# Differential Analysis
# Lecture notes for 18.155 and 156

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# Contents

## Introduction

These notes are for the graduate analysis courses (18.155 and 18.156) at MIT. They are based on various earlier similar courses. In giving the lectures I usually cut many corners!

To thank:- Austin Frakt, Philip Dorrell, Jacob Bernstein....

CHAPTER 1

# Measure and Integration

A rather quick review of measure and integration.

## 1. Continuous functions

A the beginning I want to remind you of things I think you already know and then go on to show the direction the course will be taking. Let me first try to set the context.

One basic notion I assume you are reasonably familiar with is that of a *metric space* ([**6**] p.9). This consists of a set, $X$, and a distance function

$$d : X \times X = X^2 \longrightarrow [0, \infty)\,,$$

satisfying the following three axioms:

$$\begin{aligned} &i)\ d(x,y) = 0 \Leftrightarrow x = y,\ (\text{and } d(x,y) \geq 0)\\ &ii)\ d(x,y) = d(y,x)\ \forall\ x,y \in X \\ &iii)\ d(x,y) \leq d(x,z) + d(z,y)\ \forall\ x,y,z \in X. \end{aligned} \tag{1.1}$$

The basic theory of metric spaces deals with properties of subsets (open, closed, compact, connected), sequences (convergent, Cauchy) and maps (continuous) and the relationship between these notions. Let me just remind you of one such result.

Proposition 1.1. *A map $f : X \to Y$ between metric spaces is continuous if and only if one of the three following equivalent conditions holds*

(1) $f^{-1}(O) \subset X$ *is open* $\forall\ O \subset Y$ *open.*
(2) $f^{-1}(C) \subset X$ *is closed* $\forall\ C \subset Y$ *closed.*
(3) $\lim_{n\to\infty} f(x_n) = f(x)$ *in* $Y$ *if* $x_n \to x$ *in* $X$.

The basic example of a metric space is Euclidean space. Real $n$-dimensional Euclidean space, $\mathbb{R}^n$, is the set of ordered $n$-tuples of real numbers

$$x = (x_1, \dots, x_n) \in \mathbb{R}^n\,,\ x_j \in \mathbb{R}\,,\ j = 1, \dots, n\,.$$

It is also the basic example of a vector (or linear) space with the operations

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$
$$cx = (cx_1, \dots, cx_n)\,.$$

The metric is usually taken to be given by the Euclidean metric

$$|x| = (x_1^2 + \cdots + x_n^2)^{1/2} = (\sum_{j=1}^{n} x_j^2)^{1/2}\,,$$

in the sense that

$$d(x, y) = |x - y|\ .$$

Let us abstract this immediately to the notion of a normed vector space, or normed space. This is a vector space $V$ (over $\mathbb{R}$ or $\mathbb{C}$) equipped with a *norm*, which is to say a function

$$\|\ \| : V \longrightarrow [0, \infty)$$

satisfying

$$\begin{aligned} &i)\ \|v\| = 0 \Longleftrightarrow v = 0,\\ &ii)\ \|cv\| = |c|\,\|v\|\ \forall\ c \in \mathbb{K},\\ &iii)\ \|v + w\| \le \|v\| + \|w\|. \end{aligned} \tag{1.2}$$

This means that $(V, d)$, $d(v, w) = \|v - w\|$ is a vector space; I am also using $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$ as is appropriate.

The case of a finite dimensional normed space is not very interesting because, apart from the dimension, they are all "the same". We shall say (in general) that two norms $\|\bullet\|_1$ and $\|\bullet\|_2$ on $V$ are *equivalent* of there exists $C > 0$ such that

$$\frac{1}{C}\|v\|_1 \le \|v\|_2 \le C\|v\|_1\ \forall\ v \in V\,.$$

PROPOSITION 1.2. *Any two norms on a finite dimensional vector space are equivalent.*

So, we are mainly interested in the infinite dimensional case. I will start the course, in a slightly unorthodox manner, by concentrating on one such normed space (really one class). Let $X$ be a metric space. The case of a continuous function, $f : X \to \mathbb{R}$ (or $\mathbb{C}$) is a special case of Proposition 1.1 above. We then define

$$C(X) = \{f : X \to \mathbb{R},\ f \text{ bounded and continuous}\}\,.$$

In fact the same notation is generally used for the space of complex-valued functions. If we want to distinguish between these two possibilities we can use the more pedantic notation $C(X; \mathbb{R})$ and $C(X; \mathbb{C})$.

Now, the 'obvious' norm on this linear space is the supremum (or 'uniform') norm

$$\|f\|_\infty = \sup_{x\in X} |f(x)| \ .$$

Here $X$ is an arbitrary metric space. For the moment $X$ is supposed to be a "physical" space, something like $\mathbb{R}^n$. Corresponding to the finite-dimensionality of $\mathbb{R}^n$ we often assume (or demand) that $X$ is *locally compact*. This just means that every point has a compact neighborhood, i.e., is in the interior of a compact set. Whether locally compact or not we can consider

$$(1.3)\qquad \mathcal{C}_0(X) = \left\{ f \in \mathcal{C}(X);\ \forall\ \epsilon > 0\ \exists\ K \Subset X s.t. \sup_{x\notin K} |f(x)| \le \epsilon \right\} .$$

Here the notation $K \Subset X$ means '$K$ is a compact subset of $X$'.

If $V$ is a normed linear space we are particularly interested in the continuous linear functionals on $V$. Here 'functional' just means function but $V$ is allowed to be 'large' (not like $\mathbb{R}^n$) so 'functional' is used for historical reasons.

Proposition 1.3. *The following are equivalent conditions on a linear functional* $u : V \longrightarrow \mathbb{R}$ *on a normed space* $V$.

(1) $u$ *is continuous.*
(2) $u$ *is continuous at* $0$.
(3) $\{u(f) \in \mathbb{R}\,;\ f \in V\,,\ \|f\| \le 1\}$ *is bounded.*
(4) $\exists\ C\ s.t.\ |u(f)| \le C\|f\|\ \forall\ f \in V$.

