difference

$$\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{\mathbb{R}}(\tau),\ \tau \in \mathbb{R}. \tag{3.1}$$

Since $\hat{P}(\tau) \in \mathrm{Diff}^m(M; \mathbb{E})$ depends smoothly on $\tau \in \mathbb{R}$ and $\hat{Q}(\tau)$ is a paramaterix for it, we know that

$$\hat{R}(\tau) \in \mathcal{C}^\infty(\mathbb{R}; \Psi^{-\infty}(M; \mathbb{E})) \tag{3.2}$$

is a smoothing operator on $M$ which depends smoothly on $\tau \in \mathbb{R}$ as a parameter. On the other hand, from (2.32) we also know that for large real $\tau$,

$$\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{Q}(\tau) M(\tau)$$

where $M(\tau)$ satisfies the estimates (2.27). It follows that $\hat{Q}(\tau)M(\tau)$ also satisfies these estimates and (3.2) can be strengthened to

$$\sup_{\tau \in \mathbb{R}} \|\tau^k \hat{R}(\tau, \cdot, \cdot)\|_{\mathcal{C}^p} < \infty\ \forall\ p, k. \tag{3.3}$$

That is, the kernel $\hat{R}(\tau) \in \mathcal{S}(\mathbb{R}; \mathcal{C}^\infty(M^2; \mathrm{Hom}(\mathbb{E})))$. So if we define the $t$-translation-invariant operator

$$Rf(t, z) = (2\pi)^{-1} \int e^{it\tau} \hat{R}(\tau) \hat{f}(\tau, \cdot) d\tau \tag{3.4}$$

by inverse Fourier transform then

$$R : H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2) \longrightarrow H^\infty_{\mathrm{ti}}(\mathbb{R} \times M; E_1)\ \forall\ s \in \mathbb{R}. \tag{3.5}$$

It certainly suffices to show this for $s < 0$ and then we know that the Fourier transform gives a map

$$\mathcal{F} : H^s_{\text{ti}}(\mathbb{R} \times M; E_2) \longrightarrow \langle\tau\rangle^{|s|} L^2(\mathbb{R}; H^{-|s|}(M; E_2)). \tag{3.6}$$

Since the kernel $\hat{R}(\tau)$ is rapidly decreasing in $\tau$, as well as being smooth, for every $N > 0$,

$$\hat{R}(\tau) : \langle\tau\rangle^{|s|} L^2(\mathbb{R}; H^{-|s|} M; E_2) \longrightarrow \langle\tau\rangle^{-N} L^2(\mathbb{R}; H^N(M; E_2)) \tag{3.7}$$

and inverse Fourier transform maps

$$\mathcal{F}^{-1} : \langle\tau\rangle^{-N} H^N(M; E_2) \longrightarrow H^N_{\text{ti}}(\mathbb{R} \times M; E_2)$$

which gives (3.5).

Thus $Q + R$ has the same property as $Q$ in (2.17). So it only remains to check that $Q + R$ is the two-sided version of $P$ and it is enough to do this on $\mathcal{S}(\mathbb{R} \times M; E_i)$ since these subspaces are dense in the Sobolev spaces. This in turn follows from (3.1) by taking the Fourier transform. Thus we have shown that the invertibility of $P$ follows from its ellipticity and the invertibility of $\hat{P}(\tau)$ for $\tau \in \mathbb{R}$.

The converse statement is less important but certainly worth knowing! If $P$ is an isomorphism as in (2.1), even for one value of $s$, then it must be elliptic — this follows as in the compact case since it is everywhere a local statement. Then if $\hat{P}(\tau)$ is not invertible for some $\tau \in \mathbb{R}$ we know, by ellipticity, that it is Fredholm and, by the stability of the index, of index zero (since $\hat{P}(\tau)$ *is* invertible for a dense set of $\tau \in \mathbb{C}$). There is therefore some $\tau_0 \in \mathbb{R}$ and $f_0 \in \mathcal{C}^\infty(M; E_2)$, $f_0 \neq 0$, such that

$$\hat{P}(\tau_0)^* f_0 = 0. \tag{3.8}$$

It follows that $f_0$ is *not* in the range of $\hat{P}(\tau_0)$. Then, choose a cut off function, $\rho \in \mathcal{C}^\infty_c(\mathbb{R})$ with $\rho(\tau_0) = 1$ (and supported sufficiently close to $\tau_0$) and define $f \in \mathcal{S}(\mathbb{R} \times M; E_2)$ by

$$\hat{f}(\tau, \cdot) = \rho(\tau) f_0(\cdot). \tag{3.9}$$

Then $f \notin P \cdot H^s_{\text{ti}}(\mathbb{R} \times M; E_1)$ for any $s \in \mathbb{R}$. To see this, suppose $u \in H^s_{\text{ti}}(\mathbb{R} \times M; E_1)$ has

$$Pu = f \Rightarrow \hat{P}(\tau)\hat{u}(\tau) = \hat{f}(\tau) \tag{3.10}$$

where $\hat{u}(\tau) \in \langle\tau\rangle^{|s|} L^2(\mathbb{R}; H^{-|s|}(M; E_1))$. The invertibility of $P(\tau)$ for $\tau \neq \tau_0$ on $\operatorname{supp}(\rho)$ (chosen with support close enough to $\tau_0$) shows that

$$\hat{u}(\tau) = \hat{P}(\tau)^{-1} \hat{f}(\tau) \in \mathcal{C}^\infty((\mathbb{R} \backslash \{\tau_0\}) \times M; E_1).$$

Since we know that $\hat{P}(\tau)^{-1} - \hat{Q}(\tau) = \hat{R}(\tau)$ is a meromorphic family of smoothing operators it actually follows that $\hat{u}(\iota)$ is meromorphic in $\tau$ near $\tau_0$ in the sense that

$$\hat{u}(\tau) = \sum_{j=1}^{k} (\tau - \tau_0)^{-j} u_j + v(\tau) \tag{3.11}$$

where the $u_j \in \mathcal{C}^\infty(M; E_1)$ and $v \in \mathcal{C}^\infty((\tau - \epsilon, \tau + \epsilon) \times M; E_1)$. Now, one of the $u_j$ is not identically zero, since otherwise $\hat{P}(\tau_0)v(\tau_0) = f_0$, contradicting the choice of $f_0$. However, a function such as (3.11) is *not* locally in $L^2$ with values in any Sobolev space on $M$, which contradicts the existence of $u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1)$.

This completes the proof for invertibility of $P$. To get the Fredholm version it suffices to prove that if $P$ is Fredholm then it is invertible. Since the arguments above easily show that the null space of $P$ is empty on any of the $H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1)$ spaces and the same applies to the adjoint, we easily conclude that $P$ is an isomorphism if it is Fredholm. $\square$

This result allows us to deduce similar invertibility conditions on exponentially-weighted Sobolev spaces. Set

$$e^{at} H^s_{\mathrm{ti}}(\mathbb{R} \times M; E) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R} \times M; E);\ e^{-at} u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E)\} \tag{3.12}$$

for any $\mathcal{C}^\infty$ vector bundle $E$ over $M$. The translation-invariant differential operators also act on these spaces.

LEMMA 3.2. *For any $a \in \mathbb{R}$, $P \in \mathrm{Diff}^m_{ti}(\mathbb{R} \times M; \mathbb{E})$ defines a continuous linear operator*

$$P : e^{at} H^{s+m}_{ti}(\mathbb{R} \times M; E_1) \longrightarrow e^{at} H^{s+m}_{ti}(\mathbb{R} \times M; E_2). \tag{3.13}$$

PROOF. We already know this for $a = 0$. To reduce the general case to this one, observe that (3.13) just means that

$$P \cdot e^{at} u \in e^{at} H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2)\ \forall\ u \in H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_1) \tag{3.14}$$

with continuity meaning just continuous dependence on $u$. However, (3.14) in turn means that the conjugate operator

$$P_a = e^{-at} \cdot P \cdot e^{at} : H^{s+m}_{\mathrm{ti}}(\mathbb{R} \times M; E_1) \longrightarrow H^s_{\mathrm{ti}}(\mathbb{R} \times M; E_2). \tag{3.15}$$

Conjugation by an exponential is actually an isomorphism

$$\mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M; \mathbb{E}) \ni P \longmapsto e^{-at} P e^{at} \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M; \mathbb{E}). \tag{3.16}$$

To see this, note that elements of $\mathrm{Diff}^j(M; \mathbb{E})$ commute with multiplication by $e^{at}$ and

$$e^{-at} D_t e^{at} = D_t - ia \tag{3.17}$$

which gives (3.16)).

The result now follows. □

PROPOSITION 3.3. *If* $P \in \mathrm{Diff}^m_{ti}(\mathbb{R}\times M;\mathbb{E})$ *is elliptic then as a map* (3.13) *it is invertible precisely for*

$$a \notin -\operatorname{Im}(D),\ D = D(P) \subset \mathbb{C}, \tag{3.18}$$

*that is, a is not the negative of the imaginary part of an element of D.*

Note that the set $-\operatorname{Im}(D) \subset \mathbb{R}$, for which invertibility fails, is discrete. This follows from the discreteness of $D$ and the estimate (2.6). Thus in Fig **??** invertibility on the space with weight $e^{at}$ correspond exactly to the horizonatal line with $\operatorname{Im}\tau = -a$ missing $D$.

PROOF. This is direct consequence of (**??**) and the discussion around (3.15). Namely, $P$ is invertible as a map (3.13) if and only if $P_a$ is invertible as a map (2.1) so, by Theorem 3.1, if

and only if

$$D(P_a) \cap \mathbb{R} = \emptyset.$$

From (3.17), $D(P_a) = D(P) + ia$ so this condition is just $D(P) \cap (\mathbb{R} - ia) = \emptyset$ as claimed. □

Although this is a characterization of the Fredholm properties on the standard Sobolev spaces, it is not the end of the story, as we shall see below.

One important thing to note is that $\mathbb{R}$ has *two* ends. The exponential weight $e^{at}$ treats these differently – since if it is big at one end it is small at the other – and in fact we (or rather you) can easily define doubly-exponentially weighted spaces and get similar results for those. Since this is rather an informative extended exercise, I will offer some guidance.

DEFINITION 3.4. *Set*

$$\begin{gathered} H^{s,a,b}_{ti,\exp}(\mathbb{R}\times M;E) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R}\times M;E); \\ \chi(t)e^{-at}u \in H^s_{ti}(\mathbb{R}\times M;E)(1-\chi(t))e^{bt}u \in H^s_{ti}(\mathbb{R}\times M;E)\} \end{gathered} \tag{3.19}$$

*where* $\chi \in \mathcal{C}^\infty(\mathbb{R})$, $\chi = 1$ *in* $t > 1$, $\chi = 0$ *in* $t < -1$.

**Exercises.**

(1) Show that the spaces in (3.19) are independent of the choice of $\chi$, are all Hilbertable (are complete with respect to a Hilbert norm) and show that if $a + b \geq 0$

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E) = e^{at}H^s_{\mathrm{ti}}(\mathbb{R}\times M;E) + e^{-bt}H^s_{\mathrm{ti}}(\mathbb{R}\times M;E) \tag{3.20}$$

whereas if $a+b \le 0$ then

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E) = e^{at}H^{s}_{\mathrm{ti}}(\mathbb{R}\times M;E)\cap e^{-bt}H^{s}_{\mathrm{ti}}(\mathbb{R}\times M;E). \tag{3.21}$$

(2) Show that any $P \in \mathrm{Diff}^m_{\mathrm{ti}}(\mathbb{R}\times M;\mathbb{E})$ defines a continuous linear map for any $s,\ a,\ b\in\mathbb{R}$

$$P: H^{s+m,a,b}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_1) \longrightarrow H^{s,a,b}_{\mathrm{ti\text{-}exp}}(\mathbb{R}\times M;E_2). \tag{3.22}$$

(3) Show that the standard $L^2$ pairing, with respect to $dt$, a smooth positive density on $M$ and an inner product on $E$ extends to a non-degenerate bilinear pairing

$$H^{s,a,b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)\times H^{-s,-a,-b}_{\mathrm{ti,exp}}(\mathbb{R}\times M;E)\longrightarrow\mathbb{C} \tag{3.23}$$

for any $s$, $a$ and $b$. Show that the adjoint of $P$ with respect to this pairing is $P^*$ on the 'negative' spaces – you can use this to halve the work below.

(4) Show that if $P$ is elliptic then (3.22) is Fredholm precisely when

$$a\notin -\mathrm{Im}(D) \text{ and } b\notin \mathrm{Im}(D). \tag{3.24}$$

Hint:- Assume for instance that $a+b\ge 0$ and use (3.20). Given (3.24) a parametrix for $P$ can be constructed by combining the inverses on the single exponential spaces

$$Q_{a,b} = \chi' P_a^{-1}\chi + (1-\chi'')P^{-1}_{-b}(1-\chi) \tag{3.25}$$

where $\chi$ is as in (3.19) and $\chi'$ and $\chi''$ are similar but such that $\chi'\chi = 1$, $(1-\chi'')(1-\chi) = 1-\chi$.

(5) Show that $P$ is an isomorphism if and only if

$$a+b\le 0 \text{ and } [a,-b]\cap -\mathrm{Im}(D)=\emptyset \text{ or } a+b\ge 0 \text{ and } [-b,a]\cap -\mathrm{Im}(D)=\emptyset.$$

(6) Show that if $a+b\le 0$ and (3.24) holds then

$$\mathrm{ind}(P) = \dim\operatorname{null}(P) = \sum_{\tau_i\in D\cap(\mathbb{R}\times[b,-a])}\mathrm{Mult}(P,\tau_i)$$

where $\mathrm{Mult}(P,\tau_i)$ is the *algebraic* multiplicity of $\tau$ as a 'zero' of $\hat{P}(\tau)$, namely the dimension of the generalized null space

$$\mathrm{Mult}(P,\tau_i) = \dim\left\{u=\sum_{p=0}^{N}u_p(z)D^p_\tau\delta(\tau-\tau_i); P(\tau)u(\tau)\equiv 0\right\}.$$

(7) Characterize these multiplicities in a more algebraic way. Namely, if $\tau'$ is a zero of $P(\tau)$ set $E_0=\operatorname{null}P(\tau')$ and $F_0=\mathcal{C}^\infty(M;E_2)/P(\tau')\mathcal{C}^\infty(M;E_1)$. Since $P(\tau)$ is Fredholm of index zero, these are finite dimensional vector spaces of the same dimension. Let the derivatives of $P$ be $T_i=\partial^iP/\partial\tau^i$ at $\tau=\tau'$ Then define $R_1: E_0\longrightarrow F_0$

as $T_1$ restricted to $E_0$ and projected to $F_0$. Let $E_1$ be the null space of $R_1$ and $F_1 = F_0/R_1E_0$. Now proceed inductively and define for each $i$ the space $E_i$ as the null space of $R_i$, $F_i = F_{i-1}/R_iE_{i-1}$ and $R_{i+1} : E_i \longrightarrow F_i$ as $T_i$ restricted to $E_i$ and projected to $F_i$. Clearly $E_i$ and $F_i$ have the same, finite, dimension which is non-increasing as $i$ increases. The properties of $P(\tau)$ can be used to show that for large enough $i$, $E_i = F_i = \{0\}$ and

$$\operatorname{Mult}(P,\tau') = \sum_{i=0}^{\infty} \dim(E_i) \tag{3.26}$$

where the sum is in fact finite.

(8) Derive, by duality, a similar formula for the index of $P$ when $a + b \geq 0$ and (3.24) holds, showing in particular that it is injective.

## 4. Resolvent operator

## Addenda to Chapter 7

More?

- Why – manifold with boundary later for Euclidean space, but also resolvent (Photo-C5-01)
- Hölder type estimates – Photo-C5-03. Gives interpolation.

As already noted even a result such as Proposition 3.3 and the results in the exercises above by no means exhausts the possibile realizations of an element $P \in \operatorname{Diff}^m_{\mathrm{ti}}(\mathbb{R} \times M; \mathbb{E})$ as a Fredholm operator. Necessarily these other realization cannot simply be between spaces like those in (3.19). To see what else one can do, suppose that the condition in Theorem 3.1 is violated, so

$$D(P) \cap \mathbb{R} = \{\tau_1, \dots, \tau_N\} \neq \emptyset. \tag{4.1}$$

To get a Fredholm operator we need to change either the domain or the range space. Suppose we want the range to be $L^2(\mathbb{R}\times M; E_2)$. Now, the condition (3.24) guarantees that $P$ is Fredholm as an operator (3.22). So in particular

$$P : H^{m,\epsilon,\epsilon}_{\mathrm{ti-exp}}(\mathbb{R} \times M; E_1) \longrightarrow H^{0,\epsilon,\epsilon}_{\mathrm{ti\text{-}exp}}(\mathbb{R} \times M; E_2) \tag{4.2}$$

is Fredholm for all $\epsilon > 0$ sufficiently small (becuase $D$ is discrete). The image space (which is necessarily the range in this case) just consists of the sections of the form $\exp(a|t|)f$ with $f$ in $L^2$. So, in this case the range certainly contains $L^2$ so we can define

(4.3)
$$\mathrm{Dom}_{\mathrm{AS}}(P)=\{u\in H^{m,\epsilon,\epsilon}_{\text{ti-exp}}(\mathbb{R}\times M;E_1);Pu\in L^2(\mathbb{R}\times M;E_2)\},\ \epsilon>0 \text{ sufficiently small.}$$

This space is independent of $\epsilon>0$ if it is taken smalle enough, so the same space arises by taking the intersection over $\epsilon>0$.

Proposition 4.1. *For any elliptic element $P\in\mathrm{Diff}^m_{ti}(\mathbb{R}\times M;\mathbb{E})$ the space in* (4.3) *is Hilbertable space and*

$$P:\mathrm{Dom}_{AS}(P)\longrightarrow L^2(\mathbb{R}\times M;E_2) \text{ is Fredholm.} \tag{4.4}$$

I have not made the assumption (4.1) since it is relatively easy to see that if $D\cap\mathbb{R}=\emptyset$ then the domain in (4.3) reduces again to $H^m_{\mathrm{ti}}(\mathbb{R}\times M;E_1)$ and (4.4) is just the standard realization. Conversely of course under the assumption (4.1) the domain in (4.4) is strictly larger than the standard Sobolev space. To see what it actually is requires a bit of work but if you did the exercises above you are in a position to work this out! Here is the result when there is only one pole of $\hat{P}(\tau)$ on the real line and it has order one.

Proposition 4.2. *Suppose $P\in\mathrm{Diff}^m_{ti}(\mathbb{R}\times M;\mathbb{E})$ is elliptic, $\hat{P}(\tau)$ is invertible for $\tau\in\mathbb{R}\setminus\{0\}$ and in addition $\tau\hat{P}(\tau)^{-1}$ is holomorphic near* 0. *Then the Atiyah-Singer domain in* (4.4) *is*

$$\begin{aligned}\mathrm{Dom}_{AS}(P)=\big\{u=u_1+u_2;u_1\in H^m_{ti}(\mathbb{R}\times M;E_1),\\ u_2=f(t)v,\ v\in\mathcal{C}^\infty(M;E_1),\ \hat{P}(0)v=0,\ f(t)=\int_0^t g(t)dt,\ g\in H^{m-1}(\mathbb{R})\big\}.\end{aligned}\tag{4.5}$$

Notice that the 'anomalous' term here, $u_2$, need not be square-integrable. In fact for any $\delta>0$ the power $\langle t\rangle^{\frac12-\delta}v\in\langle t\rangle^{1-\delta}L^2(\mathbb{R}\times M;E_1)$ is included and conversely

$$f\in\bigcap_{\delta>0}\langle t\rangle^{1+\delta}H^{m-1}(\mathbb{R}).\tag{4.6}$$

One can say a lot more about the growth of $f$ if desired but it is generally quite close to $\langle t\rangle L^2(\mathbb{R})$.

Domains of this sort are sometimes called 'extended $L^2$ domains' – see if you can work out what happens more generally.

CHAPTER 8

# Manifolds with boundary

- Dirac operators – Photos-C5-16, C5-17.
- Homogeneity etc Photos-C5-18, C5-19, C5-20, C5-21, C5-23, C5-24.