Proof. (1) $\Longrightarrow$ (2) by definition. Then (2) implies that $u^{-1}(-1,1)$ is a neighborhood of $0 \in V$, so for some $\epsilon > 0$, $u(\{f \in V; \|f\| < \epsilon\}) \subset (-1,1)$. By linearity of $u$, $u(\{f \in V; \|f\| < 1\}) \subset (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ is bounded, so (2) $\Longrightarrow$ (3). Then (3) implies that

$$|u(f)| \le C\ \forall\ f \in V, \|f\| \le 1$$

for some $C$. Again using linearity of $u$, if $f \ne 0$,

$$|u(f)| \le \|f\| u\left(\frac{f}{\|f\|}\right) \le C\|f\|\,,$$

giving (4). Finally, assuming (4),

$$|u(f) - u(g)| = |u(f-g)| \le C\|f-g\|$$

shows that $u$ is continuous at any point $g \in V$. □

In view of this identification, continuous linear functionals are often said to be *bounded*. One of the important ideas that we shall exploit later is that of 'duality'. In particular this suggests that it is a good idea to examine the totality of bounded linear functionals on $V$. The *dual* space is

$$V' = V^* = \{u : V \longrightarrow \mathbb{K}, \text{ linear and bounded}\}.$$

This is also a normed linear space where the linear operations are

$$\begin{matrix} (u+v)(f) = u(f) + v(f) \\ (cu)(f) = c(u(f)) \end{matrix} \ \forall\ f \in V. \tag{1.4}$$

The natural norm on $V'$ is

$$\|u\| = \sup_{\|f\| \le 1} |u(f)|.$$

This is just the 'best constant' in the boundedness estimate,

$$\|u\| = \inf \{C; |u(f)| \le C\|f\| \ \forall\ f \subset V\}.$$

One of the basic questions I wish to pursue in the first part of the course is: What is the dual of $\mathcal{C}_0(X)$ for a locally compact metric space $X$? The answer is given by Riesz' representation theorem, in terms of (Borel) measures.

Let me give you a vague picture of 'regularity of functions' which is what this course is about, even though I have not introduced most of these spaces yet. Smooth functions (and small spaces) are towards the top. Duality flips up and down and as we shall see $L^2$, the space of Lebesgue square-integrable functions, is generally 'in the middle'. What I will discuss first is the right side of the diagramme, where we have the space of continuous functions on $\mathbb{R}^n$ which vanish at infinity and its dual space, $M_{\text{fin}}(\mathbb{R}^n)$, the space of finite Borel measures. There are many other spaces that you may encounter, here I only include test functions, Schwartz functions, Sobolev spaces and their duals; $k$ is a general positive integer.

$$
\begin{tikzcd}
\mathcal{S}(\mathbb{R}^n) \arrow[d, hook] \arrow[drr, hook] & & \\
H^k(\mathbb{R}^n) \arrow[d, hook] & \mathcal{C}_c(\mathbb{R}^n) \arrow[r, hook] \arrow[dl, hook] \arrow[dd, hook] & \mathcal{C}_0(\mathbb{R}^n) \\
L^2(\mathbb{R}^b) \arrow[d, hook] \arrow[dr, hook] & & \\
H^{-k}(\mathbb{R}^n) \arrow[d, hook] & M(\mathbb{R}^n) & M_{\text{fin}}(\mathbb{R}^n) \arrow[l, hook'] \arrow[dll, hook] \\
\mathcal{S}'(\mathbb{R}^n). & &
\end{tikzcd} \tag{1.5}
$$

I have set the goal of understanding the dual space $M_{\text{fin}}(\mathbb{R}^n)$ of $\mathcal{C}_0(X)$, where $X$ is a locally compact metric space. This will force me to go through the elements of measure theory and Lebesgue integration. It does require a little forcing!

The basic case of interest is $\mathbb{R}^n$. Then an obvious example of a continuous linear functional on $\mathcal{C}_0(\mathbb{R}^n)$ is given by Riemann integration, for instance over the unit cube $[0,1]^n$:

$$u(f) = \int_{[0,1]^n} f(x)\, dx\,.$$

In some sense we must show that *all* continuous linear functionals on $\mathcal{C}_0(X)$ are given by integration. However, we have to interpret integration somewhat widely since there are also *evaluation functionals*. If $z \in X$ consider the Dirac delta

$$\delta_z(f) = f(z)\,.$$

This is also called a *point mass* of $z$. So we need a theory of measure and integration wide enough to include both of these cases.

One special feature of $\mathcal{C}_0(X)$, compared to general normed spaces, is that there is a notion of positivity for its elements. Thus $f \geq 0$ just means $f(x) \geq 0 \ \forall \ x \in X$.

Lemma 1.4. *Each $f \in \mathcal{C}_0(X)$ can be decomposed uniquely as the difference of its positive and negative parts*

$$f = f_+ - f_-\,,\ f_\pm \in \mathcal{C}_0(X)\,,\ f_\pm(x) \leq |f(x)| \ \ \forall \ x \in X\,. \tag{1.6}$$

PROOF. Simply define

$$f_{\pm}(x) = \begin{cases} \pm f(x) & \text{if} \quad \pm f(x) \geq 0 \\ 0 & \text{if} \quad \pm f(x) < 0 \end{cases}$$

for the same sign throughout. Then (3.8) holds. Observe that $f_+$ is continuous at each $y \in X$ since, with $U$ an appropriate neighborhood of $y$, in each case

$$f(y) > 0 \Longrightarrow f(x) > 0 \text{ for } x \in U \Longrightarrow f_+ = f \text{ in } U$$

$$f(y) < 0 \Longrightarrow f(x) < 0 \text{ for } x \in U \Longrightarrow f_+ = 0 \text{ in } U$$

$$f(y) = 0 \Longrightarrow \text{ given } \ \epsilon > 0 \ \exists \ U \text{ s.t. } \ |f(x)| < \epsilon \text{ in } U$$

$$\Longrightarrow |f_+(x)| < \epsilon \text{ in } U\,.$$

Thus $f_- = f - f_+ \in \mathcal{C}_0(X)$, since both $f_+$ and $f_-$ vanish at infinity. □

We can similarly split elements of the dual space into positive and negative parts although it is a little bit more delicate. We say that $u \in (\mathcal{C}_0(X))'$ is positive if

$$u(f) \geq 0 \ \forall \ 0 \leq f \in \mathcal{C}_0(X)\,. \tag{1.7}$$

For a general (real) $u \in (\mathcal{C}_0(X))'$ and for each $0 \leq f \in \mathcal{C}_0(X)$ set

$$u_+(f) = \sup\left\{u(g)\,;\, g \in \mathcal{C}_0(X)\,,\, 0 \leq g(x) \leq f(x) \ \forall \ x \in X\right\}\,. \tag{1.8}$$

This is certainly finite since $u(g) \leq C\|g\|_\infty \leq C\|f\|_\infty$. Moreover, if $0 < c \in \mathbb{R}$ then $u_+(cf) = cu_+(f)$ by inspection. Suppose $0 \leq f_i \in \mathcal{C}_0(X)$ for $i = 1, 2$. Then given $\epsilon > 0$ there exist $g_i \in \mathcal{C}_0(X)$ with $0 \leq g_i(x) \leq f_i(x)$ and

$$u_+(f_i) \leq u(g_i) + \epsilon\,.$$

It follows that $0 \leq g(x) \leq f_1(x) + f_2(x)$ if $g = g_1 + g_2$ so

$$u_+(f_1 + f_2) \geq u(g) = u(g_1) + u(g_2) \geq u_+(f_1) + u_+(f_2) - 2\epsilon\,.$$

Thus

$$u_+(f_1 + f_2) \geq u_+(f_1) + u_+(f_2).$$

Conversely, if $0 \leq g(x) \leq f_1(x) + f_2(x)$ set $g_1(x) = \min(g, f_1) \in \mathcal{C}_0(X)$ and $g_2 = g - g_1$. Then $0 \leq g_i \leq f_i$ and $u_+(f_1) + u_+(f_2) \geq u(g_1) + u(g_2) = u(g)$. Taking the supremum over $g$, $u_+(f_1 + f_2) \leq u_+(f_1) + u_+(f_2)$, so we find

$$u_+(f_1 + f_2) = u_+(f_1) + u_+(f_2)\,. \tag{1.9}$$

Having shown this effective linearity on the positive functions we can obtain a linear functional by setting

$$u_+(f) = u_+(f_+) - u_+(f_-) \ \forall \ f \in \mathcal{C}_0(X)\,. \tag{1.10}$$

Note that (1.9) shows that $u_+(f) = u_+(f_1) - u_+(f_2)$ for any decomposiiton of $f = f_1 - f_2$ with $f_i \in \mathcal{C}_0(X)$, both positive. [Since $f_1 + f_- = f_2 + f_+$ so $u_+(f_1) + u_+(f_-) = u_+(f_2) + u_+(f_+)$.] Moreover,

$$|u_+(f)| \le \max(u_+(f_+), u(f_-)) \le \|u\| \, \|f\|_\infty \\ \Longrightarrow \|u_+\| \le \|u\| \, .$$

The functional

$$u_- = u_+ - u$$

is also positive, since $u_+(f) \ge u(f)$ for all $0 \le f \in \mathcal{C}_0(x)$. Thus we have proved

Lemma 1.5. *Any element $u \in (\mathcal{C}_0(X))'$ can be decomposed,*

$$u = u_+ - u_-$$

*into the difference of positive elements with*

$$\|u_+\| \, , \, \|u_-\| \le \|u\| \, .$$

The idea behind the definition of $u_+$ is that $u$ itself is, more or less, "integration against a function" (even though we do *not* know how to interpret this yet). In defining $u_+$ from $u$ we are effectively throwing away the negative part of that 'function.' The next step is to show that a positive functional corresponds to a 'measure' meaning a function measuring the size of sets. To define this we really want to evaluate $u$ on the characteristic function of a set

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \, . \end{cases}$$

The problem is that $\chi_E$ is not continuous. Instead we use an idea similar to (15.9).

If $0 \le u \in (\mathcal{C}_0(X))'$ and $U \subset X$ is *open,* set[1]

$$\mu(U) = \sup \left\{ u(f) \, ; \, 0 \le f(x) \le 1, f \in \mathcal{C}_0(X) \, , \, \operatorname{supp}(f) \Subset U \right\} \, . \tag{1.11}$$

Here the support of $f$, $\operatorname{supp}(f)$, is the *closure* of the set of points where $f(x) \neq 0$. Thus $\operatorname{supp}(f)$ is always closed, in (15.4) we only admit $f$ if its support is a compact subset of $U$. The reason for this is that, only then do we 'really know' that $f \in \mathcal{C}_0(X)$.

Suppose we try to measure general sets in this way. We can do this by defining

$$\mu^*(E) = \inf \left\{ \mu(U) \, ; \, U \supset E \, , \, U \text{ open} \right\} \, . \tag{1.12}$$

Already with $\mu$ it may happen that $\mu(U) = \infty$, so we think of

$$\mu^* : \mathcal{P}(X) \to [0, \infty] \tag{1.13}$$

[1] See [**6**] starting p.42 or [**1**] starting p.206.

as defined on the *power set* of $X$ and taking values in the extended positive real numbers.

DEFINITION 1.6. *A positive extended function, $\mu^*$, defined on the power set of $X$ is called an* outer measure *if $\mu^*(\emptyset) = 0$, $\mu^*(A) \le \mu^*(B)$ whenever $A \subset B$ and*

$$\mu^*(\bigcup_j A_j) \le \sum_j \mu(A_j) \ \forall \quad \{A_j\}_{j=1}^{\infty} \subset \mathcal{P}(X)\,. \tag{1.14}$$

LEMMA 1.7. *If $u$ is a positive continuous linear functional on $\mathcal{C}_0(X)$ then $\mu^*$, defined by* (15.4), (15.12) *is an outer measure.*

To prove this we need to find enough continuous functions. I have relegated the proof of the following result to Problem 2.