## 1. Compactifications of $\mathbb{R}$.

As I will try to show by example later in the course, there are I believe considerable advantages to looking at compactifications of non-compact spaces. These advantages show up last in geometric and analytic considerations. Let me start with the simplest possible case, namely the real line. There are two standard compactifications which one can think of as 'exponential' and 'projective'. Since there is only one connected compact manifold with boundary compactification corresponds to the choice of a diffeomorphism onto the interior of $[0,1]$:

$$\begin{aligned} \gamma &: \mathbb{R} \longrightarrow [0,1],\ \gamma(\mathbb{R}) = (0,1), \\ \gamma^{-1} &: (0,1) \longrightarrow \mathbb{R},\ \gamma, \gamma^{-1}\mathcal{C}^\infty. \end{aligned} \tag{1.1}$$

In fact it is not particularly pleasant to have to think of the global maps $\gamma$, although we can. Rather we can think of separate maps

$$\begin{aligned} \gamma_+ &: (T_+, \infty) \longrightarrow [0,1] \\ \gamma_- &: (T_-, -\infty) \longrightarrow [0,1] \end{aligned} \tag{1.2}$$

which *both* have images $(0, x_\pm)$ and as diffeomorphism other than signs. In fact if we want the two ends to be the 'same' then we can take $\gamma_-(t) = \gamma_+(-t)$. I leave it as an exercise to show that $\gamma$ then exists with

$$\begin{cases} \gamma(t) = \gamma_+(t) & t \gg 0 \\ \gamma(t) = 1 - \gamma_-(t) & t \ll 0. \end{cases} \tag{1.3}$$

So, all we are really doing here is identifying a 'global coordinate' $\gamma_+^* x$ near $\infty$ and another near $-\infty$. Then two choices I refer to above are

$$\text{(CR.4)} \qquad \begin{array}{ll} x = e^{-t} & \text{exponential compactification} \\ x = 1/t & \text{projective compactification}\,. \end{array}$$

Note that these are *alternatives*!

Rather than just consider $\mathbb{R}$, I want to consider $\mathbb{R} \times M$, with $M$ compact, as discussed above.

LEMMA 1.1. *If $R : H \longrightarrow H$ is a compact operator on a Hilbert space then* $\mathrm{Id} - R$ *is Fredholm.*

PROOF. A compact operator is one which maps the unit ball (and hence any bounded subset) of $H$ onto a precompact set, a set with compact closure. The unit ball in the null space of $\mathrm{Id} - R$ is

$$\{u \in H; \|u\| = 1\,,\ u = Ru\} \subset R\{u \in H; \|u\| = 1\}$$

and is therefore precompact. Since it is closed, it is compact and any Hilbert space with a compact unit ball is finite dimensional. Thus the null space of $(\mathrm{Id} - R)$ is finite dimensional.

Consider a sequence $u_n = v_n - Rv_n$ in the range of $\mathrm{Id} - R$ and suppose $u_n \to u$ in $H$. We may assume $u \neq 0$, since 0 is in the range, and by passing to a subsequence suppose that of $\gamma$ on ?? fields. Clearly

$$\text{(CR.5)} \qquad \begin{array}{lll} \gamma(t) = e^{-t} & \Rightarrow & \gamma_*(\partial_t) = -x(\partial_x) \\ \tilde{\gamma}(t) = 1/t & \Rightarrow & \tilde{\gamma}_*(\partial_t) = -s^2\partial_s \end{array}$$

where I use '$s$' for the variable in the second case to try to reduce confusion, it is just a variable in $[0,1]$. Dually

$$\text{(CR.6)} \qquad \begin{gathered} \gamma^*\left(\frac{dx}{x}\right) = -dt \\ \tilde{\gamma}^*\left(\frac{ds}{s^2}\right) = -dt \end{gathered}$$

in the two cases. The minus signs just come from the fact that both $\gamma$'s reverse orientation.

PROPOSITION 1.2. *Under exponential compactification the translation-invariant Sobolev spaces on $\mathbb{R} \times M$ are identified with*

(1.4)
$$H^k_b([0,1] \times M) = \Big\{u \in L^2\left([0,1] \times M; \frac{dx}{x}V_M\right)\ ; \forall\ \ell, p \leq k \\ P_p \in \mathrm{Diff}^p(M)\,,\ (xD_x)^\ell P_p u \in L^2\big([0,1] \times M; \frac{dx}{x}V_M\big)\Big\}$$

*for $k$ a positive integer,* $\dim M = n$,

$$(1.5)\quad H^s_b([0,1]\times M) = \Big\{u \in L^2\left([0,1]\times M; \frac{dx}{x}V_M\right);$$
$$\iint \frac{|u(x,z)-u(x',z')|^2}{\big(|\log\frac{x}{x'}|^2+\rho(z,z')\big)^{\frac{n+s+1}{2}}}\frac{dx}{x}\frac{dx'}{x'}\nu\nu' < \infty\Big\} 0 < s < 1$$

*and for $s < 0$, $k \in \mathbb{N}$ s.t., $0 \le s+k < 1$,*

$$(1.6)\quad H^s_b([0,1]\times M) = \Big\{u = \sum_{0\le j+p\le k} (Xd^J_X)p_P u_{j,p},$$
$$P_p \in \mathrm{Diff}^p(M)\,,\ u_{j,p} \in H^{s+k}_b([0,1]\times M)\Big\}.$$

*Moreover the $L^2$ pairing with respect to the measure $\frac{dx}{x}\nu$ extends by continuity from the dense subspaces $\mathcal{C}^\infty_c((0,1)\times M)$ to a non-degenerate pairing*

$$(1.7)\quad H^s_b([0,1]\times M)\times H^{-s}_b([0,1]\times M) \ni (n,u) \longmapsto \int u\cdot v\frac{dx}{x}\nu \in \mathbb{C}.$$

□

Proof. This is all just translation of the properties of the space $H^s_{\mathrm{ti}}(\mathbb{R}\times M)$ to the new coordinates. □

Note that there are other properties I have not translated into this new setting. There is one additional fact which it is easy to check. Namely $\mathcal{C}^\infty([0,1]\times M)$ acts as multipliers on all the spaces $H^s_{\mathrm{b}}([0,1]\times M)$. This follows directly from Proposition 1.2;

(CR.12)
$$\mathcal{C}^\infty([0,1]\times M)\times H^s_{\mathrm{b}}([0,1]\times M) \ni (\varphi,u)\mapsto \varphi u \in H^s_{\mathrm{b}}([0,1]\times M)\,.$$

What about the '$b$' notation? Notice that $(1-x)x\partial_x$ and the smooth vector fields on $M$ span, over $\mathcal{C}^\infty(X)$, for $X=[0,1]\times M$, *all* the vector fields tangent to $\{x=0|u|x=1\}$. Thus we can define the 'boundary differential operators' as

(CR.13)
$$\mathrm{Diff}^m_{\mathrm{b}}([0,1]\times M_i)^E = \left\{P = \sum_{0\le j+p\le m} a_{j,p}(x_j)((1-x)xD_x)^j P_p\ ,\right.$$
$$\left. P_p \in \mathrm{Diff}^p(M_i)^E\right\}$$

and conclude from (CR.12) and the earlier properties that

$$
\begin{aligned}
&P \in \operatorname{Diff}_{\mathrm{b}}^m(X;E) \Rightarrow \\
&P : H_{\mathrm{b}}^{s+m}(X;E) \to H_{\mathrm{b}}^s(X;E)\, \forall s \in \mathbb{R}\,.
\end{aligned}
\tag{CR.14}
$$

THEOREM 1.3. *A differential operator as in* (1.3) *is Fredholm if and only if it is elliptic in the interior and the two "normal operators'*

$$
I_\pm(P) = \sum_{0 \le j+p \le m} a_{j,p}(x_{\pm 1})(\pm D_k)^i P_p \quad x_+ = 0\,,\ x_- = 1
\tag{CR.16}
$$

*derived from* (CR.13)*, are elliptic and invertible on the translation-invariant Sobolev spaces.*

PROOF. As usual we are more interested in the sufficiency of these conditions than the necessity. To prove this result by using the present (slightly low-tech) methods requires going back to the beginning and redoing most of the proof of the Fredholm property for elliptic operators on a compact manifold.

The first step then is *a priori* bounds. What we want to show is that if the conditions of the theorem hold then for $u \in H_{\mathrm{b}}^{s+m}(X;E)$, $x = \mathbb{R} \times M$, $\exists C > 0$ s.t.

$$
\|u\|_{m+s} \le C_s \|Pu\|_s + C_s \|x(1-x)u\|_{s-1+m}\,.
\tag{CR.17}
$$

Notice that the norm on the right has a factor, $x(1-x)$, which vanishes at the boundary. Of course this is supposed to come from the invertibility of $I_\pm(P)$ in $\mathbb{R}(0)$ and the ellipticity of $P$.

By comparison $I_\pm(P) : H_{\hbar}^{s+m}(\mathbb{R} \times M) \to H_{\hbar}^s(\mathbb{R} \times M)$ are isomorphisms — necessary and sufficient conditions for this are given in Theorem ???. We can use the compactifying map $\gamma$ to convert this to a statement as in (CR.17) for the operators

$$
P_\pm \in \operatorname{Diff}_{\mathrm{b}}^m(X)\,,\ P_\pm = I_\pm(P)(\gamma_* D_t, \cdot)\,.
\tag{CR.18}
$$

Namely

$$
\|u\|_{m+s} \le C_s \|P_\pm u\|_s
\tag{CR.19}
$$

where these norms, as in (CR.17) are in the $H_{\mathrm{b}}^s$ spaces. Note that near $x = 0$ or $x = 1$, $P_\pm$ are obtained by substituting $D_t \mapsto xD_x$ or $(1-x)D_x$ in (CR.17). Thus

$$
P - P_\pm \in (x - x_\pm) \operatorname{Diff}_{\mathrm{b}}^m(X)\,, \quad x_\pm = 0, 1
\tag{CR.20}
$$

have coefficients which *vanish* at the appropriate boundary. This is precisely how (CR.16) is derived from (CR.13). Now choose $\varphi \in$

$\mathcal{C}^\infty, (0,1) \times M$ which is equal to 1 on a sufficiently large set (and has $0 \le \varphi \le 1$) so that

$$1 - \varphi = \varphi_+ + \varphi_- \,,\ \varphi_\pm \in \mathcal{C}^\infty([0,1] \times M) \tag{CR.21}$$

have $\operatorname{supp}(\varphi_\pm) \subset \{|x - x_\pm| \le \epsilon)$, $0 \le \varphi_+ 1$.

By the interim elliptic estimate,

$$\|\varphi u\|_{s+m} \le C_s \|\varphi P u\|_s + C'_s \|\psi u\|_{s-1+m} \tag{CR.22}$$

where $\psi \in \mathcal{C}^\infty_c((0,1) \times M)$. On the other hand, because of (CR.20)

(CR.23)
$$\begin{aligned} \|\varphi_\pm u\|_{m+s} &\le C_s \|\varphi_\pm P_\pm u\|_s + C_s \|[\varphi_\pm, P_\pm u]\|_s \\ &\le C_s \|\varphi_\pm P u\|_s + C_s \varphi_\pm (P - P_\pm) u\|_s + C_s \|[\varphi_\pm, P_\pm] u\|_s \,. \end{aligned}$$

Now, if we can choose the support at $\varphi_\pm$ small enough — recalling that $C_s$ truly depends on $I_\pm(P_t)$ and $s$ — then the second term on the right in (CR.23) is bounded by $\frac{1}{4}\|u\|_{m+s}$, since *all* the coefficients of $P - P_\pm$ are small on the support off $\varphi_\pm$. Then (CR.24) ensures that the final term in (CR.17), since the coefficients vanish at $x = x_\pm$.

The last term in (CR.22) has a similar bound since $\psi$ has compact support in the interim. This combining (CR.2) and (CR.23) gives the desired bound (CR.17).

To complete the proof that $P$ is Fredholm, we need another property of these Sobolev spaces.

Lemma 1.4. *The map*

$$Xx(1-x) : H^s_b(X) \longrightarrow H^{s-1}_b(X) \tag{1.8}$$

*is compact.*

Proof. Follow it back to $\mathbb{R} \times M$!

□

Now, it follows from the *a priori* estimate (CR.17) that, as a map (CR.14), $P$ has finite dimensional null space and closed range. This is really the proof of Proposition ?? again. Moreover the adjoint of $P$ with respect to $\frac{dx}{x}V, P^*$, is again elliptic and satisfies the condition of the theorem, so it too has finite-dimensional null space. Thus the range of $P$ has finite codimension so it is Fredholm.

□

A corresponding theorem, with similar proof follows for the cusp compactification. I will formulate it later.

## 2. Basic properties

A discussion of manifolds with boundary goes here.

## 3. Boundary Sobolev spaces

Generalize results of Section 1 to arbitrary compact manifolds with boundary.

## 4. Dirac operators

Euclidean and then general Dirac operators

## 5. Homogeneous translation-invariant operators

One application of the results of Section 3 is to homogeneous constant-coefficient operators on $\mathbb{R}^n$, including the Euclidean Dirac operators introduced in Section 4. Recall from Chapter 4 that an elliptic constant-coefficient operator is Fredholm, on the standard Sobolev spaces, if and only if its characteristic polynomial has no real zeros. If $P$ is homogeneous

$$P_{ij}(t\zeta) = t^m P_{ij}(\zeta)\ \forall\ \zeta \in \mathbb{C}^n\,,\ t \in \mathbb{R}\,, \tag{5.1}$$

and elliptic, then the only real zero (of the determinant) is at $\zeta = 0$. We will proceed to discuss the radial compactification of Euclidean space to a ball, or more conveniently a half-sphere

$$\gamma_R : \mathbb{R}^n \hookrightarrow \mathbb{S}^{n,1} = \{Z \in \mathbb{R}^{n+1}\,;\, |Z| = 1\,,\ Z_0 \geq 0\}\,. \tag{5.2}$$

Transferring $P$ to $\mathbb{S}^{n,1}$ gives

$$P_R \in Z_0^m \operatorname{Diff}^m_{\mathrm{b}}(\mathbb{S}^{n,1}\,;\, \mathbb{C}^N) \tag{5.3}$$

which is elliptic and to which the discussion in Section 3 applies.

In the 1-dimensional case, the map (5.2) reduces to the second 'projective' compactification of $\mathbb{R}$ discussed above. It can be realized globally by

$$\gamma_R(z) = \left(\frac{1}{\sqrt{1+|z|^2}}\,,\ \frac{z}{\sqrt{1+|z|^2}}\right) \in \mathbb{S}^{n,1}\,. \tag{5.4}$$

Geometrically this corresponds to a form of stereographic projection. Namely, if $\mathbb{R}^n \ni z \mapsto (1, z) \in \mathbb{R}^{n+1}$ is embedded as a 'horizontal plane' which is then projected radially onto the sphere (of radius one around the origin) one arrives at (5.4). It follows easily that $\gamma_R$ is a diffeomorphism onto the open half-sphere with inverse

$$z = Z'/Z_0\,,\ Z' = (Z_1, \ldots, Z_n)\,. \tag{5.5}$$

Whilst (5.4) is concise it is not a convenient form of the compactification as far as computation is concerned. Observe that

$$x \mapsto \frac{x}{\sqrt{1+x^2}}$$

is a diffeomorphism of neighborhoods of $0 \in \mathbb{R}$. It follows that $Z_0$, the first variable in (5.4) can be replaced, near $Z_0 = 0$, by $1/|z| = x$. That is, there is a diffeomorphism

$$\{0 \le Z_0 \le \epsilon\} \cap \mathbb{S}^{n,1} \leftrightarrow [0,\delta]_x \times \mathbb{S}^{n-1}_\theta \tag{5.6}$$

which composed with (5.4) gives $x = 1/|z|$ and $\theta = z/|z|$. In other words the compactification (5.4) is equivalent to the introduction of polar coordinates near infinity on $\mathbb{R}^n$ followed by inversion of the radial variable.

LEMMA 5.1. *If $P = (P_{ij}(D_z))$ is an $N \times N$ matrix of constant coefficient operators in $\mathbb{R}^n$ which is homogeneous of degree $-m$ then (5.3) holds after radial compactification. If $P$ is elliptic then $P_R$ is elliptic.*

PROOF. This is a bit tedious if one tries to do it by direct computation. However, it is really only the homogeneity that is involved. Thus if we use the coordinates $x = 1/|z|$ and $\theta = z/|z|$ valid near the boundary of the compactification (i.e., near $\infty$ on $\mathbb{R}^n$) then

$$P_{ij} = \sum_{0\le\ell\le m} D^\ell_x P_{\ell,i,j}(x,\theta,D_\theta)\,,\ P_{\ell,i,j} \in \mathcal{C}^\infty(0,\delta)_x\,;\, \mathrm{Diff}^{m-\ell}(\mathbb{S}^{n-1}). \tag{5.7}$$

Notice that we *do* know that the coefficients are smooth in $0 < x < \delta$, since we are applying a diffeomorphism there. Moreover, the operators $P_{\ell,i,j}$ are uniquely determined by (5.7).

So we can exploit the assumed homogeneity of $P_{ij}$. This means that for any $t > 0$, the transformation $z \mapsto tz$ gives

$$P_{ij} f(tz) = t^m (P_{ij} f)(tz)\,. \tag{5.8}$$

Since $|tz| = t|z|$, this means that the transformed operator must satisfy (5.9)

$$\sum_\ell D^\ell_x P_{\ell,i,j}(x,\theta,D_\theta) f(x/t,\theta) = t^m (\sum_\ell D^\ell P_{\ell,i,j}(\cdot,\theta,D_\theta) f(\cdot,\theta))(x/t)\,.$$

Expanding this out we conclude that

$$x^{-m-\ell} P_{\ell,i,j}(x,\theta,D_\theta) = P_{\ell,i,j}(\theta,D_\theta) \tag{5.10}$$

is independent of $x$. Thus in fact (5.7) becomes

$$P_{ij} = x^m \sum_{0\le j\le\ell} x^\ell D^\ell_x P_{\ell,j,i}(\theta,D_\theta)\,. \tag{5.11}$$

Since we can rewrite

$$x^\ell D_x = \sum_{0\le j\le\ell} C_{\ell,j} (xD_x)^j \tag{5.12}$$

(with explicit coefficients if you want) this gives (5.3). Ellipticity in this sense, meaning that

$$x^{-m}P_R \in \mathrm{Diff}^m_{\mathrm{b}}(\mathbb{S}^{n,1};\mathbb{C}^N) \tag{5.13}$$

(5.11) and the original ellipticity at $P$. Namely, when expressed in terms of $xD_x$ the coefficients of 5.13 are independent of $x$ (this of course just reflects the homogeneity), ellipticity in $x > 0$ follows by the coordinate independence of ellipticity, and hence extends down to $x = 0$. □

Now the coefficient function $Z_0^{w+m}$ in (5.3) always gives an isomorphism

$$\times Z_0^m : Z_0^w H^s_{\mathrm{b}}(\mathbb{S}^{n,1}) \longrightarrow Z_0^{w+m} H^s_{\mathrm{b}}(\mathbb{S}^{n,1})\,. \tag{5.14}$$

Combining this with the results of Section 3 we find most of

Theorem 5.2. *If $P$ is an $N \times N$ matrix of constant coefficient differential operators on $\mathbb{R}^n$ which is elliptic and homogeneous of degree $-m$ then there is a discrete set $-\operatorname{Im}(D(P)) \subset \mathbb{R}$ such that*

(5.15)

$$P : Z_0^w H_b^{m+s}(\mathbb{S}^{n,1}) \longrightarrow Z_0^{w+m} H_b^s(\mathbb{S}^{n,1}) \text{ is Fredholm } \forall\ w \notin -\operatorname{Im}(D(P))$$

*where* (5.4) *is used to pull these spaces back to $\mathbb{R}^n$. Moreover,*

$$\begin{aligned} &P \text{ is injective for } w \in [0,\infty) \text{ and} \\ &P \text{ is surjective for } w \in (-\infty, n-m] \cap (-\operatorname{Im}(D)(P))\,. \end{aligned} \tag{5.16}$$

Proof. The conclusion (5.15) is exactly what we get by applying Theorem X knowing (5.3).

To see the specific restriction (5.16) on the null space and range, observe that the domain spaces in (5.15) are tempered. Thus the null space is contained in the null space on $\mathcal{S}'(\mathbb{R}^n)$. Fourier transform shows that $P(\zeta)\hat{u}(\zeta) = 0$. From the assumed ellipticity of $P$ and homogeneity it follows that $\operatorname{supp}(\hat{u}(\zeta)) \subset \{0\}$ and hence $\hat{u}$ is a sum of derivatives of delta functions and finally that $u$ itself is a polynomial. If $w \geq 0$ the domain in (5.15) contains no polynomials and the first part of (5.16) follows.