LEMMA 1.8. *Suppose $U_i$, $i = 1, \dots, N$ is ,a finite collection of open sets in a locally compact metric space and $K \Subset \bigcup_{i=1}^N U_i$ is a compact subset, then there exist continuous functions $f_i \in C(X)$ with $0 \le f_i \le 1$,* $\operatorname{supp}(f_i) \Subset U_i$ *and*

$$\sum_i f_i = 1 \text{ in a neighborhood of } K\,. \tag{1.15}$$

PROOF OF LEMMA 15.8. We have to prove (15.6). Suppose first that the $A_i$ are open, then so is $A = \bigcup_i A_i$. If $f \in C(X)$ and $\operatorname{supp}(f) \Subset A$ then $\operatorname{supp}(f)$ is covered by a finite union of the $A_i$s. Applying Lemma 15.7 we can find $f_i$'s, all but a finite number identically zero, so $\operatorname{supp}(f_i) \Subset A_i$ and $\sum_i f_i = 1$ in a neighborhood of $\operatorname{supp}(f)$.

Since $f = \sum_i f_i f$ we conclude that

$$u(f) = \sum_i u(f_i f) \Longrightarrow \mu^*(A) \le \sum_i \mu^*(A_i)$$

since $0 \le f_i f \le 1$ and $\operatorname{supp}(f_i f) \Subset A_i$.

Thus (15.6) holds when the $A_i$ are open. In the general case if $A_i \subset B_i$ with the $B_i$ open then, from the definition,

$$\mu^*(\bigcup_i A_i) \le \mu^*(\bigcup_i B_i) \le \sum_i \mu^*(B_i)\,.$$

Taking the infimum over the $B_i$ gives (15.6) in general. □

## 2. Measures and $\sigma$-algebras

An outer measure such as $\mu^*$ is a rather crude object since, even if the $A_i$ are disjoint, there is generally strict inequality in (15.6). It turns out to be unreasonable to expect equality in (15.6), for disjoint unions, for a function defined on *all* subsets of $X$. We therefore restrict attention to smaller collections of subsets.

DEFINITION 2.1. *A collection of subsets $\mathcal{M}$ of a set $X$ is a $\sigma$-algebra if*

(1) $\phi, X \in \mathcal{M}$
(2) $E \in \mathcal{M} \Longrightarrow E^C = X \backslash E \in \mathcal{M}$
(3) $\{E_i\}_{i=1}^{\infty} \subset \mathcal{M} \Longrightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

For a general outer measure $\mu^*$ we define the notion of $\mu^*$-measurability of a set.

DEFINITION 2.2. *A set $E \subset X$ is $\mu^*$-measurable (for an outer measure $\mu^*$ on $X$) if*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement}) \;\forall\; A \subset X\,. \tag{2.1}$$

PROPOSITION 2.3. *The collection of $\mu^*$-measurable sets for any outer measure is a $\sigma$-algebra.*

PROOF. Suppose $E$ is $\mu^*$-measurable, then $E^C$ is $\mu^*$-measurable by the symmetry of (3.9).

Suppose $A$, $E$ and $F$ are any three sets. Then

$$A \cap (E \cup F) = (A \cap E \cap F) \cup (A \cap E \cap F^C) \cup (A \cap E^C \cap F)$$
$$A \cap (E \cup F)^C = A \cap E^C \cap F^C\,.$$

From the subadditivity of $\mu^*$

$$\begin{aligned}\mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C)& \\ \le \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cup F^C)& \\ + \mu^*(A \cap E^C \cap F) + \mu^*(A \cap E^C \cap F^C).&\end{aligned}$$

Now, if $E$ and $F$ are $\mu^*$-measurable then applying the definition twice, for any $A$,

$$\begin{aligned}\mu^*(A) = \mu^*(A \cap E \cap F) + \mu^*(A \cap E \cap F^C)& \\ + \mu^*(A \cap E^C \cap F) + \mu^*(A \cap E^C \cap F^C)& \\ \ge \mu^*(A \cap (E \cup F)) + \mu^*(A \cap (E \cup F)^C)\,.&\end{aligned}$$

The reverse inequality follows from the subadditivity of $\mu^*$, so $E \cup F$ is also $\mu^*$-measurable.

If $\{E_i\}_{i=1}^{\infty}$ is a sequence of disjoint $\mu^*$-measurable sets, set $F_n = \bigcup_{i=1}^{n} E_i$ and $F = \bigcup_{i=1}^{\infty} E_i$. Then for any $A$,

$$\begin{aligned}\mu^*(A \cap F_n) &= \mu^*(A \cap F_n \cap E_n) + \mu^*(A \cap F_n \cap E_n^C) \\ &= \mu^*(A \cap E_n) + \mu^*(A \cap F_{n-1})\,.\end{aligned}$$

Iterating this shows that

$$\mu^*(A \cap F_n) = \sum_{j=1}^{n} \mu^*(A \cap E_j)\,.$$

From the $\mu^*$-measurability of $F_n$ and the subadditivity of $\mu^*$,

$$\begin{aligned}\mu^*(A) &= \mu^*(A \cap F_n) + \mu^*(A \cap F_n^C)\\ &\geq \sum_{j=1}^{n} \mu^*(A \cap E_j) + \mu^*(A \cap F^C)\,.\end{aligned}$$

Taking the limit as $n \to \infty$ and using subadditivity,

$$\begin{aligned}\mu^*(A) &\geq \sum_{j=1}^{\infty} \mu^*(A \cap E_j) + \mu^*(A \cap F^C)\\ &\geq \mu^*(A \cap F) + \mu^*(A \cap F^C) \geq \mu^*(A)\end{aligned} \tag{2.2}$$

proves that inequalities are equalities, so $F$ is also $\mu^*$-measurable.

In general, for *any* countable union of $\mu^*$-measurable sets,

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{j=1}^{\infty} \widetilde{A}_j\,,$$

$$\widetilde{A}_j = A_j \backslash \bigcup_{i=1}^{j-1} A_i = A_j \cap \left(\bigcup_{i=1}^{j-1} A_i\right)^C$$

is $\mu^*$-measurable since the $\widetilde{A}_j$ are disjoint. □

A *measure* (sometimes called a *positive measure*) is an extended function defined on the elements of a $\sigma$-algebra $\mathcal{M}$:

$$\mu : \mathcal{M} \to [0, \infty]$$

such that

$$\mu(\emptyset) = 0 \text{ and} \tag{2.3}$$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \tag{2.4}$$

$$\text{if } \{A_i\}_{i=1}^{\infty} \subset \mathcal{M} \text{ and } A_i \cap A_j = \phi \ i \neq j.$$

The elements of $\mathcal{M}$ with measure zero, i.e., $E \in \mathcal{M}$, $\mu(E) = 0$, are supposed to be 'ignorable'. The measure $\mu$ is said to be *complete* if

$$E \subset X \text{ and } \exists\ F \in \mathcal{M}\,,\ \mu(F) = 0\,,\ E \subset F \Rightarrow E \in \mathcal{M}\,. \tag{2.5}$$

See Problem 4.

The first part of the following important result due to Caratheodory was shown above.

THEOREM 2.4. *If $\mu^*$ is an outer measure on $X$ then the collection of $\mu^*$-measurable subsets of $X$ is a $\sigma$-algebra and $\mu^*$ restricted to $\mathcal{M}$ is a complete measure.*

PROOF. We have already shown that the collection of $\mu^*$-measurable subsets of $X$ is a $\sigma$-algebra. To see the second part, observe that taking $A = F$ in (3.11) gives

$$\mu^*(F) = \sum_j \mu^*(E_j) \text{ if } F = \bigcup_{j=1}^{\infty} E_j$$

and the $E_j$ are disjoint elements of $\mathcal{M}$. This is (3.3).

Similarly if $\mu^*(E) = 0$ and $F \subset E$ then $\mu^*(F) = 0$. Thus it is enough to show that for any subset $E \subset X$, $\mu^*(E) = 0$ implies $E \in \mathcal{M}$. For any $A \subset X$, using the fact that $\mu^*(A \cap E) = 0$, and the 'increasing' property of $\mu^*$

$$\begin{aligned} \mu^*(A) &\le \mu^*(A \cap E) + \mu^*(A \cap E^C) \\ &= \mu^*(A \cap E^C) \le \mu^*(A) \end{aligned}$$

shows that these must always be equalities, so $E \in \mathcal{M}$ (i.e., is $\mu^*$-measurable). $\square$

Going back to our primary concern, recall that we constructed the outer measure $\mu^*$ from $0 \le u \in (\mathcal{C}_0(X))'$ using (15.4) and (15.12). For the measure whose existence follows from Caratheodory's theorem to be much use we need

PROPOSITION 2.5. *If $0 \le u \in (\mathcal{C}_0(X))'$, for $X$ a locally compact metric space, then each open subset of $X$ is $\mu^*$-measurable for the outer measure defined by* (15.4) *and* (15.12) *and $\mu$ in* (15.4) *is its measure.*

PROOF. Let $U \subset X$ be open. We only need to prove (3.9) for all $A \subset X$ with $\mu^*(A) < \infty$.[2]

Suppose first that $A \subset X$ is open and $\mu^*(A) < \infty$. Then $A \cap U$ is open, so given $\epsilon > 0$ there exists $f \in C(X)$ $\operatorname{supp}(f) \Subset A \cap U$ with $0 \le f \le 1$ and

$$\mu^*(A \cap U) = \mu(A \cap U) \le u(f) + \epsilon\,.$$

Now, $A \backslash \operatorname{supp}(f)$ is also open, so we can find $g \in C(X)\,,\ 0 \le g \le 1\,,\ \operatorname{supp}(g) \Subset A \backslash \operatorname{supp}(f)$ with

$$\mu^*(A \backslash \operatorname{supp}(f)) = \mu(A \backslash \operatorname{supp}(f)) \le u(g) + \epsilon\,.$$

[2] Why?

Since

$$\begin{aligned} A\setminus \operatorname{supp}(f) \supset A \cap U^C\,,\ 0 \le f+g \le 1\,,\ \operatorname{supp}(f+g) \Subset A\,, \\ \mu(A) &\ge u(f+g) = u(f) + u(g) \\ &> \mu^*(A\cap U) + \mu^*(A \cap U^C) - 2\epsilon \\ &\ge \mu^*(A) - 2\epsilon \end{aligned}$$

using subadditivity of $\mu^*$. Letting $\epsilon \downarrow 0$ we conclude that

$$\mu^*(A) \le \mu^*(A\cap U) + \mu^*(A\cap U^C) \le \mu^*(A) = \mu(A)\,.$$

This gives (3.9) when $A$ is open.

In general, if $E \subset X$ and $\mu^*(E) < \infty$ then given $\epsilon > 0$ there exists $A \subset X$ open with $\mu^*(E) > \mu^*(A) - \epsilon$. Thus,

$$\begin{aligned} \mu^*(E) &\ge \mu^*(A\cap U) + \mu^*(A\cap U^C) - \epsilon \\ &\ge \mu^*(E\cap U) + \mu^*(E\cap U^C) - \epsilon \\ &\ge \mu^*(E) - \epsilon\,. \end{aligned}$$

This shows that (3.9) always holds, so $U$ is $\mu^*$-measurable if it is open. We have already observed that $\mu(U) = \mu^*(U)$ if $U$ is open. □

Thus we have shown that the $\sigma$-algebra given by Caratheodory's theorem contains all open sets. You showed in Problem 3 that the intersection of any collection of $\sigma$-algebras on a given set is a $\sigma$-algebra. Since $\mathcal{P}(X)$ is always a $\sigma$-algebra it follows that for *any* collection $\mathcal{E} \subset \mathcal{P}(X)$ there is always a smallest $\sigma$-algebra containing $\mathcal{E}$, namely

$$\mathcal{M}_{\mathcal{E}} = \bigcap \{\mathcal{M} \supset \mathcal{E}\,;\, \mathcal{M} \text{ is a } \sigma\text{-algebra}\ , \mathcal{M} \subset \mathcal{P}(X)\}\ .$$

The elements of the smallest $\sigma$-algebra containing the *open sets* are called 'Borel sets'. A measure defined on the $\sigma$-algebra of all Borel sets is called a *Borel measure.* This we have shown:

Proposition 2.6. *The measure defined by* (15.4)*,* (15.12) *from* $0 \le u \in (\mathcal{C}_0(X))'$ *by Caratheodory's theorem is a Borel measure.*

Proof. This is what Proposition 3.14 says! See how easy proofs are. □

We can even continue in the same vein. A Borel measure is said to be *outer regular* on $E \subset X$ if

$$\mu(E) = \inf\{\mu(U)\,;\, U \supset E\,,\, U \text{ open}\}\ . \tag{2.6}$$

Thus the measure constructed in Proposition 3.14 is outer regular on all Borel sets! A Borel measure is *inner regular* on $E$ if

$$\mu(E) = \sup\{\mu(K)\,;\, K \subset E\,,\, K \text{ compact}\}\ . \tag{2.7}$$

Here we need to know that compact sets are Borel measurable. This is Problem 5.