The second part of (5.16) follows by a duality argument. Namely, the adjoint of $P$ with respect to $L^2(\mathbb{R}^n)$, the usual Lebesgue space, is $P^*$ which is another elliptic homogeneous differential operator with constant coefficients. Thus the first part of (5.16) applies to $P^*$. Using the homogeneity of Lebesgue measure,

$$|dz| = \frac{dx}{x^{n+1}} \cdot \nu_\theta \text{ near } \infty \tag{5.17}$$

and the shift in weight in (5.15), the second part of (5.16) follows. □

One important consequence of this is a result going back to Nirenberg and Walker (although expressed in different language).

COROLLARY 5.3. *If $P$ is an elliptic $N \times N$ matrix constant coefficient differential operator which is homogeneous of degree $m$, with $n > m$, the the map* (5.15) *is an isomorphism for $w \in (0, n-m)$.*

In particular this applies to the Laplacian in dimensions $n > 2$ and to the constant coefficient Dirac operators discussed above in dimensions $n > 1$. In these cases it is also straightforward to compute the index and to identify the surjective set. Namely, for a constant coefficient Dirac operator

$$D(P) = i\mathbb{N}_0 \cup i(n - m + \mathbb{N}_0)\,. \tag{5.18}$$

Figure goes here.

## 6. Scattering structure

Let me briefly review how the main result of Section 5 was arrived at. To deal with a constant coefficient Dirac operator we first radially compactified $\mathbb{R}^n$ to a ball, then peeled off a multiplicative factor $Z_0$ from the operator showed that the remaining operator was Fredholm by identifing a neighbourhood of the boundary with part of $\mathbb{R}\times\mathbb{S}^{n-1}$ using the exponential map to exploit the results of Section 1 near infinity. Here we will use a similar, but different, procedure to treat a different class of operators which are Fredholm on the *standard* Sobolev spaces.

Although we will only apply this in the case of a ball, coming from $\mathbb{R}^n$, I cannot resist carrying out the discussed for a general compact manifolds — since $I$ think the generality clarifies what is going on. Starting from a compact manifold with boundary, $M$, the first step is essentially the reverse of the radial compactification of $\mathbb{R}^n$.

Near any point on the boundary, $p \in \partial M$, we can introduce 'admissible' coordinates, $x, y_1, \dots, y_{n-1}$ where $\{x = 0\|$ is the local form of the boundary and $y_1, \dots, y_{n-1}$ are tangential coordinates; we normalize $y_1 = \cdots = y_{n-1} = 0$ at $p$. By reversing the radial compactification of $\mathbb{R}^n$ I mean we can introduce a diffeomorphism of a neighbourhood of $p$ to a conic set in $\mathbb{R}^n$ :

$$z_n = 1/x\,,\ z_j = y_j/x\,,\ j = 1, \dots, n-1\,. \tag{6.1}$$

Clearly the 'square' $|y| < \epsilon$, $0 < x < \epsilon$ is mapped onto the truncated conic set

$$z_n \geq 1/\epsilon\,,\ |z'| < \epsilon|z_n|\,, z' = (z_1, \dots, z_{n-1})\,. \tag{6.2}$$

Definition 6.1. *We define spaces $H^s_{sc}(M)$ for any compact manifold with boundary $M$ by the requirements*

$$u \in H^s_{sc}(M) \Longleftrightarrow u \in H^s_{\mathrm{loc}}(M \setminus \partial M) \text{ and } R^*_j(\varphi_j u) \in H^s(\mathbb{R}^n) \tag{6.3}$$

*for $\varphi_j \in \mathcal{C}^\infty(M)$, $0 \le \varphi_i \le 1$, $\sum \varphi_i = 1$ in a neighbourhood of the boundary and where each $\varphi_j$ is supported in a coordinate patch* (**??**), (6.2) *with $R$ given by* (6.1).

Of course such a definition would not make much sense if it depended on the choice of the partition of unity near the boundary $\{\varphi_i\|$ or the choice of coordinate. So really (6.1) should be preceded by such an invariance statement. The key to this is the following observation.

Proposition 6.2. *If we set $\mathcal{V}_{\mathrm{sc}}(M) = x\mathcal{V}_{\mathrm{b}}(M)$ for any compact manifold with boundary then for any $\psi \in \mathcal{C}^\infty(M)$ supported in a coordinate patch* (**??**), *and any $\mathcal{C}^\infty$ vector field $V$ on $M$*

$$\psi V \in \mathcal{V}_{\mathrm{sc}}(M) \Longleftrightarrow \psi V = \sum_{j=1}^n \mu_j (R^{-1})_*(D_{z_j}),\ \mu_j \in \mathcal{C}^\infty(M). \tag{6.4}$$

Proof. The main step is to compute the form of $D_{z_j}$ in terms of the coordinate obtained by inverting (6.1). Clearly

$$D_{z_n} = x^2 D_x\,,\ D_{z_j} = xD_{y_j} - y_i x^2 D_x\,,\ j < n\,. \tag{6.5}$$

Now, as discussed in Section 3, $xD_x$ and $D_{y_j}$ locally span $\mathcal{V}_{\mathrm{b}}(M)$, so $x^2D_x$, $xD_{y_j}$ locally span $\mathcal{V}_{\mathrm{sc}}(M)$. Thus (6.5) shows that in the singular coordinates (6.1), $\mathcal{V}_{\mathrm{sc}}(M)$ is spanned by the $D_{z_\ell}$, which is exactly what (6.4) claims. ☐

Next let's check what happens to Euclidean measure under $R$, actually we did this before:

$$|dz| = \frac{|dx|}{x^{n+1}}\nu_y\,. \tag{SS.9}$$

Thus we can first identify what (6.3) means in the case of $s = 0$.

Lemma 6.3. *For $s = 0$, Definition* (6.1) *unambiguously defines*

$$H^0_{sc}(M) = \left\{u \in L^2_{\mathrm{loc}}(M)\,;\ \int |u|^2 \frac{\nu_M}{x^{n+1}} < \infty\right\} \tag{6.6}$$

*where $\nu_M$ is a positive smooth density on $M$ (smooth up to the boundary of course) and $x \in \mathcal{C}^\infty(M)$ is a boundary defining function.*

Proof. This is just what (6.3) and (SS.9) mean. ☐

Combining this with Proposition 6.2 we can see directly what (6.3) means for $kin\mathbb{N}$.

LEMMA 6.4. *If* (6.3) *holds for* $s = k \in \mathbb{N}$ *for any one such partition of unity then* $u \in H^0_{sc}(M)$ *in the sense of* (6.6) *and*

$$V_1 \dots V_j u \in H^0_{sc}(M) \ \forall \ V_i \in \mathcal{V}_{\mathrm{sc}}(M) \ \text{if } j \le k\,, \tag{6.7}$$

*and conversely.*

PROOF. For clarity we can proceed by induction on $k$ and replace (6.7) by the statements that $u \in H^{k-1}_{\mathrm{sc}}(M)$ and $Vu \in H^{k-1}_{\mathrm{sc}}(M)$ $\forall V \in \mathcal{V}_{\mathrm{sc}}(M)$. In the interior this is clear and follows immediately from Proposition 6.2 provided we carry along the inductive statement that

$$\mathcal{C}^\infty(M) \text{ acts by multiplication on } H^k_{\mathrm{sc}}(M)\,. \tag{6.8}$$

□

As usual we can pass to general $s \in \mathbb{R}$ by treating the cases $0 < s < 1$ first and then using the action of the vector fields.

PROPOSITION 6.5. *For* $0 < s < 1$ *the condition* (6.3) *(for any one partition of unity) is equivalent to requiring* $u \in H^0_{sc}(M)$ *and*

$$\iint_{M\times M} \frac{|u(p) - u(p')|^2}{\rho_{sc}^{n+2s}} \frac{\nu_M}{x^{n+1}} \frac{\nu'_M}{(x')^{n+1}} < \infty \tag{6.9}$$

*where* $\rho_{sc}(p, p') = \chi\chi' p(p, p') + \sum_j \varphi_j \varphi'_j \langle z - z' \rangle$.

PROOF. Use local coordinates. □

Then for $s \ge 1$ if $k$ is the integral part of $s$, so $0 \le s - k < 1$, $k \in \mathbb{N}$,

$$u \in H^s_{\mathrm{sc}}(M) \Longleftrightarrow V_1, \dots, V_j u \in H^{s-k}_{\mathrm{sc}}(M)\,, V_i \in \mathcal{V}_{\mathrm{sc}}(M)\,,\ j \le k \tag{6.10}$$

and for $s < 0$ if $k \in \mathbb{N}$ is chosen so that $0 \le k + s < 1$, then

$$\begin{gathered} u \in H^s_{\mathrm{sc}}(M) \Leftrightarrow \exists\ V_j \in H^{s+k}_{\mathrm{sc}}(M)\,,\ j = 1, \dots, \mathbb{N}\,, \\ u_j \in H^{s-k}_{\mathrm{sc}}(M)\,, V_{j,i}(M)\,,\ 1 \le i \le \ell_j \le k \text{ s.t.} \\ u = u_0 + \sum_{j=1}^N V_{j,i} \cdots V_{j,\ell_j} u_j. \end{gathered} \tag{6.11}$$

All this complexity is just because we are preceding in such a 'low-tech' fashion. The important point is that these Sobolev spaces are determined by the choice of 'structure vector fields', $V \in \mathcal{V}_{\mathrm{sc}}(M)$. I leave it as an important exercise to check that

LEMMA 6.6. *For the ball, or half-sphere,*

$$\gamma_R^* H^s_{sc}(\mathbb{S}^{n,1}) = H^s(\mathbb{R}^n)\,.$$

Thus on Euclidean space we have done *nothing*. However, my claim is that we understand things better by doing this! The idea is that we should Fourier analysis on $\mathbb{R}^n$ to analyse differential operators which are made up out of $\mathcal{V}_{\mathrm{sc}}(M)$ on any compact manifold with boundary $M$, and this includes $\mathbb{S}^{n,1}$ as the radial compactification of $\mathbb{R}^n$. Thus set

$$\begin{aligned}(6.12)\quad \mathrm{Diff}^m_{\mathrm{sc}}(M) = \big\{P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M);\ \exists\ f \in \mathcal{C}^\infty(M) \text{ and}\\ V_{i,j} \in \mathcal{V}_{\mathrm{sc}}(M) \text{ s.t. } P = f + \sum_{i,1\le j\le m} V_{i,1}\dots V_{i,j}\big\}.\end{aligned}$$

In local coordinates this is just a differential operator and it is smooth up to the boundary. Since only scattering vector fields are allowed in the definition such an operator is quite degenerate at the boundary. It always looks like

$$P = \sum_{k+|\alpha|\le m} a_{k,\alpha}(x,y)(x^2D_x)^k(xD_y)^\alpha, \tag{6.13}$$

with smooth coefficients in terms of local coordinates (**??**).

Now, if we *freeze* the coefficients at a point, $p$, on the boundary of $M$ we get a polynomial

$$\sigma_{\mathrm{sc}}(P)(p) = \sum_{k+|\alpha|\le m} a_{k,\alpha}(p)\tau^k\eta^\alpha. \tag{6.14}$$

Note that this is *not* in general homogeneous since the lower order terms are retained. Despite this one gets essentially the same polynomial at each point, independent of the admissible coordinates chosen, as will be shown below. Let's just assume this for the moment so that the condition in the following result makes sense.

THEOREM 6.7. *If $P \in \mathrm{Diff}^m_{sc}(M;\mathbb{E})$ acts between vector bundles over $M$, is elliptic in the interior and each of the polynomials (matrices)* (6.14) *is elliptic and has no real zeros then*

$$P : H^{s+m}_{sc}(M,E_1) \longrightarrow H^s_{sc}(M;E_2) \text{ is Fredholm} \tag{6.15}$$

*for each $s \in \mathbb{R}$ and conversely.*

Last time at the end I gave the following definition and theorem.

DEFINITION 6.8. *We define weighted (non-standard) Sobolev spaces for $(m,w) \in \mathbb{R}^2$ on $\mathbb{R}^n$ by*

$$\tilde H^{m,w}(\mathbb{R}^n) = \{u \in M^m_{\mathrm{loc}}(\mathbb{R}^n); F^*\left((1-\chi)r^{-w}u\right) \in H^m_{\mathrm{ti}}(\mathbb{R}\times\mathbb{S}^{n-1})\} \tag{6.16}$$

*where $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$, $\chi(y) = 1$ in $|y| < 1$ and*

$$F : \mathbb{R}\times\mathbb{S}^{n-1} \ni (t,\theta) \longrightarrow (e^t, e^t\theta) \in \mathbb{R}^n\setminus\{0\}. \tag{6.17}$$

THEOREM 6.9. *If* $P = \sum_{i=1}^{n} \Gamma_i D_i$, $\Gamma_i \in M(N, \mathbb{C})$, *is an elliptic, constant coefficient, homogeneous differential operator of first order then*

$$P : \tilde{H}^{m,w}(\mathbb{R}^n) \longrightarrow \tilde{H}^{m-1,w+1}(\mathbb{R}^n) \ \forall \ (m, w) \in \mathbb{R}^2 \tag{6.18}$$

*is continuous and is Fredholm for* $w \in \mathbb{R} \setminus \tilde{D}$ *where* $\tilde{D}$ *is discrete.*

*If* $P$ *is a Dirac operators, which is to say explicitly here that the coefficients are 'Pauli matrices' in the sense that*

$$\Gamma_i^* = \Gamma_i, \ \Gamma_i^2 = \mathrm{Id}_{N \times N}, \ \forall \ i, \ \Gamma_i\Gamma_j + \Gamma_j\Gamma_i = 0, \ i \neq j, \tag{6.19}$$

*then*

$$\tilde{D} = -\mathbb{N}_0 \cup (n - 2 + \mathbb{N}_0) \tag{6.20}$$

*and if* $n > 2$ *then for* $w \in (0, n-2)$ *the operator* $P$ *in* (6.18) *is an isomorphism.*

I also proved the following result from which this is derived

LEMMA 6.10. *In polar coordinates on* $\mathbb{R}^n$ *in which* $\mathbb{R}^n \setminus \{0\} \simeq (0, \infty) \times \mathbb{S}^{n-1}$, $y = r\theta$,

$$D_{y_j} = \tag{6.21}$$

CHAPTER 9

# Electromagnetism

## 1. Maxwell's equations

Maxwell's equations in a vacuum take the standard form

$$\begin{aligned} &\operatorname{div}\mathbf{E} = \rho \qquad &&\operatorname{div}\mathbf{B} = 0 \\ &\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad &&\operatorname{curl}\mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \end{aligned} \tag{1.1}$$

where $\mathbf{E}$ is the electric and $\mathbf{B}$ the magnetic field strength, both are 3-vectors depending on position $z \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. The external quantities are $\rho$, the charge density which is a scalar, and $\mathbf{J}$, the current density which is a vector.

We will be interested here in stationary solutions for which $\mathbf{E}$ and $\mathbf{B}$ are independent of time and with $\mathbf{J} = 0$, since this also represents motion in the standard description. Thus we arrive at

$$\begin{aligned} &\operatorname{div}\mathbf{E} = \rho \qquad &&\operatorname{div}\mathbf{B} = 0 \\ &\operatorname{curl}\mathbf{E} = 0 \qquad &&\operatorname{curl}\mathbf{B} = 0. \end{aligned} \tag{1.2}$$

The simplest interesting solutions represent charged particles, say with the charge at the origin, $\rho = c\delta_0(z)$, and with no magnetic field, $\mathbf{B} = 0$. By identifying $\mathbf{E}$ with a 1-form, instead of a vector field on $\mathbb{R}^3$,

$$\mathbf{E} = (E_1, E_2, E_3) \Longrightarrow e = E_1 dz_1 + E_2 dz_2 + E_3 dz_3 \tag{1.3}$$

we may identify $\operatorname{curl}\mathbf{E}$ with the 2-form $de$,

$$de = \left(\frac{\partial E_2}{\partial z_1} - \frac{\partial E_1}{\partial z_2}\right) dz_1 \wedge dz_2 + \left(\frac{\partial E_3}{\partial z_2} - \frac{\partial E_2}{\partial z_3}\right) dz_2 \wedge dz_3 + \left(\frac{\partial E_1}{\partial z_3} - \frac{\partial E_3}{\partial z_1}\right) dz_3 \wedge dz_1. \tag{1.4}$$

Thus (1.2) implies that $e$ is a closed 1-form, satisfying

$$\frac{\partial E_1}{\partial z_1} + \frac{\partial E_2}{\partial z_2} + \frac{\partial E_3}{\partial z_3} = c\delta_0(z). \tag{1.5}$$

By the Poincaré Lemma, a closed 1-form on $\mathbb{R}^3$ is exact, $e = dp$, with $p$ determined up to an additive constant. If $e$ is smooth (which it cannot be, because of (1.5)), then

$$p(z) - p(z') = \int_0^1 \gamma^* e \qquad \text{along } \gamma : [0,1] \longrightarrow \mathbb{R}^3, \gamma(0) = z', \gamma(1) = z. \tag{1.6}$$

It is reasonable to look for a particular $p$ and 1-form $e$ which satisfy (1.5) and are smooth outside the origin. Then (1.6) gives a potential which is well defined, up to an additive constant, outside 0, once $z'$ is fixed, since $de = 0$ implies that the integral of $\gamma^* e$ along a closed curve vanishes. This depends on the fact that $\mathbb{R}^3 \backslash \{0\}$ is simply connected.

So, modulo confirmation of these simple statements, it suffices to look for $p \in \mathcal{C}^\infty(\mathbb{R}^3 \backslash \{0\})$ satisfying $e = dp$ and (1.5), so

$$\Delta p = -\left( \frac{\partial^2 p}{\partial z_1^2} + \frac{\partial^2 p}{\partial z_2^2} + \frac{\partial^2 p}{\partial z_3^2} \right) = -c\delta_0(z). \tag{1.7}$$

Then $\mathbf{E}$ is recovered from $e = dp$.

The operator 'div' can also be understood in terms of de Rham $d$ together with the Hodge star $*$. If we take $\mathbb{R}^3$ to have the standard orientation and Euclidean metric $dz_1^2 + dz_2^2 + dz_3^2$, the Hodge star operator is given on 1-forms by

$$*dz_1 = dz_2 \wedge dz_3, \quad *dz_2 = dz_3 \wedge dz_1, \quad *dz_3 = dz_1 \wedge dz_2. \tag{1.8}$$

Thus $*e$ is a 2-form,

$$\begin{gathered} * \, e = E_1 \, dz_2 \wedge dz_3 + E_2 \, dz_3 \wedge dz_1 + E_3 \, dz_1 \wedge dz_2 \\ \Longrightarrow d{*}e = \left( \frac{\partial E_1}{\partial z_1} + \frac{\partial E_2}{\partial z_2} + \frac{\partial E_3}{\partial z_3} \right) dz_1 \wedge dz_2 \wedge dz_3 = (\operatorname{div} \mathbf{E}) \, dz_1 \wedge dz_2 \wedge dz_3. \end{gathered} \tag{1.9}$$

The stationary Maxwell's equations on $e$ become

$$d * e = \rho \, dz_1 \wedge dz_2 \wedge dz_3, \ de = 0. \tag{1.10}$$

There is essential symmetry in (1.1) except for the appearance of the "source" terms, $\rho$ and $\mathbf{J}$. To reduce (1.1) to two equations, analogous to (1.10) but in 4-dimensional (Minkowski) space requires $\mathbf{B}$ to be identified with a 2-form on $\mathbb{R}^3$, rather than a 1-form. Thus, set

$$\beta = B_1 \, dz_2 \wedge dz_3 + B_2 \, dz_3 \wedge dz_1 + B_3 \, dz_1 \wedge dz_2. \tag{1.11}$$

Then

$$d\beta = \operatorname{div} \mathbf{B} \, dz_1 \wedge dz_2 \wedge dz_3 \tag{1.12}$$

as follows from (1.9) and the second equation in (1.1) implies $\beta$ is closed.