Definition 2.7. *A* Radon measure *(on a metric space) is a Borel measure which is outer regular on all Borel sets, inner regular on open sets and finite on compact sets.*

Proposition 2.8. *The measure defined by* (15.4), (15.12) *from* $0 \le u \in (\mathcal{C}_0(X))'$ *using Caratheodory's theorem is a Radon measure.*

Proof. Suppose $K \subset X$ is compact. Let $\chi_K$ be the characteristic function of $K$, $\chi_K = 1$ on $K$, $\chi_K = 0$ on $K^C$. Suppose $f \in \mathcal{C}_0(X)$, $\operatorname{supp}(f) \Subset X$ and $f \ge \chi_K$. Set

$$U_\epsilon = \{x \in X \,;\, f(x) > 1 - \epsilon\}$$

where $\epsilon > 0$ is small. Thus $U_\epsilon$ is open, by the continuity of $f$ and contains $K$. Moreover, we can choose $g \in C(X)$, $\operatorname{supp}(g) \Subset U_\epsilon$, $0 \le g \le 1$ with $g = 1$ near[3] $K$. Thus, $g \le (1-\epsilon)^{-1} f$ and hence

$$\mu^*(K) \le u(g) = (1-\epsilon)^{-1} u(f)\,.$$

Letting $\epsilon \downarrow 0$, and using the measurability of $K$,

$$\mu(K) \le u(f)$$
$$\Rightarrow \mu(K) = \inf\{u(f)\,;\, f \in C(X)\,,\, \operatorname{supp}(f) \Subset X\ ,\, f \ge \chi_K\}\ .$$

In particular this implies that $\mu(K) < \infty$ if $K \Subset X$, but is also proves (3.17). $\square$

Let me now review a little of what we have done. We used the positive functional $u$ to define an outer measure $\mu^*$, hence a measure $\mu$ and then checked the properties of the latter.

This is a pretty nice scheme; getting ahead of myself a little, let me suggest that we try it on something else.

Let us say that $Q \subset \mathbb{R}^n$ is 'rectangular' if it is a product of finite intervals (open, closed or half-open)

$$Q = \prod_{i=1}^{n} (\text{or}[a_i\,,\, b_i]\text{or})\ a_i \le b_i \tag{2.8}$$

we all agree on its standard volume:

$$v(Q) = \prod_{i=1}^{n} (b_i - a_i) \in [0, \infty)\,. \tag{2.9}$$

[3] Meaning in a neighborhood of $K$.

Clearly if we have two such sets, $Q_1 \subset Q_2$, then $v(Q_1) \leq v(Q_2)$. Let us try to define an outer measure on subsets of $\mathbb{R}^n$ by

$$v^*(A) = \inf \left\{ \sum_{i=1}^{\infty} v(Q_i)\,;\, A \subset \bigcup_{i=1}^{\infty} Q_i,\ Q_i \text{ rectangular} \right\}. \tag{2.10}$$

We want to show that (3.22) does define an outer measure. This is pretty easy; certainly $v(\emptyset) = 0$. Similarly if $\{A_i\}_{i=1}^{\infty}$ are (disjoint) sets and $\{Q_{ij}\}_{i=1}^{\infty}$ is a covering of $A_i$ by open rectangles then all the $Q_{ij}$ together cover $A = \bigcup_i A_i$ and

$$\begin{aligned} v^*(A) &\leq \sum_i \sum_j v(Q_{ij}) \\ \Rightarrow v^*(A) &\leq \sum_i v^*(A_i)\,. \end{aligned}$$

So we have an outer measure. We also want

Lemma 2.9. *If $Q$ is rectangular then $v^*(Q) = v(Q)$.*

Assuming this, the measure defined from $v^*$ using Caratheodory's theorem is called Lebesgue measure.

Proposition 2.10. *Lebesgue measure is a Borel measure.*

To prove this we just need to show that (open) rectangular sets are $v^*$-measurable.

## 3. Measureability of functions

Suppose that $\mathcal{M}$ is a $\sigma$-algebra on a set $X$[4] and $\mathcal{N}$ is a $\sigma$-algebra on another set $Y$. A map $f : X \to Y$ is said to be *measurable* with respect to these given $\sigma$-algebras on $X$ and $Y$ if

$$f^{-1}(E) \in \mathcal{M}\ \forall\ E \in \mathcal{N}\,. \tag{3.1}$$

Notice how similar this is to one of the characterizations of continuity for maps between metric spaces in terms of open sets. Indeed this analogy yields a useful result.

Lemma 3.1. *If $G \subset \mathcal{N}$ generates $\mathcal{N}$, in the sense that*

$$\mathcal{N} = \bigcap \{\mathcal{N}'; \mathcal{N}' \supset G,\ \mathcal{N}' \text{ a } \sigma\text{-algebra}\} \tag{3.2}$$

*then $f : X \longrightarrow Y$ is measurable iff $f^{-1}(A) \in \mathcal{M}$ for all $A \in G$.*

[4] Then $X$, or if you want to be pedantic $(X, \mathcal{M})$, is often said to be a *measure space* or even a *measurable space.*

PROOF. The main point to note here is that $f^{-1}$ as a map on power sets, is very well behaved for *any* map. That is if $f : X \to Y$ then $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ satisfies:

$$\begin{aligned} f^{-1}(E^C) &= (f^{-1}(E))^C \\ f^{-1}\left(\bigcup_{j=1}^{\infty} E_j\right) &= \bigcup_{j=1}^{\infty} f^{-1}(E_j) \\ f^{-1}\left(\bigcap_{j=1}^{\infty} E_j\right) &= \bigcap_{j=1}^{\infty} f^{-1}(E_j) \\ f^{-1}(\phi) = \phi\,,\ & f^{-1}(Y) = X\,. \end{aligned} \tag{3.3}$$

Putting these things together one sees that if $\mathcal{M}$ is any $\sigma$-algebra on $X$ then

$$f_*(\mathcal{M}) = \left\{E \subset Y; f^{-1}(E) \in \mathcal{M}\right\} \tag{3.4}$$

is always a $\sigma$-algebra on $Y$.

In particular if $f^{-1}(A) \in \mathcal{M}$ for all $A \in G \subset \mathcal{N}$ then $f_*(\mathcal{M})$ is a $\sigma$-algebra containing $G$, hence containing $\mathcal{N}$ by the generating condition. Thus $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$ so $f$ is measurable. $\square$

PROPOSITION 3.2. *Any continuous map $f : X \to Y$ between metric spaces is measurable with respect to the Borel $\sigma$-algebras on $X$ and $Y$.*

PROOF. The continuity of $f$ shows that $f^{-1}(E) \subset X$ is open if $E \subset Y$ is open. By definition, the open sets generate the Borel $\sigma$-algebra on $Y$ so the preceeding Lemma shows that $f$ is Borel measurable i.e.,

$$f^{-1}(\mathcal{B}(Y)) \subset \mathcal{B}(X).$$

$\square$

We are mainly interested in functions on $X$. If $\mathcal{M}$ is a $\sigma$-algebra on $X$ then $f : X \to \mathbb{R}$ is *measurable* if it is measurable with respect to the Borel $\sigma$-algebra on $\mathbb{R}$ and $\mathcal{M}$ on $X$. More generally, for an extended function $f : X \to [-\infty, \infty]$ we take as the 'Borel' $\sigma$-algebra in $[-\infty, \infty]$ the smallest $\sigma$-algebra containing all open subsets of $\mathbb{R}$ and all sets $(a, \infty]$ and $[-\infty, b)$; in fact it is generated by the sets $(a, \infty]$. (See Problem 6.)

Our main task is to define the integral of a measurable function: we start with *simple functions*. Observe that the characteristic function of a set

$$\chi_E = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is measurable if and only if $E \in \mathcal{M}$. More generally a simple function,

$$f = \sum_{i=1}^{N} a_i \chi_{E_i}, \ a_i \in \mathbb{R} \tag{3.5}$$

is measurable if the $E_i$ are measurable. The presentation, (3.5), of a simple function is not unique. We can make it so, getting the minimal presentation, by insisting that all the $a_i$ are non-zero and

$$E_i = \{x \in E\,;\ f(x) = a_i\}$$

then $f$ in (3.5) is measurable iff all the $E_i$ are measurable.

The Lebesgue integral is based on approximation of functions by simple functions, so it is important to show that this is possible.

Proposition 3.3. *For any non-negative $\mu$-measurable extended function $f : X \longrightarrow [0,\infty]$ there is an increasing sequence $f_n$ of simple measurable functions such that $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$ and this limit is uniform on any measurable set on which $f$ is finite.*

Proof. Folland [**1**] page 45 has a nice proof. For each integer $n > 0$ and $0 \le k \le 2^{2n} - 1$, set

$$\begin{aligned} E_{n,k} &= \{x \in X; 2^{-n}k \le f(x) < 2^{-n}(k+1)\}, \\ E'_n &= \{x \in X; f(x) \ge 2^n\}. \end{aligned}$$

These are measurable sets. On increasing $n$ by one, the interval in the definition of $E_{n,k}$ is divided into two. It follows that the sequence of simple functions

$$f_n = \sum_k 2^{-n}k\chi_{E_{k,n}} + 2^n \chi_{E'_n} \tag{3.6}$$

is increasing and has limit $f$ and that this limit is uniform on any measurable set where $f$ is finite. □

## 4. Integration

The $(\mu)$-*integral* of a non-negative simple function is by definition

$$\int_Y f\, d\mu = \sum_i a_i \mu(Y \cap E_i)\,,\ Y \in \mathcal{M}\,. \tag{4.1}$$

Here the convention is that if $\mu(Y \cap E_i) = \infty$ but $a_i = 0$ then $a_i \cdot \mu(Y \cap E_i) = 0$. Clearly this integral takes values in $[0,\infty]$. More significantly, if $c \geq 0$ is a constant and $f$ and $g$ are two non-negative ($\mu$-measurable) simple functions then

$$\begin{aligned} \int_Y cf d\mu &= c \int_Y f d\mu \\ \int_Y (f+g) d\mu &= \int_Y f d\mu + \int_Y g d\mu \\ 0 \leq f \leq g \Rightarrow \int_Y f\, d\mu &\leq \int_Y g\, d\mu\,. \end{aligned} \tag{4.2}$$

(See [**1**] Proposition 2.13 on page 48.)

To see this, observe that (4.1) holds for *any* presentation (3.5) of $f$ with all $a_i \geq 0$. Indeed, by restriction to $E_i$ and division by $a_i$ (which can be assumed non-zero) it is enough to consider the special case

$$\chi_E = \sum_j b_j \chi_{F_j}.$$

The $F_j$ can always be written as the union of a finite number, $N'$, of disjoint measurable sets, $F_j = \cup_{l \in S_j} G_l$ where $j = 1, \dots, N$ and $S_j \subset \{1, \dots, N'\}$. Thus

$$\sum_j b_j \mu(F_j) = \sum_j b_j \sum_{l \in S_j} \mu(G_l) = \mu(E)$$

since $\sum_{\{j; l \in S_j\}} b_j = 1$ for each $j$.

From this all the statements follow easily.