Thus $e$ and $\beta$ are respectively a closed 1-form and a closed 2-form on $\mathbb{R}^3$. If we return to the general time-dependent setting then we may define a 2-form on $\mathbb{R}^4$ by

$$\lambda = e \wedge dt + \beta \tag{1.13}$$

where $e$ and $\beta$ are pulled back by the projection $\pi : \mathbb{R}^4 \to \mathbb{R}^3$. Computing directly,

$$d\lambda = d'e \wedge dt + d'\beta + \frac{\partial \beta}{\partial t} \wedge dt \tag{1.14}$$

where $d'$ is now the differential on $\mathbb{R}^3$. Thus

$$d\lambda = 0 \Leftrightarrow d'e + \frac{\partial \beta}{\partial t} = 0, \quad d'\beta = 0 \tag{1.15}$$

recovers two of Maxwell's equations. On the other hand we can define a 4-dimensional analogue of the Hodge star but corresponding to the Minkowski metric, not the Euclidean one. Using the natural analogue of the 3-dimensional Euclidean Hodge by formally inserting an $i$ into the $t$-component, gives

$$\begin{cases} *_4 dz_1 \wedge dz_2 = i dz_3 \wedge dt \\ *_4 dz_1 \wedge dz_3 = i dt \wedge dz_2 \\ *_4 dz_1 \wedge dt = -i dz_2 \wedge dz_3 \\ *_4 dz_2 \wedge dz_3 = i dz_1 \wedge dt \\ *_4 dz_2 \wedge dt = -i dz_3 \wedge dz_1 \\ *_4 dz_3 \wedge dt = -i dz_1 \wedge dz_2. \end{cases} \tag{1.16}$$

The other two of Maxwell's equations then become

$$d *_4 \lambda = d(-i * e + i(*\beta) \wedge dt) = -i(\rho \, dz_1 \wedge dz_2 \wedge dz_3 + j \wedge dt) \tag{1.17}$$

where $j$ is the 1-form associated to $\mathbf{J}$ as in (1.3). For our purposes this is really just to confirm that it is best to think of $\mathbf{B}$ as the 2-form $\beta$ rather than try to make it into a 1-form. There are other good reasons for this, related to behaviour under linear coodinate changes.

Returning to the stationary setting, note that (1.7) has a 'preferred' solution

$$p = \frac{1}{4\pi |z|}. \tag{1.18}$$

This is in fact the only solution which vanishes at infinity.

Proposition 1.1. *The only tempered solutions of* (1.7) *are of the form*

$$p = \frac{1}{4\pi |z|} + q, \;\; \Delta q = 0, \;\; q \text{ a polynomial.} \tag{1.19}$$

PROOF. The only solutions are of the form (1.19) where $q \in \mathcal{S}'(\mathbb{R}^3)$ is harmonic. Thus $\widehat{q} \in \mathcal{S}'(\mathbb{R}^3)$ satisfies $|\xi|^2\widehat{q} = 0$, which implies that $q$ is a polynomial. $\square$

## 2. Hodge Theory

The Hodge $*$ operator discussed briefly above in the case of $\mathbb{R}^3$ (and Minkowski 4-space) makes sense in any oriented real vector space, $V$, with a Euclidean inner product—that is, on a finite dimensional real Hilbert space. Namely, if $e_1, \dots, e_n$ is an oriented orthonormal basis then

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \operatorname{sgn}(i_*) e_{i_{k+1}} \wedge \cdots e_{i_n} \tag{2.1}$$

extends by linearity to

$$* : \bigwedge^k V \longrightarrow \bigwedge^{n-k} V. \tag{2.2}$$

PROPOSITION 2.1. *The linear map* (2.2) *is independent of the oriented orthonormal basis used to define it and so depends only on the choice of inner product and orientation of* $V$. *Moreover,*

$$*^2 = (-1)^{k(n-k)}, \text{ on } \bigwedge^k V. \tag{2.3}$$

PROOF. Note that $\operatorname{sgn}(i_*)$, the sign of the permutation defined by $\{i_1, \dots, i_n\}$ is fixed by

$$e_{i_1} \wedge \cdots \wedge e_{i_n} = \operatorname{sgn}(i_*) e_1 \wedge \cdots \wedge e_n. \tag{2.4}$$

Thus, on the basis $e_{i_1} \wedge \ldots \wedge e_{i_n}$ of $\bigwedge^k V$ given by strictly increasing sequences $i_1 < i_2 < \cdots < i_k$ in $\{1, \dots, n\}$,

$$e_* \wedge *e_* = \operatorname{sgn}(i_*)^2 e_1 \wedge \cdots \wedge e_n = e_1 \wedge \cdots \wedge e_n. \tag{2.5}$$

The standard inner product on $\bigwedge^k V$ is chosen so that this basis is orthonormal. Then (2.5) can be rewritten

$$e_I \wedge *e_J = \langle e_I, e_J \rangle e_1 \wedge \cdots \wedge e_n. \tag{2.6}$$

This in turn fixes $*$ uniquely since the pairing given by

$$\bigwedge^k V \times \bigwedge^{k-1} V \ni (u, v) \mapsto (u \wedge v)/_{e_1 \wedge \cdots \wedge e_n} \tag{2.7}$$

is non-degenerate, as can be checked on these bases.

Thus it follows from (2.6) that $*$ depends only on the choice of inner product and orientation as claimed, provided it is shown that the inner product on $\bigwedge^k V$ only depends on that of $V$. This is a standard fact following from the embedding

$$\bigwedge^k V \hookrightarrow V^{\otimes k} \tag{2.8}$$

as the totally antisymmetric part, the fact that $V^{\otimes k}$ has a natural inner product and the fact that this induces one on $\bigwedge^k V$ after normalization (depending on the convention used in (2.8). These details are omitted. □

Since $*$ is uniquely determined in this way, it necessarily depends smoothly on the data, in particular the inner product. On an oriented Riemannian manifold the induced inner product on $T^*_pM$ varies smoothly with $p$ (by assumption) so

$$*:\bigwedge\nolimits^k_p M \longrightarrow \bigwedge\nolimits^{n-k}_p M,\ \bigwedge\nolimits^k_p M = \bigwedge\nolimits^k_p(T^*_pM) \tag{2.9}$$

varies smoothly and so defines a smooth bundle map

$$* \in \mathcal{C}^\infty(M;\bigwedge\nolimits^k M, \bigwedge\nolimits^{n-k} M). \tag{2.10}$$

An oriented Riemannian manifold carries a natural volume form $\nu \in \mathcal{C}^\infty(M, \bigwedge^n M)$, and this allows (2.6) to be written in integral form:

$$\int_M \langle\alpha,\beta\rangle\,\nu = \int_M \alpha\wedge *\beta\ \forall\,\alpha,\beta\in\mathcal{C}^\infty(M,\bigwedge\nolimits^k M). \tag{2.11}$$

Lemma 2.2. *On an oriented, (compact) Riemannian manifold the adjoint of $d$ with respect to the Riemannian inner product and volume form is*

$$d^* = \delta = (-1)^{k+n(n-k+1)} *d* \ \text{ on } \bigwedge\nolimits^k M. \tag{2.12}$$

Proof. By definition,

$$\begin{gathered} d:\mathcal{C}^\infty(M,\bigwedge\nolimits^k M)\longrightarrow \mathcal{C}^\infty(M,\bigwedge\nolimits^{k+1} M) \\ \Longrightarrow \delta:\mathcal{C}^\infty(M,\bigwedge\nolimits^{k+1} M)\longrightarrow \mathcal{C}^\infty(M,\bigwedge\nolimits^k M), \\ \int_M\langle d\alpha,\alpha'\rangle\,\nu = \int_M\langle\alpha,\delta\alpha'\rangle\,\nu\,\forall\,\alpha\in\mathcal{C}^\infty(M,\bigwedge\nolimits^k M), \alpha'\in\mathcal{C}^\infty(M,\bigwedge\nolimits^{k+1} M). \end{gathered} \tag{2.13}$$

Applying (2.11) and using Stokes' theorem, (and compactness of either M or the support of at least one of $\alpha,\alpha'$),

$$\int_M\langle\delta\alpha,\alpha'\rangle\,\nu = \int_M d\alpha\wedge *\alpha'$$
$$= \int_M d(\alpha\wedge*\alpha') + (-1)^{k+1}\int_M \alpha\wedge d*\alpha' = 0 + (-1)^{k+1}\int_M\langle\alpha, *^{-1}d*\alpha'\rangle\,\nu.$$

Taking into account (2.3) to compute $*^{-1}$ on $n-k$ forms shows that

$$\delta\alpha' = (-1)^{k+1+n(n-k)} *d* \quad \text{on } (k+1)\text{-forms} \tag{2.14}$$

which is just (2.12) on $k$-forms. □

Notice that changing the orientation simply changes the sign of $*$ on all forms. Thus (2.12) does not depend on the orientation and as a local formula is valid even if $M$ is not orientable — since the existence of $\delta = d^*$ does *not* require $M$ to be orientable.

THEOREM 2.3 (Hodge/Weyl). *On any compact Riemannian manifold there is a canonical isomorphism*

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{Ho}}(M) = \Big\{u \in L^2(M; \textstyle\bigwedge^k M);\ (d+\delta)u = 0\Big\} \tag{2.15}$$

*where the left-hand side is either the $\mathcal{C}^\infty$ or the distributional de Rham cohomology*

$$\begin{aligned}\Big\{u \in \mathcal{C}^\infty(M; \textstyle\bigwedge^k M);\ du = 0\Big\} \Big/ d\,\mathcal{C}^\infty(M; \textstyle\bigwedge^k M) \\ \cong \Big\{u \in \mathcal{C}^{-\infty}(M; \textstyle\bigwedge^k M);\ du = 0\Big\} \Big/ d\,\mathcal{C}^{-\infty}(M; \textstyle\bigwedge^k M).\end{aligned} \tag{2.16}$$

PROOF. The critical point of course is that

$$d + \delta \in \mathrm{Diff}^1(M; \textstyle\bigwedge^* M) \text{ is elliptic.} \tag{2.17}$$

We know that the symbol of $d$ at a point $\zeta \in T^*_p M$ is the map

$$\textstyle\bigwedge^k M \ni \alpha \mapsto i\zeta \wedge \alpha. \tag{2.18}$$

We are only interested in $\zeta \neq 0$ and by homogeneity it is enough to consider $|\zeta| = 1$. Let $e_1 = \zeta,\ e_2,\ \dots,\ e_n$ be an orthonormal basis of $T^*_p M$, then from (2.12) with a fixed sign throughout:

$$\sigma(\delta, \zeta)\alpha = \pm * (i\zeta \wedge \cdot) * \alpha. \tag{2.19}$$

Take $\alpha = e_I$, $*\alpha = \pm e_{I'}$ where $I \cup I' = \{1, \dots, n\}$. Thus

$$\sigma(\delta, \zeta)\alpha = \begin{cases} 0 & 1 \notin I \\ \pm i\alpha_{I\setminus\{1\}} & 1 \in I \end{cases}. \tag{2.20}$$

In particular, $\sigma(d+\delta)$ is an isomorphism since it satisfies

$$\sigma(d+\delta)^2 = |\zeta|^2 \tag{2.21}$$

as follows from (2.18) and (2.20) or directly from the fact that

$$(d+\delta)^2 = d^2 + d\delta + \delta d + \delta^2 = d\delta + \delta d \tag{2.22}$$

again using (2.18) and (2.20).

Once we know that $d + \delta$ is elliptic we conclude from the discussion of Fredholm properties above that the distributional null space

$$\Big\{u \in \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M);\ (d+\delta)u = 0\Big\} \subset \mathcal{C}^\infty(M, \textstyle\bigwedge^* M) \tag{2.23}$$

is finite dimensional. From this it follows that

$$\begin{aligned} H^k_{\mathrm{Ho}} =& \{u \in \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^k M);\ (d+\delta)u = 0\} \\ =& \{u \in \mathcal{C}^{\infty}(M, \textstyle\bigwedge^k M);\ du = \delta u = 0\} \end{aligned} \tag{2.24}$$

and that the null space in (2.23) is simply the direct sum of these spaces over $k$. Indeed, from (2.23) the integration by parts in

$$0 = \int \langle du, (d+\delta)u\rangle\, \nu = \|du\|^2_{L^2} + \int \langle u, \delta^2 u\rangle\, \nu = \|du\|^2_{L^2}$$

is justified.

Thus we can consider $d+\delta$ as a Fredholm operator in three forms

$$\begin{aligned} d+\delta :& \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M) \longrightarrow \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M), \\ d+\delta :& H^1(M, \textstyle\bigwedge^* M) \longrightarrow H^1(M, \textstyle\bigwedge^* M), \\ d+\delta :& \mathcal{C}^{\infty}(M, \textstyle\bigwedge^* M) \longrightarrow \mathcal{C}^{\infty}(M, \textstyle\bigwedge^* M) \end{aligned} \tag{2.25}$$

and obtain the three direct sum decompositions

$$\begin{aligned} \mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)\mathcal{C}^{-\infty}(M, \textstyle\bigwedge^* M), \\ L^2(M, \textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)L^2(M, \textstyle\bigwedge^* M), \\ \mathcal{C}^{\infty}(M, \textstyle\bigwedge^* M) &= H^*_{\mathrm{Ho}} \oplus (d+\delta)\mathcal{C}^{\infty}(M, \textstyle\bigwedge^* M). \end{aligned} \tag{2.26}$$

The same complement occurs in all three cases in view of (2.24).

From (2.24) directly, all the "harmonic" forms in $H^k_{\mathrm{Ho}}(M)$ are closed and so there is a natural map

$$H^k_{\mathrm{Ho}}(M) \longrightarrow H^k_{\mathrm{dR}}(M) \longrightarrow H^k_{\mathrm{dR},\mathcal{C}^{-\infty}}(M) \tag{2.27}$$

where the two de Rham spaces are those in (2.16), not yet shown to be equal.

We proceed to show that the maps in (2.27) are isomorphisms. First to show injectivity, suppose $u \in H^k_{\mathrm{Ho}}(M)$ is mapped to zero in either space. This means $u = dv$ where $v$ is either $\mathcal{C}^\infty$ or distributional, so it suffices to suppose $v \in \mathcal{C}^{-\infty}(M, \bigwedge^{k-1} M)$. Since $u$ is smooth the integration by parts in the distributional pairing

$$\|u\|^2_{L^2} = \int_M \langle u, dv\rangle\, \nu = \int_M \langle \delta u, v\rangle\, \nu = 0$$

is justified, so $u = 0$ and the maps are injective.

To see surjectivity, use the Hodge decomposition (2.26). If $u' \in \mathcal{C}^{-\infty}(M, \bigwedge^k M)$ or $\mathcal{C}^{\infty}(M, \bigwedge^k M)$, we find

$$u' = u_0 + (d+\delta)v \tag{2.28}$$

where correspondingly, $v \in \mathcal{C}^{-\infty}(M, \bigwedge^* M)$ or $\mathcal{C}^{\infty}(M, \bigwedge^* M)$ and $u_0 \in H^k_{\mathrm{Ho}}(M)$. If $u'$ is closed, $du' = 0$, then $d\delta v = 0$ follows from applying

$d$ to (2.28) and hence $(d+\delta)\delta v = 0$, since $\delta^2 = 0$. Thus $\delta v \in H^*_{\mathrm{Ho}}(M)$ and in particular, $\delta v \in \mathcal{C}^\infty(M, \bigwedge^* M)$. Then the integration by parts in

$$\|\delta v\|^2_{L^2} = \int \langle \delta v, \delta v \rangle \, \nu = \int \langle v, (d+\delta)\delta v \rangle \, \nu = 0$$

is justified, so $\delta v = 0$. Then (2.28) shows that any closed form, smooth or distributional, is cohomologous in the same sense to $u_0 \in H^k_{\mathrm{Ho}}(M)$. Thus the natural maps (2.27) are isomorphisms and the Theorem is proved. $\square$

Thus, on a compact Riemannian manifold (whether orientable or not), each de Rham class has a unique harmonic representative.

## 3. Coulomb potential

## 4. Dirac strings

## Addenda to Chapter 9

CHAPTER 10

# Monopoles

## 1. Gauge theory

## 2. Bogomolny equations

(1) Compact operators, spectral theorem
(2) Families of Fredholm operators(*)
(3) Non-compact self-adjoint operators, spectral theorem
(4) Spectral theory of the Laplacian on a compact manifold
(5) Pseudodifferential operators(*)
(6) Invertibility of the Laplacian on Euclidean space
(7) Lie groups(‡), bundles and gauge invariance
(8) Bogomolny equations on $\mathbb{R}^3$
(9) Gauge fixing
(10) Charge and monopoles
(11) Monopole moduli spaces

* I will drop these if it looks as though time will become an issue.
†,‡ I will provide a brief and elementary discussion of manifolds and Lie groups if that is found to be necessary.

## 3. Problems

PROBLEM 1. Prove that $u_+$, defined by (15.10) is linear.

PROBLEM 2. Prove Lemma 15.7.

Hint(s). All functions here are supposed to be continuous, I just don't bother to keep on saying it.

(1) Recall, or check, that the local compactness of a metric space $X$ means that for each point $x \in X$ there is an $\epsilon > 0$ such that the ball $\{y \in X; d(x, y) \leq \delta\}$ is compact for $\delta \leq \epsilon$.
(2) First do the case $n = 1$, so $K \Subset U$ is a compact set in an open subset.
    (a) Given $\delta > 0$, use the local compactness of $X$, to cover $K$ with a finite number of compact closed balls of radius at most $\delta$.

(b) Deduce that if $\epsilon > 0$ is small enough then the set $\{x \in X; d(x, K) \leq \epsilon\}$, where
$$d(x, K) = \inf_{y \in K} d(x, y),$$
is compact.

(c) Show that $d(x, K)$, for $K$ compact, is continuous.

(d) Given $\epsilon > 0$ show that there is a continuous function $g_\epsilon : \mathbb{R} \longrightarrow [0, 1]$ such that $g_\epsilon(t) = 1$ for $t \leq \epsilon/2$ and $g_\epsilon(t) = 0$ for $t > 3\epsilon/4$.

(e) Show that $f = g_\epsilon \circ d(\cdot, K)$ satisfies the conditions for $n = 1$ if $\epsilon > 0$ is small enough.

(3) Prove the general case by induction over $n$.

(a) In the general case, set $K' = K \cap U_1^{\complement}$ and show that the inductive hypothesis applies to $K'$ and the $U_j$ for $j > 1$; let $f'_j$, $j = 2, \dots, n$ be the functions supplied by the inductive assumption and put $f' = \sum_{j \geq 2} f'_j$.

(b) Show that $K_1 = K \cap \{f' \leq \frac{1}{2}\}$ is a compact subset of $U_1$.

(c) Using the case $n = 1$ construct a function $F$ for $K_1$ and $U_1$.

(d) Use the case $n = 1$ again to find $G$ such that $G = 1$ on $K$ and $\operatorname{supp}(G) \Subset \{f' + F > \frac{1}{2}\}$.

(e) Make sense of the functions
$$f_1 = F \frac{G}{f' + F}, \ f_j = f'_j \frac{G}{f' + F}, \ j \geq 2$$
and show that they satisfies the inductive assumptions.

Problem 3. Show that $\sigma$-algebras are closed under countable intersections.

Problem 4. (Easy) Show that if $\mu$ is a complete measure and $E \subset F$ where $F$ is measurable and has measure 0 then $\mu(E) = 0$.

Problem 5. Show that compact subsets are measurable for any Borel measure. (This just means that compact sets are Borel sets if you follow through the tortuous terminology.)

Problem 6. Show that the smallest $\sigma$-algebra containing the sets
$$(a, \infty] \subset [-\infty, \infty]$$
for all $a \in \mathbb{R}$, generates what is called above the 'Borel' $\sigma$-algebra on $[-\infty, \infty]$.

Problem 7. Write down a careful proof of Proposition 1.1.

PROBLEM 8. Write down a careful proof of Proposition 1.2.

PROBLEM 9. Let $X$ be the metric space

$$X = \{0\} \cup \{1/n; n \in \mathbb{N} = \{1, 2, \ldots\}\} \subset \mathbb{R}$$

with the induced metric (i.e. the same distance as on $\mathbb{R}$). Recall why $X$ is compact. Show that the space $\mathcal{C}_0(X)$ and its dual are infinite dimensional. Try to describe the dual space in terms of sequences; at least *guess* the answer.

PROBLEM 10. For the space $Y = \mathbb{N} = \{1, 2, \ldots\} \subset \mathbb{R}$, describe $\mathcal{C}_0(Y)$ and guess a description of its dual in terms of sequences.

PROBLEM 11. Let $(X, \mathcal{M}, \mu)$ be any measure space (so $\mu$ is a measure on the $\sigma$-algebra $\mathcal{M}$ of subsets of $X$). Show that the set of equivalence classes of $\mu$-integrable functions on $X$, with the equivalence relation given by (4.8), is a normed linear space with the usual linear structure and the norm given by

$$\|f\| = \int_X |f| d\mu.$$

PROBLEM 12. Let $(X, \mathcal{M})$ be a set with a $\sigma$-algebra. Let $\mu : \mathcal{M} \to \mathbb{R}$ be a finite measure in the sense that $\mu(\phi) = 0$ and for any $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M}$ with $E_i \cap E_j = \phi$ for $i \neq j$,

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \tag{3.1}$$

with the series on the right *always* absolutely convergenct (i.e., this is part of the requirement on $\mu$). Define

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \tag{3.2}$$

for $E \in \mathcal{M}$, with the supremum over *all* measurable decompositions $E = \bigcup_{i=1}^{\infty} E_i$ with the $E_i$ disjoint. Show that $|\mu|$ is a finite, positive measure.

**Hint 1.** You must show that $|\mu|(E) = \sum_{i=1}^{\infty} |\mu|(A_i)$ if $\bigcup_i A_i = E$, $A_i \in \mathcal{M}$ being disjoint. Observe that if $A_j = \bigcup_l A_{jl}$ is a measurable decomposition of $A_j$ then together the $A_{jl}$ give a decomposition of $E$. Similarly, if $E = \bigcup_j E_j$ is any such decomposition of $E$ then $A_{jl} = A_j \cap E_l$ gives such a decomposition of $A_j$.

**Hint 2.** See [**6**] p. 117!

PROBLEM 13. (Hahn Decomposition)

With assumptions as in Problem 12:

(1) Show that $\mu_+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu_- = \frac{1}{2}(|\mu| - \mu)$ are positive measures, $\mu = \mu_+ - \mu_-$. Conclude that the definition of a measure based on (4.16) is the *same* as that in Problem 12.
(2) Show that $\mu_\pm$ so constructed are orthogonal in the sense that there is a set $E \in \mathcal{M}$ such that $\mu_-(E) = 0$, $\mu_+(X \setminus E) = 0$.

**Hint.** Use the definition of $|\mu|$ to show that for any $F \in \mathcal{M}$ and any $\epsilon > 0$ there is a subset $F' \in \mathcal{M}$, $F' \subset F$ such that $\mu_+(F') \geq \mu_+(F) - \epsilon$ and $\mu_-(F') \leq \epsilon$. Given $\delta > 0$ apply this result repeatedly (say with $\epsilon = 2^{-n}\delta$) to find a decreasing sequence of sets $F_1 = X$, $F_n \in \mathcal{M}$, $F_{n+1} \subset F_n$ such that $\mu_+(F_n) \geq \mu_+(F_{n-1}) - 2^{-n}\delta$ and $\mu_-(F_n) \leq 2^{-n}\delta$. Conclude that $G = \bigcap_n F_n$ has $\mu_+(G) \geq \mu_+(X) - \delta$ and $\mu_-(G) = 0$. Now let $G_m$ be chosen this way with $\delta = 1/m$. Show that $E = \bigcup_m G_m$ is as required.

Problem 14. Now suppose that $\mu$ is a finite, positive Radon measure on a locally compact metric space $X$ (meaning a finite positive Borel measure outer regular on Borel sets and inner regular on open sets). Show that $\mu$ is inner regular on all Borel sets and hence, given $\epsilon > 0$ and $E \in \mathcal{B}(X)$ there exist sets $K \subset E \subset U$ with $K$ compact and $U$ open such that $\mu(K) \geq \mu(E) - \epsilon$, $\mu(E) \geq \mu(U) - \epsilon$.

**Hint.** First take $U$ open, then use *its* inner regularity to find $K$ with $K' \Subset U$ and $\mu(K') \geq \mu(U) - \epsilon/2$. How big is $\mu(E \backslash K')$? Find $V \supset K' \backslash E$ with $V$ open and look at $K = K' \backslash V$.

Problem 15. Using Problem 14 show that if $\mu$ is a finite Borel measure on a locally compact metric space $X$ then the following three conditions are equivalent

(1) $\mu = \mu_1 - \mu_2$ with $\mu_1$ and $\mu_2$ both positive finite Radon measures.
(2) $|\mu|$ is a finite positive Radon measure.
(3) $\mu_+$ and $\mu_-$ are finite positive Radon measures.

Problem 16. Let $\| \ \|$ be a norm on a vector space $V$. Show that $\|u\| = (u, u)^{1/2}$ for an inner product satisfying (1.1) - (1.4) if and only if the parallelogram law holds for every pair $u, v \in V$.

Hint (From Dimitri Kountourogiannis)

If $\| \cdot \|$ comes from an inner product, then it must satisfy the polarisation identity:

$$(x, y) = 1/4(\|x + y\|^2 - \|x - y\|^2 - i\|x + iy\|^2 - i\|x - iy\|^2)$$

i.e, the inner product is recoverable from the norm, so use the RHS (right hand side) to define an inner product on the vector space. You will need the paralellogram law to verify the additivity of the RHS. Note the polarization identity is a bit more transparent for real vector spaces. There we have

$$(x, y) = 1/2(\|x+y\|^2 - \|x-y\|^2)$$

both are easy to prove using $\|a\|^2 = (a, a)$.

Problem 17. Show (Rudin does it) that if $u : \mathbb{R}^n \to \mathbb{C}$ has continuous partial derivatives then it is differentiable at each point in the sense of (6.19).

Problem 18. Consider the function $f(x) = \langle x\rangle^{-1} = (1+|x|^2)^{-1/2}$. Show that

$$\frac{\partial f}{\partial x_j} = l_j(x) \cdot \langle x\rangle^{-3}$$

with $l_j(x)$ a linear function. Conclude by *induction* that $\langle x\rangle^{-1} \in \mathcal{C}_0^k(\mathbb{R}^n)$ for *all* $k$.

Problem 19. Show that $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$.

Problem 20. Prove (2.8), probably by induction over $k$.

Problem 21. Prove Lemma 2.4.

*Hint.* Show that a set $U \ni 0$ in $\mathcal{S}(\mathbb{R}^n)$ is a neighbourhood of 0 if and only if for some $k$ and $\epsilon > 0$ it contains a set of the form

$$\left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) \,;\, \sum_{\substack{|\alpha|\le k,\\ |\beta|\le k}} \sup \left| x^\alpha D^\beta \varphi \right| < \epsilon \right\} .$$

Problem 22. Prove (3.7), by estimating the integrals.

Problem 23. Prove (3.9) where

$$\psi_j(z; x') = \int_0^{\prime} \frac{\partial \psi}{\partial z_j}(z + tx')\, dt \,.$$

Problem 24. Prove (3.20). You will probably have to go back to first principles to do this. Show that it is enough to assume $u \ge 0$ has compact support. Then show it is enough to assume that $u$ is a simple, and integrable, function. Finally look at the definition of Lebesgue measure and show that if $E \subset \mathbb{R}^n$ is Borel and has finite Lebesgue measure then

$$\lim_{|t|\to\infty} \mu(E \backslash (E + t)) = 0$$

where $\mu =$ Lebesgue measure and

$$E + t = \{p \in \mathbb{R}^n \,;\, p' + t \,,\, p' \in E\} \,.$$

PROBLEM 25. Prove Leibniz' formula

$$D^\alpha{}_x(\varphi\psi) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta} D^\alpha{}_x\varphi \cdot d_x^{\alpha-\beta}\psi$$

for any $\mathcal{C}^\infty$ functions and $\varphi$ and $\psi$. Here $\alpha$ and $\beta$ are multiindices, $\beta\le\alpha$ means $\beta_j\le\alpha_j$ for each $j_?$ and

$$\binom{\alpha}{\beta} = \prod_j \binom{\alpha_j}{\beta_j}.$$

I suggest induction!

PROBLEM 26. Prove the generalization of Proposition 3.10 that $u\in\mathcal{S}'(\mathbb{R}^n)$, $\operatorname{supp}(w)\subset\{0\}$ implies there are constants $c\alpha$, $|\alpha|\le m$, for some $m$, such that

$$u = \sum_{|\alpha|\le m} c_\alpha D^\alpha \delta\,.$$

*Hint* This is not so easy! I would be happy if you can show that $u\in M(\mathbb{R}^n)$, $\operatorname{supp} u\subset\{0\}$ implies $u = c\delta$. To see this, you can show that

$$\begin{aligned}\varphi\in\mathcal{S}(\mathbb{R}^n),\ &\varphi(0)=0\\ &\Rightarrow \exists\varphi_j\in\mathcal{S}(\mathbb{R}^n)\,,\ \varphi_j(x)=0 \text{ in } |x|\le\epsilon_j>0(\downarrow 0)\,,\\ &\qquad\qquad \sup|\varphi_j-\varphi|\to 0 \text{ as } j\to\infty\,.\end{aligned}$$

To prove the general case you need something similar — that given $m$, if $\varphi\in\mathcal{S}(\mathbb{R}^n)$ and $D^\alpha{}_x\varphi(0)=0$ for $|\alpha|\le m$ then $\exists\,\varphi_j\in\mathcal{S}(\mathbb{R}^n)$, $\varphi_j=0$ in $|x|\le\epsilon_j$, $\epsilon_j\downarrow 0$ such that $\varphi_j\to\varphi$ in the $\mathcal{C}^m$ norm.

PROBLEM 27. If $m\in\mathbb{N}$, $m'>0$ show that $u\in H^m(\mathbb{R}^n)$ and $D^\alpha u\in H^{m'}(\mathbb{R}^n)$ for all $|\alpha|\le m$ implies $u\in H^{m+m'}(\mathbb{R}^n)$. Is the converse true?

PROBLEM 28. Show that every element $u\in L^2(\mathbb{R}^n)$ can be written as a sum

$$u = u_0 + \sum_{j=1}^n D_j u_j\,,\ u_j\in H^1(\mathbb{R}^n)\,,\ j=0,\dots,n\,.$$

PROBLEM 29. Consider for $n=1$, the locally integrable function (the Heaviside function),

$$H(x) = \begin{cases} 0 & x\le 0\\ 1 & x>1\,.\end{cases}$$

Show that $D_x H(x) = c\delta$; what is the constant $c$?

PROBLEM 30. For what range of orders $m$ is it true that $\delta \in H^m(\mathbb{R}^n)$ , $\delta(\varphi) = \varphi(0)$?

PROBLEM 31. Try to write the Dirac measure explicitly (as possible) in the form (5.8). How many derivatives do you think are necessary?

PROBLEM 32. Go through the computation of $\overline{\partial}E$ again, but cutting out a disk $\{x^2 + y^2 \le \epsilon^2\}$ instead.

PROBLEM 33. Consider the Laplacian, (6.4), for $n = 3$. Show that $E = c(x^2 + y^2)^{-1/2}$ is a fundamental solution for some value of $c$.

PROBLEM 34. Recall that a topology on a set $X$ is a collection $\mathcal{F}$ of subsets (called the *open* sets) with the properties, $\phi \in \mathcal{F}$, $X \in \mathcal{F}$ and $\mathcal{F}$ is closed under finite intersections and arbitrary unions. Show that the following definition of an open set $U \subset \mathcal{S}'(\mathbb{R}^n)$ defines a topology:

$$\begin{aligned} \forall\ u \in U \text{ and all } \varphi \in \mathcal{S}(\mathbb{R}^n)\ \exists \epsilon > 0 \text{ st.} \\ |(u' - u)(\varphi)| < \epsilon \Rightarrow u' \in U\,. \end{aligned}$$

This is called the weak topology (because there are very few open sets). Show that $u_j \to u$ weakly in $\mathcal{S}'(\mathbb{R}^n)$ means that for every open set $U \ni u$ $\exists N$ st. $u_j \in U\ \forall\ j \ge N$.

PROBLEM 35. Prove (6.18) where $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$.

PROBLEM 36. Show that for fixed $v \in \mathcal{S}'(\mathbb{R}^n)$ with compact support

$$\mathcal{S}(\mathbb{R}^n) \ni \varphi \mapsto v * \varphi \in \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear map.

PROBLEM 37. Prove the ?? to properties in Theorem 6.6 for $u * v$ where $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^n)$ with at least one of them having compact support.

PROBLEM 38. Use Theorem 6.9 to show that if $P(D)$ is hypoelliptic then *every* parametrix $F \in \mathcal{S}(\mathbb{R}^n)$ has $\operatorname{sing\,supp}(F) = \{0\}$.

PROBLEM 39. Show that if $P(D)$ is an ellipitic differential operator of order $m$, $u \in L^2(\mathbb{R}^n)$ and $P(D)u \in L^2(\mathbb{R}^n)$ then $u \in H^m(\mathbb{R}^n)$.

PROBLEM 40 (Taylor's theorem). . Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued function which is $k$ times continuously differentiable. Prove that there is a polynomial $p$ and a continuous function $v$ such that

$$u(x) = p(x) + v(x) \text{ where } \lim_{|x|\downarrow 0} \frac{|v(x)|}{|x|^k} = 0.$$

PROBLEM 41. Let $\mathcal{C}(\mathbb{B}^n)$ be the space of continuous functions on the (closed) unit ball, $\mathbb{B}^n = \{x \in \mathbb{R}^n; |x| \leq 1\}$. Let $\mathcal{C}_0(\mathbb{B}^n) \subset \mathcal{C}(\mathbb{B}^n)$ be the subspace of functions which vanish at each point of the boundary and let $\mathcal{C}(\mathbb{S}^{n-1})$ be the space of continuous functions on the unit sphere. Show that inclusion and restriction to the boundary gives a short exact sequence

$$\mathcal{C}_0(\mathbb{B}^n) \hookrightarrow \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathcal{C}(\mathbb{S}^{n-1})$$

(meaning the first map is injective, the second is surjective and the image of the first is the null space of the second.)

PROBLEM 42 (Measures). A measure on the ball is a continuous linear functional $\mu : \mathcal{C}(\mathbb{B}^n) \longrightarrow \mathbb{R}$ where continuity is with respect to the supremum norm, i.e. there must be a constant $C$ such that

$$|\mu(f)| \leq C \sup_{x \in \mathbb{R}^n} |f(x)| \ \forall \ f \in \mathcal{C}(\mathbb{B}^n).$$

Let $M(\mathbb{B}^n)$ be the linear space of such measures. The space $M(\mathbb{S}^{n-1})$ of measures on the sphere is defined similarly. Describe an injective map

$$M(\mathbb{S}^{n-1}) \longrightarrow M(\mathbb{B}^n).$$

Can you define another space so that this can be extended to a short exact sequence?

PROBLEM 43. Show that the Riemann integral defines a measure

$$\mathcal{C}(\mathbb{B}^n) \ni f \longmapsto \int_{\mathbb{B}^n} f(x)dx. \tag{3.3}$$

PROBLEM 44. If $g \in \mathcal{C}(\mathbb{B}^n)$ and $\mu \in M(\mathbb{B}^n)$ show that $g\mu \in M(\mathbb{B}^n)$ where $(g\mu)(f) = \mu(fg)$ for all $f \in \mathcal{C}(\mathbb{B}^n)$. Describe all the measures with the property that

$$x_j\mu = 0 \text{ in } M(\mathbb{B}^n) \text{ for } j = 1, \dots, n.$$

PROBLEM 45 (Hörmander, Theorem 3.1.4). Let $I \subset \mathbb{R}$ be an open, non-empty interval.

i) Show (you may use results from class) that there exists $\psi \in \mathcal{C}^\infty_c(I)$ with $\int_{\mathbb{R}} \psi(x)ds = 1$.
ii) Show that any $\phi \in \mathcal{C}^\infty_c(I)$ may be written in the form
$$\phi = \tilde{\phi} + c\psi, \ c \in \mathbb{C}, \ \tilde{\phi} \in \mathcal{C}^\infty_c(I) \text{ with } \int_{\mathbb{R}} \tilde{\phi} = 0.$$
iii) Show that if $\tilde{\phi} \in \mathcal{C}^\infty_c(I)$ and $\int_{\mathbb{R}} \tilde{\phi} = 0$ then there exists $\mu \in \mathcal{C}^\infty_c(I)$ such that $\frac{d\mu}{dx} = \tilde{\phi}$ in $I$.

iv) Suppose $u \in \mathcal{C}^{-\infty}(I)$ satisfies $\frac{du}{dx} = 0$, i.e.

$$u(-\frac{d\phi}{dx}) = 0 \ \forall\ \phi \in \mathcal{C}_c^\infty(I),$$

show that $u = c$ for some constant $c$.

v) Suppose that $u \in \mathcal{C}^{-\infty}(I)$ satisfies $\frac{du}{dx} = c$, for some constant $c$, show that $u = cx + d$ for some $d \in \mathbb{C}$.

Problem 46. [Hörmander Theorem 3.1.16]

i) Use Taylor's formula to show that there is a fixed $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ can be written in the form

$$\phi = c\psi + \sum_{j=1}^n x_j \psi_j$$

where $c \in \mathbb{C}$ and the $\psi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ depend on $\phi$.

ii) Recall that $\delta_0$ is the distribution defined by

$$\delta_0(\phi) = \phi(0) \ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n);$$

explain why $\delta_0 \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$.

iii) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $u(x_j\phi) = 0$ for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $j = 1, \dots, n$ then $u = c\delta_0$ for some $c \in \mathbb{C}$.

iv) Define the 'Heaviside function'

$$H(\phi) = \int_0^\infty \phi(x)dx \ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R});$$

show that $H \in \mathcal{C}^{-\infty}(\mathbb{R})$.

v) Compute $\frac{d}{dx}H \in \mathcal{C}^{-\infty}(\mathbb{R})$.

Problem 47. Using Problems 45 and 46, find all $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfying the differential equation

$$x\frac{du}{dx} = 0 \text{ in } \mathbb{R}.$$

These three problems are all about homogeneous distributions on the line, extending various things using the fact that

$$x_+^z = \begin{cases} \exp(z\log x) & x > 0 \\ 0 & x \le 0 \end{cases}$$

is a continuous function on $\mathbb{R}$ if $\operatorname{Re} z > 0$ and is differentiable if $\operatorname{Re} z > 1$ and then satisfies

$$\frac{d}{dx}x_+^z = zx_+^{z-1}.$$

We used this to define

$$x_+^z = \frac{1}{z+k}\frac{1}{z+k-1}\cdots\frac{1}{z+1}\frac{d^k}{dx^k}x_+^{z+k} \text{ if } z \in \mathbb{C}\setminus -\mathbb{N}. \tag{3.4}$$

PROBLEM 48. [Hadamard regularization]

i) Show that (3.4) just means that for each $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$

$$x_+^z(\phi) = \frac{(-1)^k}{(z+k)\cdots(z+1)}\int_0^\infty \frac{d^k\phi}{dx^k}(x)x^{z+k}dx,\ \operatorname{Re} z > -k,\ z \notin -\mathbb{N}.$$

ii) Use integration by parts to show that

$$x_+^z(\phi) = \lim_{\epsilon\downarrow 0}\left[\int_\epsilon^\infty \phi(x)x^z dx - \sum_{j=1}^k C_j(\phi)\epsilon^{z+j}\right],\ \operatorname{Re} z > -k,\ z \notin -\mathbb{N} \tag{3.5}$$

for certain constants $C_j(\phi)$ which you should give explicitly. [This is called Hadamard regularization after Jacques Hadamard, feel free to look at his classic book [**3**].]

iii) Assuming that $-k+1 \geq \operatorname{Re} z > -k$, $z \neq -k+1$, show that there can only be one set of the constants with $j < k$ (for each choice of $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$) such that the limit in (3.5) exists.

iiv) Use ii), and maybe iii), to show that

$$\frac{d}{dx}x_+^z = zx_+^{z-1} \text{ in } \mathcal{C}^{-\infty}(\mathbb{R})\ \forall\ z \notin -\mathbb{N}_0 = \{0,1,\dots\}.$$

v) Similarly show that $xx_+^z = x_+^{z+1}$ for all $z \notin -\mathbb{N}$.

vi) Show that $x_+^z = 0$ in $x < 0$ for all $z \notin -\mathbb{N}$. (Duh.)

PROBLEM 49. [Null space of $x\frac{d}{dx} - z$]

i) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ then $\tilde{u}(\phi) = u(\tilde{\phi})$, where $\tilde{\phi}(x) = \phi(-x)\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R})$, defines an element of $\mathcal{C}^{-\infty}(\mathbb{R})$. What is $\tilde{u}$ if $u \in \mathcal{C}^0(\mathbb{R})$? Compute $\widetilde{\delta_0}$.

ii) Show that $\frac{d}{dx}\tilde{u} = -\widetilde{\frac{d}{dx}u}$.

iii) Define $x_-^z = \widetilde{x_+^z}$ for $z \notin -\mathbb{N}$ and show that $\frac{d}{dx}x_-^z = -zx_-^{z-1}$ and $xx_-^z = -x_-^{z+1}$.

iv) Suppose that $u \in \mathcal{C}^{-\infty}(\mathbb{R})$ satisfies the distributional equation $(x\frac{d}{dx} - z)u = 0$ (meaning of course, $x\frac{du}{dx} = zu$ where $z$ is a constant). Show that

$$u\big|_{x>0} = c_+x_-^z\big|_{x>0} \text{ and } u\big|_{x<0} = c_-x_-^z\big|_{x<0}$$

for some constants $c_\pm$. Deduce that $v = u - c_+x_+^z - c_-x_-^z$ satisfies

$$(x\frac{d}{dx} - z)v = 0 \text{ and } \operatorname{supp}(v) \subset \{0\}. \tag{3.6}$$

v) Show that for each $k \in \mathbb{N}$, $(x\frac{d}{dx} + k + 1)\frac{d^k}{dx^k}\delta_0 = 0$.

vi) Using the *fact* that any $v \in \mathcal{C}^{-\infty}(\mathbb{R})$ with $\operatorname{supp}(v) \subset \{0\}$ is a finite sum of constant multiples of the $\frac{d^k}{dx^k}\delta_0$, show that, for $z \notin -\mathbb{N}$, the only solution of (3.6) is $v = 0$.

vii) Conclude that for $z \notin -\mathbb{N}$

$$\left\{u \in \mathcal{C}^{-\infty}(\mathbb{R}); (x\frac{d}{dx} - z)u = 0\right\} \tag{3.7}$$

is a two-dimensional vector space.

Problem 50. [Negative integral order] To do the same thing for negative integral order we need to work a little differently. Fix $k \in \mathbb{N}$.

i) We define *weak convergence* of distributions by saying $u_n \to u$ in $\mathcal{C}_c^\infty(X)$, where $u_n, u \in \mathcal{C}^{-\infty}(X)$, $X \subset \mathbb{R}^n$ being open, if $u_n(\phi) \to u(\phi)$ for each $\phi \in \mathcal{C}_c^\infty(X)$. Show that $u_n \to u$ implies that $\frac{\partial u_n}{\partial x_j} \to \frac{\partial u}{\partial x_j}$ for each $j = 1, \dots, n$ and $fu_n \to fu$ if $f \in \mathcal{C}^\infty(X)$.

ii) Show that $(z + k)x_+^z$ is weakly continuous as $z \to -k$ in the sense that for any sequence $z_n \to -k$, $z_n \notin -\mathbb{N}$, $(z_n + k)x_+^{z_n} \to v_k$ where

$$v_k = \frac{1}{-1} \cdots \frac{1}{-k+1}\frac{d^{k+1}}{dx^{k+1}}x_+, \ x_+ = x_+^1.$$

iii) Compute $v_k$, including the constant factor.

iv) Do the same thing for $(z + k)x_-^z$ as $z \to -k$.

v) Show that there is a linear combination $(k + z)(x_+^z + c(k)x_-^z)$ such that as $z \to -k$ the limit is zero.

vi) If you get this far, show that in fact $x_+^z + c(k)x_-^z$ also has a weak limit, $u_k$, as $z \to -k$. [This may be the hardest part.]

vii) Show that this limit distribution satisfies $(x\frac{d}{dx} + k)u_k = 0$.

viii) Conclude that (3.7) does in fact hold for $z \in -\mathbb{N}$ as well. [There are still some things to prove to get this.]

Problem 51. Show that for any set $G \subset \mathbb{R}^n$

$$v^*(G) = \inf \sum_{i=1}^{\infty} v(A_i)$$

where the infimum is taken over coverings of $G$ by rectangular sets (products of intervals).

PROBLEM 52. Show that a $\sigma$-algebra is closed under countable intersections.

PROBLEM 53. Show that compact sets are Lebesgue measurable and have finite volume and also show the inner regularity of the Lebesgue measure on open sets, that is if $E$ is open then

$$v(E) = \sup\{v(K); K \subset E,\ K \text{ compact}\}. \tag{3.8}$$

PROBLEM 54. Show that a set $B \subset \mathbb{R}^n$ is Lebesgue measurable if and only if

$$v^*(E) = v^*(E \cap B) + v^*(E \cap B^{\complement})\ \forall \text{ open } E \subset \mathbb{R}^n.$$

[The definition is this for all $E \subset \mathbb{R}^n$.]

PROBLEM 55. Show that a real-valued continuous function $f : U \longrightarrow \mathbb{R}$ on an open set, is Lebesgue measurable, in the sense that $f^{-1}(I) \subset U \subset \mathbb{R}^n$ is measurable for each interval $I$.

PROBLEM 56. Hilbert space and the Riesz representation theorem. If you need help with this, it can be found in lots of places – for instance [**7**] has a nice treatment.

i) A pre-Hilbert space is a vector space $V$ (over $\mathbb{C}$) with a 'positive definite sesquilinear inner product' i.e. a function

$$V \times V \ni (v, w) \mapsto \langle v, w\rangle \in \mathbb{C}$$

satisfying

- $\langle w, v\rangle = \overline{\langle v, w\rangle}$
- $\langle a_1 v_1 + a_2 v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$
- $\langle v, v\rangle \geq 0$
- $\langle v, v\rangle = 0 \Rightarrow v = 0.$

Prove Schwarz' inequality, that

$$|\langle u, v\rangle| \leq \langle u\rangle^{\frac{1}{2}}\langle v\rangle^{\frac{1}{2}}\ \forall\ u, v \in V.$$

Hint: Reduce to the case $\langle v, v\rangle = 1$ and then expand

$$\langle u - \langle u, v\rangle v, u - \langle u, v\rangle v\rangle \geq 0.$$

ii) Show that $\|v\| = \langle v, v\rangle^{1/2}$ is a norm and that it satisfies the parallelogram law:

$$\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2\ \forall\ v_1, v_2 \in V. \tag{3.9}$$

iii) Conversely, suppose that $V$ is a linear space over $\mathbb{C}$ with a norm which satisfies (3.9). Show that

$$4\langle v, w\rangle = \|v + w\|^2 - \|v - w\|^2 + i\|v + iw\|^2 - i\|v - iw\|^2$$

defines a pre-Hilbert inner product which gives the original norm.

iv) Let $V$ be a Hilbert space, so as in (i) but complete as well. Let $C \subset V$ be a closed non-empty convex subset, meaning $v, w \in C \Rightarrow (v+w)/2 \in C$. Show that there exists a unique $v \in C$ minimizing the norm, i.e. such that

$$\|v\| = \inf_{w \in C} \|w\|.$$

*Hint:* Use the parallelogram law to show that a norm minimizing sequence is Cauchy.

v) Let $u : H \to \mathbb{C}$ be a continuous linear functional on a Hilbert space, so $|u(\varphi)| \le C\|\varphi\| \ \forall \ \varphi \in H$. Show that $N = \{\varphi \in H; u(\varphi) = 0\}$ is closed and that if $v_0 \in H$ has $u(v_0) \neq 0$ then each $v \in H$ can be written uniquely in the form

$$v = cv_0 + w, \ c \in \mathbb{C}, \ w \in N.$$

vi) With $u$ as in v), not the zero functional, show that there exists a unique $f \in H$ with $u(f) = 1$ and $\langle w, f \rangle = 0$ for all $w \in N$.
*Hint:* Apply iv) to $C = \{g \in V; u(g) = 1\}$.

vii) Prove the Riesz Representation theorem, that every continuous linear functional on a Hilbert space is of the form

$$u_f : H \ni \varphi \mapsto \langle \varphi, f \rangle \text{ for a unique } f \in H.$$

Problem 57. Density of $\mathcal{C}_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$.

i) Recall in a few words why simple integrable functions are dense in $L^1(\mathbb{R}^n)$ with respect to the norm $\|f\|_{L^1} = \int_{\mathbb{R}^n} |f(x)|dx$.

ii) Show that simple functions $\sum_{j=1}^N c_j \chi(U_j)$ where the $U_j$ are open and bounded are also dense in $L^1(\mathbb{R}^n)$.

iii) Show that if $U$ is open and bounded then $F(y) = v(U \cap U_y)$, where $U_y = \{z \in \mathbb{R}^n; z = y + y', y' \in U\}$ is continuous in $y \in \mathbb{R}^n$ and that

$$v(U \cap U_y^{\complement}) + v(U^{\complement} \cap U_y) \to 0 \text{ as } y \to 0.$$

iv) If $U$ is open and bounded and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ show that

$$f(x) = \int_U \varphi(x-y)dy \in \mathcal{C}_c^\infty(\mathbb{R}^n).$$

v) Show that if $U$ is open and bounded then

$$\sup_{|y| \le \delta} \int |\chi_U(x) - \chi_U(x-y)|dx \to 0 \text{ as } \delta \downarrow 0.$$

vi) If $U$ is open and bounded and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\varphi \geq 0$, $\int \varphi = 1$ then

$$f_\delta \to \chi_U \text{ in } L^1(\mathbb{R}^n) \text{ as } \delta \downarrow 0$$

where

$$f_\delta(x) = \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \chi_U(x-y)dy.$$

*Hint:* Write $\chi_U(x) = \delta^{-n} \int \varphi\left(\frac{y}{\delta}\right) \chi_U(x)$ and use v).

vii) Conclude that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$.

viii) Show that $\mathcal{C}_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ for any $1 \leq p < \infty$.

PROBLEM 58. Schwartz representation theorem. Here we (well you) come to grips with the general structure of a tempered distribution.

i) Recall briefly the proof of the Sobolev embedding theorem and the corresponding estimate

$$\sup_{x \in \mathbb{R}^n} |\phi(x)| \leq C\|\phi\|_{H^m}, \ \frac{n}{2} < m \in \mathbb{R}.$$

ii) For $m = n+1$ write down a(n equivalent) norm on the right in a form that does not involve the Fourier transform.

iii) Show that for any $\alpha \in \mathbb{N}_0$

$$|D^\alpha\left((1+|x|^2)^N\phi\right)| \leq C_{\alpha,N} \sum_{\beta \leq \alpha} (1+|x|^2)^N |D^\beta \phi|.$$

iv) Deduce the general estimates

$$\sup_{\substack{|\alpha| \leq N \\ x \in \mathbb{R}^n}} (1+|x|^2)^N |D^\alpha \phi(x)| \leq C_N \|(1+|x|^2)^N \phi\|_{H^{N+n+1}}.$$

v) Conclude that for each tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ there is an integer $N$ and a constant $C$ such that

$$|u(\phi)| \leq C\|(1+|x|^2)^N \phi\|_{H^{2N}} \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

vi) Show that $v = (1+|x|^2)^{-N} u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies

$$|v(\phi)| \leq C\|(1+|D|^2)^N \phi\|_{L^2} \ \forall\ \phi \in \mathcal{S}(\mathbb{R}^n).$$

vi) Recall (from class or just show it) that if $v$ is a tempered distribution then there is a unique $w \in \mathcal{S}'(\mathbb{R}^n)$ such that $(1+|D|^2)^N w = v$.

vii) Use the Riesz Representation Theorem to conclude that for each tempered distribution $u$ there exists $N$ and $w \in L^2(\mathbb{R}^n)$ such that

$$u = (1+|D|^2)^N (1+|x|^2)^N w. \tag{3.10}$$

viii) Use the Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ (and the fact that it is an isomorphism on $L^2(\mathbb{R}^n)$) to show that any tempered distribution can be written in the form

$$u = (1 + |x|^2)^N (1 + |D|^2)^N w \text{ for some } N \text{ and some } w \in L^2(\mathbb{R}^n).$$

ix) Show that any tempered distribution can be written in the form

$$u = (1+|x|^2)^N (1+|D|^2)^{N+n+1} \tilde{w} \text{ for some } N \text{ and some } \tilde{w} \in H^{2(n+1)}(\mathbb{R}^n).$$

x) Conclude that any tempered distribution can be written in the form

$$u = (1 + |x|^2)^N (1 + |D|^2)^M U \text{ for some } N, M \quad \text{and a bounded continuous function } U$$

Problem 59. Distributions of compact support.

i) Recall the definition of the support of a distribution, defined in terms of its complement

$$\mathbb{R}^n \backslash \operatorname{supp}(u) = \left\{ p \in \mathbb{R}^n ; \exists\ U \subset \mathbb{R}^n, \text{ open, with } p \in U \text{ such that } u\big|_U = 0 \right\}$$

ii) Show that if $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ satisfy

$$\operatorname{supp}(u) \cap \operatorname{supp}(\phi) = \emptyset$$

then $u(\phi) = 0$.

iii) Consider the space $\mathcal{C}^\infty(\mathbb{R}^n)$ of all smooth functions on $\mathbb{R}^n$, without restriction on supports. Show that for each $N$

$$\|f\|_{(N)} = \sup_{|\alpha| \le N,\ |x| \le N} |D^\alpha f(x)|$$

is a seminorn on $\mathcal{C}^\infty(\mathbb{R}^n)$ (meaning it satisfies $\|f\| \ge 0$, $\|cf\| = |c|\|f\|$ for $c \in \mathbb{C}$ and the triangle inequality but that $\|f\| = 0$ does not necessarily imply that $f = 0$.)

iv) Show that $\mathcal{C}^\infty_c(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ is dense in the sense that for each $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ there is a sequence $f_n$ in $\mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $\|f - f_n\|_{(N)} \to 0$ for each $N$.

v) Let $\mathcal{E}'(\mathbb{R}^n)$ temporarily (or permanantly if you prefer) denote the dual space of $\mathcal{C}^\infty(\mathbb{R}^n)$ (which is also written $\mathcal{E}(\mathbb{R}^n)$), that is, $v \in \mathcal{E}'(\mathbb{R}^n)$ is a linear map $v : \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathbb{C}$ which is continuous in the sense that for some $N$

$$|v(f)| \le C\|f\|_{(N)}\ \forall\ f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{3.11}$$

Show that such a $v$ 'is' a distribution and that the map $\mathcal{E}'(\mathbb{R}^n) \longrightarrow \mathcal{C}^{-\infty}(\mathbb{R}^n)$ is injective.

vi) Show that if $v \in \mathcal{E}'(\mathbb{R}^n)$ satisfies (3.11) and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ has $f = 0$ in $|x| < N + \epsilon$ for some $\epsilon > 0$ then $v(f) = 0$.

vii) Conclude that each element of $\mathcal{E}'(\mathbb{R}^n)$ has compact support when considered as an element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$.

viii) Show the converse, that each element of $\mathcal{C}^{-\infty}(\mathbb{R}^n)$ with compact support is an element of $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and hence conclude that $\mathcal{E}'(\mathbb{R}^n)$ 'is' the space of distributions of compact support.

I will denote the space of distributions of compact support by $\mathcal{C}_c^{-\infty}(\mathbb{R})$.

Problem 60. Hypoellipticity of the heat operator $H = iD_t + \Delta = iD_t + \sum\limits_{j=1}^{n} D_{x_j}^2$ on $\mathbb{R}^{n+1}$.

(1) Using $\tau$ to denote the 'dual variable' to $t$ and $\xi \in \mathbb{R}^n$ to denote the dual variables to $x \in \mathbb{R}^n$ observe that $H = p(D_t, D_x)$ where $p = i\tau + |\xi|^2$.

(2) Show that $|p(\tau, \xi)| > \frac{1}{2}\left(|\tau| + |\xi|^2\right)$.

(3) Use an inductive argument to show that, in $(\tau, \xi) \neq 0$ where it makes sense,

$$D_\tau^k D_\xi^\alpha \frac{1}{p(\tau,\xi)} = \sum_{j=1}^{|\alpha|} \frac{q_{k,\alpha,j}(\xi)}{p(\tau,\xi)^{k+j+1}} \tag{3.12}$$

where $q_{k,\alpha,j}(\xi)$ is a polynomial of degree (at most) $2j - |\alpha|$.

(4) Conclude that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ is identically equal to 1 in a neighbourhood of 0 then the function

$$g(\tau, \xi) = \frac{1 - \phi(\tau, \xi)}{i\tau + |\xi|^2}$$

is the Fourier transform of a distribution $F \in \mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{sing\,supp}(F) \subset \{0\}$. [Remember that $\operatorname{sing\,supp}(F)$ is the complement of the largest open subset of $\mathbb{R}^n$ the restriction of $F$ to which is smooth].

(5) Show that $F$ is a parametrix for the heat operator.

(6) Deduce that $iD_t + \Delta$ is *hypoelliptic* – that is, if $U \subset \mathbb{R}^n$ is an open set and $u \in \mathcal{C}^{-\infty}(U)$ satisfies $(iD_t + \Delta)u \in \mathcal{C}^\infty(U)$ then $u \in \mathcal{C}^\infty(U)$.

(7) Show that $iD_t - \Delta$ is also hypoelliptic.

Problem 61. Wavefront set computations and more – all pretty easy, especially if you use results from class.

i) Compute $\mathrm{WF}(\delta)$ where $\delta \in \mathcal{S}'(\mathbb{R}^n)$ is the Dirac delta function at the origin.

ii) Compute $\mathrm{WF}(H(x))$ where $H(x) \in \mathcal{S}'(\mathbb{R})$ is the Heaviside function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}.$$

Hint: $D_x$ is elliptic in one dimension, hit $H$ with it.

iii) Compute $\mathrm{WF}(E)$, $E = iH(x_1)\delta(x')$ which is the Heaviside in the first variable on $\mathbb{R}^n$, $n > 1$, and delta in the others.

iv) Show that $D_{x_1}E = \delta$, so $E$ is a fundamental solution of $D_{x_1}$.

v) If $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ show that $u = E \star f$ solves $D_{x_1}u = f$.

vi) What does our estimate on $\mathrm{WF}(E \star f)$ tell us about $\mathrm{WF}(u)$ in terms of $\mathrm{WF}(f)$?

Problem 62. The wave equation in two variables (or one spatial variable).

i) Recall that the Riemann function

$$E(t,x) = \begin{cases} -\frac{1}{4} & \text{if } t > x \text{ and } t > -x \\ 0 & \text{otherwise} \end{cases}$$

is a fundamental solution of $D_t^2 - D_x^2$ (check my constant).

ii) Find the singular support of $E$.

iii) Write the Fourier transform (dual) variables as $\tau, \xi$ and show that

$$\mathrm{WF}(E) \subset \{0\} \times \mathbb{S}^1 \cup \{(t,x,\tau,\xi); x = t > 0 \text{ and } \xi + \tau = 0\} \\ \cup \{(t,x,\tau,\xi); -x = t > 0 \text{ and } \xi = \tau\}.$$

iv) Show that if $f \in \mathcal{C}_c^{-\infty}(\mathbb{R}^2)$ then $u = E \star f$ satisfies $(D_t^2 - D_x^2)u = f$.

v) With $u$ defined as in iv) show that

$$\mathrm{supp}(u) \subset \{(t,x);\ \exists \\ (t',x') \in \mathrm{supp}(f) \text{ with } t' + x' \leq t + x \text{ and } t' - x' \leq t - x\}.$$

vi) Sketch an illustrative example of v).

vii) Show that, still with $u$ given by iv),

$$\operatorname{sing\,supp}(u) \subset \{(t,x);\ \exists\ (t',x') \in \operatorname{sing\,supp}(f) \text{ with} \\ t \geq t' \text{ and } t + x = t' + x' \text{ or } t - x = t' - x'\}.$$

viii) Bound $\mathrm{WF}(u)$ in terms of $\mathrm{WF}(f)$.

Problem 63. A little uniqueness theorems. Suppose $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ recall that the Fourier transform $\hat{u} \in \mathcal{C}^\infty(\mathbb{R}^n)$. Now, suppose $u \in$

$\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfies $P(D)u = 0$ for some non-trivial polynomial $P$, show that $u = 0$.

PROBLEM 64. Work out the elementary behavior of the heat equation.

i) Show that the function on $\mathbb{R} \times \mathbb{R}^n$, for $n \geq 1$,
$$F(t,x) = \begin{cases} t^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right) & t > 0 \\ 0 & t \leq 0 \end{cases}$$
is measurable, bounded on the any set $\{|(t,x)| \geq R\}$ and is integrable on $\{|(t,x)| \leq R\}$ for any $R > 0$.

ii) Conclude that $F$ defines a tempered distribution on $\mathbb{R}^{n+1}$.

iii) Show that $F$ is $\mathcal{C}^\infty$ outside the origin.

iv) Show that $F$ satisfies the heat equation
$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t,x) = 0 \text{ in } (t,x) \neq 0.$$

v) Show that $F$ satisfies
$$F(s^2 t, sx) = s^{-n} F(t,x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1}) \tag{3.13}$$
where the left hand side is defined by duality "$F(s^2t, sx) = F_s$" where
$$F_s(\phi) = s^{-n-2}F(\phi_{1/s}),\ \ \phi_{1/s}(t,x) = \phi(\frac{t}{s^2}, \frac{x}{s}).$$

vi) Conclude that
$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t,x) = G(t,x)$$
where $G(t,x)$ satisfies
$$G(s^2t, sx) = s^{-n-2}G(t,x) \text{ in } \mathcal{S}'(\mathbb{R}^{n+1}) \tag{3.14}$$
in the same sense as above and has support at most $\{0\}$.

vii) Hence deduce that
$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2)F(t,x) = c\delta(t)\delta(x) \tag{3.15}$$
for some real constant $c$.

Hint: Check which distributions with support at $(0,0)$ satisfy (3.14).

viii) If $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ show that $u = F \star \psi$ satisfies

$$u \in \mathcal{C}^\infty(\mathbb{R}^{n+1}) \text{ and}$$
$$\sup_{x\in\mathbb{R}^n,\ t\in[-S,S]} (1+|x|)^N |D^\alpha u(t,x)| < \infty \ \forall\ S > 0, \alpha \in \mathbb{N}^{n+1},\ N. \tag{3.16}$$

ix) Supposing that $u$ satisfies (3.16) and is a real-valued solution of

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2) u(t,x) = 0$$

in $\mathbb{R}^{n+1}$, show that

$$v(t) = \int_{\mathbb{R}^n} u^2(t,x)$$

is a non-increasing function of $t$.

Hint: Multiply the equation by $u$ and integrate over a slab $[t_1, t_2] \times \mathbb{R}^n$.

x) Show that $c$ in (3.15) is non-zero by arriving at a contradiction from the assumption that it is zero. Namely, show that if $c = 0$ then $u$ in viii) satisfies the conditions of ix) and also vanishes in $t < T$ for some $T$ (depending on $\psi$). Conclude that $u = 0$ for all $\psi$. Using properties of convolution show that this in turn implies that $F = 0$ which is a contradiction.

xi) So, finally, we know that $E = \frac{1}{c} F$ is a fundamental solution of the heat operator which vanishes in $t < 0$. Explain why this allows us to show that for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ there is a solution of

$$(\partial_t - \sum_{j=1}^n \partial_{x_j}^2) u = \psi,\ u = 0 \text{ in } t < T \text{ for some } T. \tag{3.17}$$

What is the largest value of $T$ for which this holds?

xii) Can you give a heuristic, or indeed a rigorous, explanation of why

$$c = \int_{\mathbb{R}^n} \exp(-\frac{|x|^2}{4}) dx?$$

xiii) Explain why the argument we used for the wave equation to show that there is *only one* solution, $u \in \mathcal{C}^\infty(\mathbb{R}^{n+1})$, of (3.17) does not apply here. (Indeed such uniqueness does not hold without some growth assumption on $u$.)

Problem 65. (Poisson summation formula) As in class, let $L \subset \mathbb{R}^n$ be an integral lattice of the form

$$L = \left\{ v = \sum_{j=1}^{n} k_j v_j,\ k_j \in \mathbb{Z} \right\}$$

where the $v_j$ form a basis of $\mathbb{R}^n$ and using the dual basis $w_j$ (so $w_j \cdot v_i = \delta_{ij}$ is 0 or 1 as $i \neq j$ or $i = j$) set

$$L^\circ = \left\{ w = 2\pi \sum_{j=1}^{n} k_j w_j,\ k_j \in \mathbb{Z} \right\}.$$

Recall that we defined

$$\mathcal{C}^\infty(\mathbb{T}_L) = \{u \in \mathcal{C}^\infty(\mathbb{R}^n); u(z+v) = u(z)\ \forall\ z \in \mathbb{R}^n,\ v \in L\}. \tag{3.18}$$

i) Show that summation over shifts by lattice points:

$$A_L : \mathcal{S}(\mathbb{R}^n) \ni f \longmapsto A_L f(z) = \sum_{v \in L} f(z-v) \in \mathcal{C}^\infty(\mathbb{T}_L). \tag{3.19}$$

defines a map into smooth periodic functions.

ii) Show that there exists $f \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ such that $A_L f \equiv 1$ is the costant function on $\mathbb{R}^n$.

iii) Show that the map (3.19) is surjective. Hint: Well obviously enough use the $f$ in part ii) and show that if $u$ is periodic then $A_L(uf) = u$.

iv) Show that the infinite sum

$$F = \sum_{v \in L} \delta(\cdot - v) \in \mathcal{S}'(\mathbb{R}^n) \tag{3.20}$$

does indeed define a tempered distribution and that $F$ is $L$-periodic and satisfies $\exp(iw \cdot z)F(z) = F(z)$ for each $w \in L^\circ$ with equality in $\mathcal{S}'(\mathbb{R}^n)$.

v) Deduce that $\hat{F}$, the Fourier transform of $F$, is $L^\circ$ periodic, conclude that it is of the form

$$\hat{F}(\xi) = c \sum_{w \in L^\circ} \delta(\xi - w) \tag{3.21}$$

vi) Compute the constant $c$.

vii) Show that $A_L(f) = F \star f$.

viii) Using this, or otherwise, show that $A_L(f) = 0$ in $\mathcal{C}^\infty(\mathbb{T}_L)$ if and only if $\hat{f} = 0$ on $L^\circ$.

PROBLEM 66. For a measurable set $\Omega \subset \mathbb{R}^n$, with non-zero measure, set $H = L^2(\Omega)$ and let $\mathcal{B} = \mathcal{B}(H)$ be the algebra of bounded linear operators on the Hilbert space $H$ with the norm on $\mathcal{B}$ being

$$\|B\|_{\mathcal{B}} = \sup\{\|Bf\|_H; f \in H,\ \|f\|_H = 1\}. \tag{3.22}$$

i) Show that $\mathcal{B}$ is complete with respect to this norm. Hint (probably not necessary!) For a Cauchy sequence $\{B_n\}$ observe that $B_n f$ is Cauchy for each $f \in H$.
ii) If $V \subset H$ is a finite-dimensional subspace and $W \subset H$ is a closed subspace with a finite-dimensional complement (that is $W + U = H$ for some finite-dimensional subspace $U$) show that there is a closed subspace $Y \subset W$ with finite-dimensional complement (in $H$) such that $V \perp Y$, that is $\langle v, y\rangle = 0$ for all $v \in V$ and $y \in Y$.
iii) If $A \in \mathcal{B}$ has finite rank (meaning $AH$ is a finite-dimensional vector space) show that there is a finite-dimensional space $V \subset H$ such that $AV \subset V$ and $AV^\perp = \{0\}$ where
$$V^\perp = \{f \in H; \langle f, v\rangle = 0\ \forall\ v \in V\}.$$
Hint: Set $R = AH$, a finite dimensional subspace by hypothesis. Let $N$ be the null space of $A$, show that $N^\perp$ is finite dimensional. Try $V = R + N^\perp$.
iv) If $A \in \mathcal{B}$ has finite rank, show that $(\mathrm{Id} - zA)^{-1}$ exists for all but a finite set of $\lambda \in \mathbb{C}$ (just quote some matrix theory). What might it mean to say in this case that $(\mathrm{Id} - zA)^{-1}$ is meromorphic in $z$? (No marks for this second part).
v) Recall that $\mathcal{K} \subset \mathcal{B}$ is the algebra of compact operators, defined as the closure of the space of finite rank operators. Show that $\mathcal{K}$ is an ideal in $\mathcal{B}$.
vi) If $A \in \mathcal{K}$ show that
$$\mathrm{Id} + A = (\mathrm{Id} + B)(\mathrm{Id} + A')$$
where $B \in \mathcal{K}$, $(\mathrm{Id} + B)^{-1}$ exists and $A'$ has finite rank. Hint: Use the invertibility of $\mathrm{Id} + B$ when $\|B\|_{\mathcal{B}} < 1$ proved in class.
vii) Conclude that if $A \in \mathcal{K}$ then
$$\{f \in H; (\mathrm{Id} + A)f = 0\} \text{ and } \big((\mathrm{Id} + A)H\big)^\perp \text{ are finite dimensional.}$$

PROBLEM 67. [Separable Hilbert spaces]

i) (Gramm-Schmidt Lemma). Let $\{v_i\}_{i\in\mathbb{N}}$ be a sequence in a Hilbert space $H$. Let $V_j \subset H$ be the span of the first $j$ elements and set $N_j = \dim V_j$. Show that there is an orthonormal sequence $e_1, \dots, e_j$ (finite if $N_j$ is bounded above) such that $V_j$ is the span of the first $N_j$ elements. Hint: Proceed by induction over $N$ such that the result is true for all $j$ with $N_j < N$. So, consider what happens for a value of $j$ with $N_j = N_{j-1}+1$ and add element $e_{N_j} \in V_j$ which is orthogonal to all the previous $e_k$'s.

ii) A Hilbert space is separable if it has a countable dense subset (sometimes people say Hilbert space when they mean separable Hilbert space). Show that every separable Hilbert space has a complete orthonormal sequence, that is a sequence $\{e_j\}$ such that $\langle u, e_j\rangle = 0$ for all $j$ implies $u = 0$.

iii) Let $\{e_j\}$ an orthonormal sequence in a Hilbert space, show that for any $a_j \in \mathbb{C}$,

$$\|\sum_{j=1}^{N} a_j e_j\|^2 = \sum_{j=1}^{N} |a_j|^2.$$

iv) (Bessel's inequality) Show that if $e_j$ is an orthormal sequence in a Hilbert space and $u \in H$ then

$$\|\sum_{j=1}^{N}\langle u, e_j\rangle e_j\|^2 \leq \|u\|^2$$

and conclude (assuming the sequence of $e_j$'s to be infinite) that the series

$$\sum_{j=1}^{\infty}\langle u, e_j\rangle e_j$$

converges in $H$.

v) Show that if $e_j$ is a complete orthonormal basis in a separable Hilbert space then, for each $u \in H$,

$$u = \sum_{j=1}^{\infty}\langle u, e_j\rangle e_j.$$

Problem 68. [Compactness] Let's agree that a compact set in a metric space is one for which every open cover has a finite subcover. You may use the compactness of closed bounded sets in a finite dimensional vector space.

i) Show that a compact subset of a Hilbert space is closed and bounded.

ii) If $e_j$ is a complete orthonormal subspace of a separable Hilbert space and $K$ is compact show that given $\epsilon > 0$ there exists $N$

such that

$$\sum_{j\ge N}|\langle u,e_j\rangle|^2\le\epsilon\ \forall\ u\in K. \tag{3.23}$$

iii) Conversely show that any closed bounded set in a separable Hilbert space for which (3.23) holds for some orthonormal basis is indeed compact.

iv) Show directly that any sequence in a compact set in a Hilbert space has a convergent subsequence.

v) Show that a subspace of $H$ which has a precompact unit ball must be finite dimensional.

vi) Use the existence of a complete orthonormal basis to show that any bounded sequence $\{u_j\}$, $\|u_j\|\le C$, has a weakly convergent subsequence, meaning that $\langle v,u_j\rangle$ converges in $\mathbb{C}$ along the subsequence for each $v\in H$. Show that the subsequnce can be chosen so that $\langle e_k,u_j\rangle$ converges for each $k$, where $e_k$ is the complete orthonormal sequence.

Problem 69. [Spectral theorem, compact case] Recall that a bounded operator $A$ on a Hilbert space $H$ is compact if $A\{\|u\|\le 1\}$ is precompact (has compact closure). Throughout this problem $A$ will be a compact operator on a separable Hilbert space, $H$.

i) Show that if $0\ne\lambda\in\mathbb{C}$ then

$$E_\lambda=\{u\in H; Au=\lambda u\}.$$

is finite dimensional.

ii) If $A$ is self-adjoint show that all eigenvalues (meaning $E_\lambda\ne\{0\}$) are real and that different eigenspaces are orthogonal.

iii) Show that $\alpha_A=\sup\{|\langle Au,u\rangle|^2\};\|u\|=1\}$ is attained. Hint: Choose a sequence such that $|\langle Au_j,u_j\rangle|^2$ tends to the supremum, pass to a weakly convergent sequence as discussed above and then using the compactness to a furhter subsequence such that $Au_j$ converges.

iv) If $v$ is such a maximum point and $f\perp v$ show that $\langle Av,f\rangle+\langle Af,v\rangle=0$.

v) If $A$ is also self-adjoint and $u$ is a maximum point as in iii) deduce that $Au=\lambda u$ for some $\lambda\in\mathbb{R}$ and that $\lambda=\pm\alpha$.

vi) Still assuming $A$ to be self-adjoint, deduce that there is a finite-dimensional subspace $M\subset H$, the sum of eigenspaces with eigenvalues $\pm\alpha$, containing all the maximum points.

vii) Continuing vi) show that $A$ restricts to a self-adjoint bounded operator on the Hilbert space $M^\perp$ and that the supremum in iii) for this new operator is smaller.

viii) Deduce that for any compact self-adjoint operator on a separable Hilbert space there is a complete orthonormal basis of eigenvectors. Hint: Be careful about the null space – it could be big.

PROBLEM 70. Show that a (complex-valued) square-integrable function $u \in L^2(\mathbb{R}^n)$ is continuous in the mean, in the sense that

$$\lim_{\epsilon\downarrow 0} \sup_{|y|<\epsilon} \int |u(x+y) - u(x)|^2 dx = 0. \tag{3.24}$$

Hint: Show that it is enough to prove this for non-negative functions and then that it suffices to prove it for non-negative simple functions and finally that it is enough to check it for the characteristic function of an open set of finite measure. Then use Problem 57 to show that it is true in this case.

PROBLEM 71. [Ascoli-Arzela] Recall the proof of the theorem of Ascoli and Arzela, that a subset of $\mathcal{C}^0_0(\mathbb{R}^n)$ is precompact (with respect to the supremum norm) if and only if it is equicontinuous and equi-small at infinity, i.e. given $\epsilon > 0$ there exists $\delta > 0$ such that for all elements $u \in B$

$$|y| < \delta \Longrightarrow \sup_{x\in\mathbb{R}^n} |u(x+y) = u(x)| < \epsilon \text{ and } |x| > 1/\delta \Longrightarrow |u(x)| < \epsilon. \tag{3.25}$$

PROBLEM 72. [Compactness of sets in $L^2(\mathbb{R}^n)$.] Show that a subset $B \subset L^2(\mathbb{R}^n)$ is *precompact* in $L^2(\mathbb{R}^n)$ if and only if it satisfies the following two conditions:

i) (Equi-continuity in the mean) For each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_{\mathbb{R}^n} |u(x+y) - u(x)|^2 dx < \epsilon \ \forall\ |y| < \delta,\ u \in B. \tag{3.26}$$

ii) (Equi-smallness at infinity) For each $\epsilon > 0$ there exists $R$ such that

$$\int_{|x|>R|} |u|^2 dx < \epsilon \ \forall\ u \in B. \tag{3.27}$$

Hint: Problem 70 shows that (3.26) holds for each $u \in L^2(\mathbb{R}^n)$; check that (3.27) also holds for each function. Then use a covering argument to prove that both these conditions must hold for a compact subset of $L^2(\mathbb{R})$ and hence for a precompact set. One method to prove the converse is to show that if (3.26) and (3.27) hold then $B$ is bounded and to use this to extract a weakly convergent sequence from any given sequence in $B$. Next show that (3.26) is equivalent to (3.27) for the set $\mathcal{F}(B)$, the image of $B$ under the Fourier transform. Show, possibly using Problem 71, that if $\chi_R$ is cut-off to a ball of radius $R$ then $\chi_R \mathcal{G}(\chi_R \hat{u}_n)$ converges strongly if $u_n$ converges weakly. Deduce from this that the weakly convergent subsequence in fact converges strongly so $\bar{B}$ is sequently compact, and hence is compact.

PROBLEM 73. Consider the space $\mathcal{C}_{\mathrm{c}}(\mathbb{R}^n)$ of all continuous functions on $\mathbb{R}^n$ with compact support. Thus each element vanishes in $|x| > R$ for some $R$, depending on the function. We want to give this a toplogy in terms of which is complete. We will use the *inductive limit* topology. Thus the whole space can be written as a countable union

$$\mathcal{C}_{\mathrm{c}}(\mathbb{R}^n) = \bigcup_n \{u : \mathbb{R}^n; u \text{ is continuous and } u(x) = 0 \text{ for } |x| > R\}. \tag{3.28}$$

Each of the space on the right is a Banach space for the supremum norm.

(1) Show that the supreumum norm is not complete on the whole of this space.
(2) Define a subset $U \subset \mathcal{C}_{\mathrm{c}}(\mathbb{R}^n)$ to be open if its intersection with each of the subspaces on the right in (3.28) is open w.r.t. the supremum norm.
(3) Show that this definition does yield a topology.
(4) Show that any sequence $\{f_n\}$ which is 'Cauchy' in the sense that for any open neighbourhood $U$ of 0 there exists $N$ such that $f_n - f_m \in U$ for all $n, m \geq N$, is convergent (in the corresponding sense that there exists $f$ in the space such that $f - f_n \in U$ eventually).
(5) If you are determined, discuss the corresponding issue for nets.

PROBLEM 74. Show that the continuity of a linear functional $u : \mathcal{C}^\infty_c(\mathbb{R}^n) \longrightarrow \mathbb{C}$ with respect to the inductive limit topology defined in (1.17) means precisely that for each $n \in \mathbb{N}$ there exists $k = k(n)$ and $C = C_n$ such that

$$|u(\varphi)| \leq C\|\varphi\|_{\mathcal{C}^k},\ \forall\ \varphi \in \dot{\mathcal{C}}^\infty(B(n)). \tag{3.29}$$

The point of course is that the 'order' $k$ and the constnat $C$ can both increase as $n$, measuring the size of the support, increases.

PROBLEM 75. [Restriction from Sobolev spaces] The Sobolev embedding theorem shows that a function in $H^m(\mathbb{R}^n)$, for $m > n/2$ is continuous – and hence can be restricted to a subspace of $\mathbb{R}^n$. In fact this works more generally. Show that there is a well defined *restriction* *map*

$$H^m(\mathbb{R}^n) \longrightarrow H^{m-\frac{1}{2}}(\mathbb{R}^n) \text{ if } m > \frac{1}{2} \tag{3.30}$$

with the following properties:

(1) On $\mathcal{S}(\mathbb{R}^n)$ it is given by $u \longmapsto u(0, x')$, $x' \in \mathbb{R}^{n-1}$.

(2) It is continuous and linear.

Hint: Use the usual method of finding a weak version of the map on smooth Schwartz functions; namely show that in terms of the Fourier transforms on $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$

$$\widehat{u(0,\cdot)}(\xi') = (2\pi)^{-1} \int_{\mathbb{R}} \hat{u}(\xi_1, \xi') d\xi_1, \ \forall \ \xi' \in \mathbb{R}^{n-1}. \tag{3.31}$$

Use Cauchy's inequality to show that this is continuous as a map on Sobolev spaces as indicated and then the density of $\mathcal{S}(\mathbb{R}^n)$ in $H^m(\mathbb{R}^n)$ to conclude that the map is well-defined and unique.

Problem 76. [Restriction by WF] From class we know that the product of two distributions, one with compact support, is defined provided they have no 'opposite' directions in their wavefront set:

$$(x, \omega) \in \mathrm{WF}(u) \Longrightarrow (x, -\omega) \notin \mathrm{WF}(v) \text{ then } uv \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n). \tag{3.32}$$

Show that this product has the property that $f(uv) = (fu)v = u(fv)$ if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$. Use this to define a restriction map to $x_1 = 0$ for distributions of compact support satisfying $((0, x'), (\omega_1, 0)) \notin \mathrm{WF}(u)$ as the product

$$u_0 = u\delta(x_1). \tag{3.33}$$

[Show that $u_0(f)$, $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ only depends on $f(0, \cdot) \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$.

Problem 77. [Stone's theorem] For a bounded self-adjoint operator $A$ show that the spectral measure can be obtained from the resolvent in the sense that for $\phi, \psi \in H$

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \langle [(A - t - i\epsilon)^{-1} - (A + t + i\epsilon)^{-1}]\phi, \psi\rangle \longrightarrow \mu_{\phi,\psi} \tag{3.34}$$

in the sense of distributions – or measures if you are prepared to work harder!

Problem 78. If $u \in \mathcal{S}(\mathbb{R}^n)$ and $\psi' = \psi_R + \mu$ is, as in the proof of Lemma 7.5, such that

$$\mathrm{supp}(\psi') \cap \mathrm{Css}(u) = \emptyset$$

show that

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto \phi\psi' u \in \mathcal{S}(\mathbb{R}^n)$$

is continuous and hence (or otherwise) show that the functional $u_1 u_2$ defined by (7.20) is an element of $\mathcal{S}'(\mathbb{R}^n)$.

Problem 79. Under the conditions of Lemma 7.10 show that

$$\operatorname{Css}(u*v)\cap\mathbb{S}^{n-1}\subset\{\frac{sx+ty}{|sx+ty|},\ |x|=|y|=1,\ x\in\operatorname{Css}(u),\ y\in\operatorname{Css}(v),\ 0\le s,t\le 1\}. \tag{3.35}$$

Notice that this make sense exactly because $sx+ty=0$ implies that $t/s=1$ but $x+y\neq 0$ under these conditions by the assumption of Lemma 7.10.

Problem 80. Show that the pairing $u(v)$ of two distributions $u,v\in {}^{\mathrm{b}}S'(\mathbb{R}^n)$ may be defined under the hypothesis (7.50).

Problem 81. Show that under the hypothesis (7.51)

$$\begin{gathered}\operatorname{WF}_{\mathrm{sc}}(u*v)\subset\{(x+y,p);(x,p)\in\operatorname{WF}_{\mathrm{sc}}(u)\cap(\mathbb{R}^n\times\mathbb{S}^{n-1}),\ (y,p)\in\operatorname{WF}_{\mathrm{sc}}(v)\cap(\mathbb{R}^n\times\mathbb{S}^{n-1})\}\\ \cup\{(\theta,q)\in\mathbb{S}^{n-1}\times\mathbb{B}^n;\theta=\frac{s'\theta'+s''\theta''}{|s'\theta'+s''\theta''|},\ 0\le s',s''\le 1,\\ (\theta',q)\in\operatorname{WF}_{\mathrm{sc}}(u)\cap(\mathbb{S}^{n-1}\times\mathbb{B}^n),\ (\theta'',q)\in\operatorname{WF}_{\mathrm{sc}}(v)\cap(\mathbb{S}^{n-1}\times\mathbb{B}^n)\}.\end{gathered} \tag{3.36}$$

Problem 82. Formulate and prove a bound similar to (3.36) for $\operatorname{WF}_{\mathrm{sc}}(uv)$ when $u,\ v\in\mathcal{S}'(\mathbb{R}^n)$ satisfy (7.50).

Problem 83. Show that for convolution $u*v$ defined under condition (7.51) it is still true that

$$P(D)(u*v)=(P(D)u)*v=u*(P(D)v). \tag{3.37}$$

Problem 84. Using Problem 80 (or otherwise) show that integration is defined as a functional

$$\{u\in\mathcal{S}'(\mathbb{R}^n);(\mathbb{S}^{n-1}\times\{0\})\cap\operatorname{WF}_{\mathrm{sc}}(u)=\emptyset\}\longrightarrow\mathbb{C}. \tag{3.38}$$

If $u$ satisfies this condition, show that $\int P(D)u=c\int u$ where $c$ is the constant term in $P(D)$, i.e. $P(D)1=c$.

Problem 85. Compute $\operatorname{WF}_{\mathrm{sc}}(E)$ where $E=C/|x-y|$ is the standard fundamental solution for the Laplacian on $\mathbb{R}^3$. Using Problem 83 give a condition on $\operatorname{WF}_{\mathrm{sc}}(f)$ under which $u=E*f$ is defined and satisfies $\Delta u=f$. Show that under this condition $\int f$ is defined using Problem 84. What can you say about $\operatorname{WF}_{\mathrm{sc}}(u)$? Why is it not the case that $\int\Delta u=0$, even though this is true if $u$ has compact support?

## 4. Solutions to (some of) the problems

SOLUTION 4.1 (To Problem 10). (by Matjaž Konvalinka).

Since the topology on $\mathbb{N}$, inherited from $\mathbb{R}$, is discrete, a set is compact if and only if it is finite. If a sequence $\{x_n\}$ (i.e. a function $\mathbb{N} \to \mathbb{C}$) is in $\mathcal{C}_0(\mathbb{N})$ if and only if for any $\epsilon > 0$ there exists a compact (hence finite) set $F_\epsilon$ so that $|x_n| < \epsilon$ for any $n$ not in $F_\epsilon$. We can assume that $F_\epsilon = \{1, \ldots, n_\epsilon\}$, which gives us the condition that $\{x_n\}$ is in $\mathcal{C}_0(\mathbb{N})$ if and only if it converges to 0. We denote this space by $c_0$, and the supremum norm by $\|\cdot\|_0$. A sequence $\{x_n\}$ will be abbreviated to $x$.

Let $l^1$ denote the space of (real or complex) sequences $x$ with a finite 1-norm

$$\|x\|_1 = \sum_{n=1}^{\infty} |x_n|.$$

We can define pointwise summation and multiplication with scalars, and $(l^1, \|\cdot\|_1)$ is a normed (in fact Banach) space. Because the functional

$$y \mapsto \sum_{n=1}^{\infty} x_n y_n$$

is linear and bounded ($|\sum_{n=1}^{\infty} x_n y_n| \leq \sum_{n=1}^{\infty} |x_n||y_n| \leq \|x\|_0 \, \|y\|_1$) by $\|x\|_0$, the mapping

$$\Phi \colon l^1 \longmapsto c_0^*$$

defined by

$$x \mapsto \left( y \mapsto \sum_{n=1}^{\infty} x_n y_n \right)$$

is a (linear) well-defined mapping with norm at most 1. In fact, $\Phi$ is an isometry because if $|x_j| = \|x\|_0$ then $|\Phi(x)(e_j)| = 1$ where $e_j$ is the $j$-th unit vector. We claim that $\Phi$ is also surjective (and hence an isometric isomorphism). If $\varphi$ is a functional on $c_0$ let us denote $\varphi(e_j)$ by $x_j$. Then $\Phi(x)(y) = \sum_{n=1}^{\infty} \varphi(e_n) y_n = \sum_{n=1}^{\infty} \varphi(y_n e_n) = \varphi(y)$ (the last equality holds because $\sum_{n=1}^{\infty} y_n e_n$ converges to $y$ in $c_0$ and $\varphi$ is continuous with respect to the topology in $c_0$), so $\Phi(x) = \varphi$.

SOLUTION 4.2 (To Problem 29). (Matjaž Konvalinka) Since

$$D_x H(\varphi) = H(-D_x \varphi) = i \int_{-\infty}^{\infty} H(x) \varphi'(x)\, dx = i \int_0^{\infty} \varphi'(x)\, dx = i(0 - \varphi(0)) = -i\delta(\varphi),$$

we get $D_x H = C\delta$ for $C = -i$.

SOLUTION 4.3 (To Problem 40). (Matjaž Konvalinka) Let us prove this in the case where $n = 1$. Define (for $b \neq 0$)

$$U(x) = u(b) - u(x) - (b-x)u'(x) - \ldots - \frac{(b-x)^{k-1}}{(k-1)!}u^{(k-1)}(x);$$

then

$$U'(x) = -\frac{(b-x)^{k-1}}{(k-1)!}u^{(k)}(x).$$

For the continuously differentiable function $V(x) = U(x) - (1-x/b)^k U(0)$ we have $V(0) = V(b) = 0$, so by Rolle's theorem there exists $\zeta$ between 0 and $b$ with

$$V'(\zeta) = U'(\zeta) + \frac{k(b-\zeta)^{k-1}}{b^k}U(0) = 0$$

Then

$$U(0) = -\frac{b^k}{k(b-\zeta)^{k-1}}U'(\zeta),$$

$$u(b) = u(0) + u'(0)b + \ldots + \frac{u^{(k-1)}(0)}{(k-1)!}b^{k-1} + \frac{u^{(k)}(\zeta)}{k!}b^k.$$

The required decomposition is $u(x) = p(x) + v(x)$ for

$$p(x) = u(0) + u'(0)x + \frac{u''(0)}{2}x^2 + \ldots + \frac{u^{(k-1)}(0)}{(k-1)!}x^{k-1} + \frac{u^{(k)}(0)}{k!}x^k,$$

$$v(x) = u(x) - p(x) = \frac{u^{(k)}(\zeta) - u^{(k)}(0)}{k!}x^k$$

for $\zeta$ between 0 and $x$, and since $u^{(k)}$ is continuous, $(u(x) - p(x))/x^k$ tends to 0 as $x$ tends to 0.

The proof for general $n$ is not much more difficult. Define the function $w_x \colon I \to \mathbb{R}$ by $w_x(t) = u(tx)$. Then $w_x$ is $k$-times continuously differentiable,

$$w_x'(t) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(tx)x_i,$$

$$w_x''(t) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}(tx)x_i x_j,$$

$$w_x^{(l)}(t) = \sum_{l_1+l_2+\ldots+l_i=l} \frac{l!}{l_1!l_2!\cdots l_i!}\frac{\partial^l u}{\partial x_1^{l_1}\partial x_2^{l_2}\cdots\partial x_i^{l_i}}(tx)x_1^{l_1}x_2^{l_2}\cdots x_i^{l_i}$$

so by above $u(x) = w_x(1)$ is the sum of some polynomial $p$ (od degree $k$), and we have

$$\frac{u(x) - p(x)}{|x|^k} = \frac{v_x(1)}{|x|^k} = \frac{w_x^{(k)}(\zeta_x) - w_x^{(k)}(0)}{k!|x|^k},$$

so it is bounded by a positive combination of terms of the form

$$\left| \frac{\partial^l u}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_i^{l_i}}(\zeta_x x) - \frac{\partial^l u}{\partial x_1^{l_1} \partial x_2^{l_2} \cdots \partial x_i^{l_i}}(0) \right|$$

with $l_1 + \ldots + l_i = k$ and $0 < \zeta_x < 1$. This tends to zero as $x \to 0$ because the derivative is continuous.

Solution 4.4 (Solution to Problem 41). (Matjž Konvalinka) Obviously the map $\mathcal{C}_0(\mathbb{B}^n) \to \mathcal{C}(\mathbb{B}^n)$ is injective (since it is just the inclusion map), and $f \in \mathcal{C}(\mathbb{B}^n)$ is in $\mathcal{C}_0(\mathbb{B}^n)$ if and only if it is zero on $\partial\mathbb{B}^n$, ie. if and only if $f|_{\mathbb{S}^{n-1}} = 0$. It remains to prove that any map $g$ on $\mathbb{S}^{n-1}$ is the restriction of a continuous function on $\mathbb{B}^n$. This is clear since

$$f(x) = \begin{cases} |x|g(x/|x|) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is well-defined, coincides with $f$ on $\mathbb{S}^{n-1}$, and is continuous: if $M$ is the maximum of $|g|$ on $\mathbb{S}^{n-1}$, and $\epsilon > 0$ is given, then $|f(x)| < \epsilon$ for $|x| < \epsilon/M$.

Solution 4.5. (partly Matjaž Konvalinka)
For any $\varphi \in \mathcal{S}(\mathbb{R})$ we have

$$|\int_{-\infty}^{\infty} \varphi(x)dx| \leq \int_{-\infty}^{\infty} |\varphi(x)|dx \leq \sup((1+x|^2)|\varphi(x)|) \int_{-\infty}^{\infty} (1+|x|^2)^{-1}dx$$
$$\leq C \sup((1 + x|^2)|\varphi(x)|).$$

Thus $\mathcal{S}(\mathbb{R}) \ni \varphi \longmapsto \int_{\mathbb{R}} \varphi dx$ is continous.

Now, choose $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R})$ with $\int_{\mathbb{R}} \phi(x)dx = 1$. Then, for $\psi \in \mathcal{S}(\mathbb{R})$, set

$$A\psi(x) = \int_{-\infty}^{x} (\psi(t) - c(\psi)\phi(t))\,dt,\ c(\psi) = \int_{-\infty}^{\infty} \psi(s)\,ds. \tag{4.1}$$

Note that the assumption on $\phi$ means that

$$A\psi(x) = -\int_{x}^{\infty} (\psi(t) - c(\psi)\phi(t))\,dt \tag{4.2}$$

Clearly $A\psi$ is smooth, and in fact it is a Schwartz function since

$$\frac{d}{dx}(A\psi(x)) = \psi(x) - c\phi(x) \in \mathcal{S}(\mathbb{R}) \tag{4.3}$$

so it suffices to show that $x^k A\psi$ is bounded for any $k$ as $|x| \to \pm\infty$. Since $\psi(t) - c\phi(t) \le C_k t^{-k-1}$ in $t \ge 1$ it follows from (4.2) that

$$|x^k A\psi(x)| \le Cx^k \int_x^\infty t^{-k-1} dt \le C',\ k > 1,\ \text{in } x > 1.$$

A similar estimate as $x \to -\infty$ follows from (4.1). Now, $A$ is clearly linear, and it follows from the estimates above, including that on the integral, that for any $k$ there exists $C$ and $j$ such that

$$\sup_{\alpha,\beta\le k} |x^\alpha D^\beta A\psi| \le C \sum_{\alpha',\beta'\le j} \sup_{x\in\mathbb{R}} |x^{\alpha'} D^{\beta'}\psi|.$$

Finally then, given $u \in \mathcal{S}'(\mathbb{R})$ define $v(\psi) = -u(A\psi)$. From the continuity of $A$, $v \in \mathcal{S}(\mathbb{R})$ and from the definition of $A$, $A(\psi') = \psi$. Thus

$$dv/dx(\psi) = v(-\psi') = u(A\psi') = u(\psi) \Longrightarrow \frac{dv}{dx} = u.$$

Solution 4.6. We have to prove that $\langle\xi\rangle^{m+m'}\widehat{u} \in L_2(\mathbb{R}^n)$, in other words, that

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2(m+m')}|\widehat{u}|^2\, d\xi < \infty.$$

But that is true since

$$\int_{\mathbb{R}^n} \langle\xi\rangle^{2(m+m')}|\widehat{u}|^2\, d\xi = \int_{\mathbb{R}^n} \langle\xi\rangle^{2m'}(1+\xi_1^2+\ldots+\xi_n^2)^m|\widehat{u}|^2\, d\xi =$$

$$= \int_{\mathbb{R}^n} \langle\xi\rangle^{2m'} \left(\sum_{|\alpha|\le m} C_\alpha \xi^{2\alpha}\right) |\widehat{u}|^2\, d\xi = \sum_{|\alpha|\le m} C_\alpha \left(\int_{\mathbb{R}^n} \langle\xi\rangle^{2m'}\xi^{2\alpha}|\widehat{u}|^2\, d\xi\right)$$

and since $\langle\xi\rangle^{m'}\xi^\alpha\widehat{u} = \langle\xi\rangle^{m'}\widehat{D^\alpha u}$ is in $L^2(\mathbb{R}^n)$ (note that $u \in H^m(\mathbb{R}^n)$ follows from $D^\alpha u \in H^{m'}(\mathbb{R}^n)$, $|\alpha| \le m$). The converse is also true since $C_\alpha$ in the formula above are strictly positive.

Solution 4.7. Take $v \in L^2(\mathbb{R}^n)$, and define subsets of $\mathbb{R}^n$ by

$$E_0 = \{x\colon |x| \le 1\},$$
$$E_i = \{x\colon |x| \ge 1, |x_i| = \max_j |x_j|\}.$$

Then obviously we have $1 = \sum_{i=0}^n \chi_{E_j}$ a.e., and $v = \sum_{j=0}^n v_j$ for $v_j = \chi_{E_j} v$. Then $\langle x\rangle$ is bounded by $\sqrt{2}$ on $E_0$, and $\langle x\rangle v_0 \in L^2(\mathbb{R}^n)$; and on $E_j$, $1 \le j \le n$, we have

$$\frac{\langle x\rangle}{|x_j|} \le \frac{(1+n|x_j|^2)^{1/2}}{|x_j|} = \bigl(n + 1/|x_j|^2\bigr)^{1/2} \le (2n)^{1/2},$$

so $\langle x\rangle v_j = x_j w_j$ for $w_j \in L^2(\mathbb{R}^n)$. But that means that $\langle x\rangle v = w_0 + \sum_{j=1}^n x_j w_j$ for $w_j \in L^2(\mathbb{R}^n)$.
If $u$ is in $L^2(\mathbb{R}^n)$ then $\widehat{u} \in L^2(\mathbb{R}^n)$, and so there exist $w_0, \dots, w_n \in L^2(\mathbb{R}^n)$ so that

$$\langle\xi\rangle\widehat{u} = w_0 + \sum_{j=1}^n \xi_j w_j,$$

in other words

$$\widehat{u} = \widehat{u}_0 + \sum_{j=1}^n \xi_j \widehat{u}_j$$

where $\langle\xi\rangle\widehat{u}_j \in L^2(\mathbb{R}^n)$. Hence

$$u = u_0 + \sum_{j=1}^n D_j u_j$$

where $u_j \in H^1(\mathbb{R}^n)$.

Solution 4.8. Since

$$D_x H(\varphi) = H(-D_x\varphi) = i\int_{-\infty}^{\infty} H(x)\varphi'(x)\,dx = i\int_0^{\infty}\varphi'(x)\,dx = i(0-\varphi(0)) = -i\delta(\varphi),$$

we get $D_x H = C\delta$ for $C = -i$.

Solution 4.9. It is equivalent to ask when $\langle\xi\rangle^m\widehat{\delta_0}$ is in $L^2(\mathbb{R}^n)$. Since

$$\widehat{\delta_0}(\psi) = \delta_0(\widehat{\psi}) = \widehat{\psi}(0) = \int_{\mathbb{R}^n}\psi(x)\,dx = 1(\psi),$$

this is equivalent to finding $m$ such that $\langle\xi\rangle^{2m}$ has a finite integral over $\mathbb{R}^n$. One option is to write $\langle\xi\rangle = (1+r^2)^{1/2}$ in spherical coordinates, and to recall that the Jacobian of spherical coordinates in $n$ dimensions has the form $r^{n-1}\Psi(\varphi_1, \dots, \varphi_{n-1})$, and so $\langle\xi\rangle^{2m}$ is integrable if and only if

$$\int_0^{\infty}\frac{r^{n-1}}{(1+r^2)^m}\,dr$$

converges. It is obvious that this is true if and only if $n-1-2m < -1$, ie. if and only if $m > n/2$.

Solution 4.10 (Solution to Problem31). We know that $\delta \in H^m(\mathbb{R}^n)$ for any $m < -n/1$. Thus is just because $\langle\xi\rangle^p \in L^2(\mathbb{R}^n)$ when $p < -n/2$. Now, divide $\mathbb{R}^n$ into $n+1$ regions, as above, being $A_0 = \{\xi; |\xi| \le 1$ and $A_i = \{\xi; |\xi_i| = \sup_j |\xi_j|,\ |\xi| \ge 1\}$. Let $v_0$ have Fourier transform $\chi_{A_0}$ and for $i = 1, \dots, n$, $v_i \in \mathcal{S};(\mathbb{R}^n)$ have Fourier transforms $\xi_i^{-n-1}\chi_{A_i}$. Since $|\xi_i| > c\langle\xi\rangle$ on the support of $\widehat{v_i}$ for each $i = 1, \dots, n$, each term is in $H^m$ for any $m < 1 + n/2$ so, by the Sobolev embedding theorem, each $v_i \in \mathcal{C}_0^0(\mathbb{R}^n)$ and

$$1 = \hat{v}_0 \sum_{i=1}^{n} \xi_i^{n+1} \widehat{v_i} \Longrightarrow \delta = v_0 + \sum_i D_i^{n+1} v_i. \tag{4.4}$$

How to see that this cannot be done with $n$ or less derivatives? For the moment I do not have a proof of this, although I believe it is true. Notice that we are actually proving that $\delta$ can be written

$$\delta = \sum_{|\alpha| \le n+1} D^\alpha u_\alpha, \ u_\alpha \in H^{n/2}(\mathbb{R}^n). \tag{4.5}$$

This cannot be improved to $n$ from $n+1$ since this would mean that $\delta \in H^{-n/2}(\mathbb{R}^n)$, which it isn't. However, what I am asking is a little more subtle than this.

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