Definition 4.1. *For a non-negative $\mu$-measurable extended function $f : X \longrightarrow [0, \infty]$ the integral (with respect to $\mu$) over any measurable set $E \subset X$ is*

$$\int_E f d\mu = \sup\{\int_E h d\mu;\ 0 \leq h \leq f,\ h \text{ simple and measurable.}\} \tag{4.3}$$

By taking suprema, $\int_E f d\mu$ has the first and last properties in (4.2). It also has the middle property, but this is less obvious. To see this, we shall prove the basic 'Monotone convergence theorem' (of Lebesgue). Before doing so however, note what the vanishing of the integral means.

Lemma 4.2. *If $f : X \longrightarrow [0, \infty]$ is measurable then $\int_E f d\mu = 0$ for a measurable set $E$ if and only if*

$$\{x \in E; f(x) > 0\} \text{ has measure zero.} \tag{4.4}$$

Proof. If (4.4) holds, then any positive simple function bounded above by $f$ must also vanish outside a set of measure zero, so its integral must be zero and hence $\int_E f d\mu = 0$. Conversely, observe that the set in (4.4) can be written as

$$E_n = \bigcup_n \{x \in E; f(x) > 1/n\}.$$

Since these sets increase with $n$, if (4.4) does not hold then one of these must have positive measure. In that case the simple function $n^{-1}\chi_{E_n}$ has positive integral so $\int_E f d\mu > 0$. □

Notice the fundamental difference in approach here between Riemann and Lebesgue integrals. The Lebesgue integral, (4.3), uses approximation by functions constant on possibly quite nasty measurable sets, not just intervals as in the Riemann lower and upper integrals.

THEOREM 4.3 (Monotone Convergence). *Let $f_n$ be an increasing sequence of non-negative measurable (extended) functions, then $f(x) = \lim_{n\to\infty} f_n(x)$ is measurable and*

$$\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu \tag{4.5}$$

*for any measurable set $E \subset X$.*

PROOF. To see that $f$ is measurable, observe that

$$f^{-1}(a, \infty] = \bigcup_n f_n^{-1}(a, \infty]. \tag{4.6}$$

Since the sets $(a, \infty]$ generate the Borel $\sigma$-algebra this shows that $f$ is measurable.

So we proceed to prove the main part of the proposition, which is (4.5). Rudin has quite a nice proof of this, [**6**] page 21. Here I paraphrase it. We can easily see from (4.1) that

$$\alpha = \sup \int_E f_n d\mu = \lim_{n\to\infty} \int_E f_n d\mu \le \int_E f d\mu.$$

Given a simple measurable function $g$ with $0 \le g \le f$ and $0 < c < 1$ consider the sets $E_n = \{x \in E; f_n(x) \ge cg(x)\}$. These are measurable and increase with $n$. Moreover $E = \bigcup_n E_n$. It follows that

$$\int_E f_n d\mu \ge \int_{E_n} f_n d\mu \ge c \int_{E_n} g d\mu = \sum_i a_i \mu(E_n \cap F_i) \tag{4.7}$$

in terms of the natural presentation of $g = \sum_i a_i \chi_{F_i}$. Now, the fact that the $E_n$ are measurable and increase to $E$ shows that

$$\mu(E_n \cap F_i) \to \mu(E \cap F_i)$$

as $n \to \infty$. Thus the right side of (4.7) tends to $c \int_E g d\mu$ as $n \to \infty$. Hence $\alpha \geq c \int_E g d\mu$ for all $0 < c < 1$. Taking the supremum over $c$ and then over all such $g$ shows that

$$\alpha = \lim_{n\to\infty} \int_E f_n d\mu \geq \sup \int_E g d\mu = \int_E f d\mu.$$

They must therefore be equal. $\square$

Now for instance the additivity in (4.1) for $f \geq 0$ and $g \geq 0$ any measurable functions follows from Proposition 3.3. Thus if $f \geq 0$ is measurable and $f_n$ is an approximating sequence as in the Proposition then $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$. So if $f$ and $g$ are two non-negative measurable functions then $f_n(x) + g_n(x) \uparrow f + g(x)$ which shows not only that $f + g$ is measurable by also that

$$\int_E (f+g) d\mu = \int_E f d\mu + \int_E g d\mu.$$

As with the definition of $u_+$ long ago, this allows us to extend the definition of the integral to any *integrable* function.

Definition 4.4. *A measurable extended function $f : X \longrightarrow [-\infty, \infty]$ is said to be integrable on $E$ if its positive and negative parts both have finite integrals over $E$, and then*

$$\int_E f d\mu = \int_E f_+ d\mu - \int_E f_- d\mu.$$

Notice if $f$ is $\mu$-integrable then so is $|f|$. One of the objects we wish to study is the space of integrable functions. The fact that the integral of $|f|$ can vanish encourages us to look at what at first seems a much more complicated object. Namely we consider an equivalence relation between integrable functions

$$f_1 \equiv f_2 \Longleftrightarrow \mu(\{x \in X; f_1(x) \neq f_2(x)\}) = 0. \tag{4.8}$$

That is we identify two such functions if they are equal 'off a set of measure zero.' Clearly if $f_1 \equiv f_2$ in this sense then

$$\int_X |f_1| d\mu = \int_X |f_2| d\mu = 0, \ \int_X f_1 d\mu = \int_X f_2 d\mu.$$

A necessary condition for a measurable function $f \geq 0$ to be integrable is

$$\mu\{x \in X; f(x) = \infty\} = 0.$$

Let $E$ be the (necessarily measureable) set where $f = \infty$. Indeed, if this does not have measure zero, then the sequence of simple functions

$n\chi_E \le f$ has integral tending to infinity. It follows that each equivalence class under (4.8) has a representative which is an honest function, i.e. which is finite everywhere. Namely if $f$ is one representative then

$$f'(x) = \begin{cases} f(x) & x \notin E \\ 0 & x \in E \end{cases}$$

is also a representative.

We shall denote by $L^1(X, \mu)$ the space consisting of such equivalence classes of integrable functions. This is a normed linear space as I ask you to show in Problem 11.

The monotone convergence theorem often occurrs in the slightly disguised form of Fatou's Lemma.

LEMMA 4.5 (Fatou). *If $f_k$ is a sequence of non-negative integrable functions then*

$$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu \,.$$

PROOF. Set $F_k(x) = \inf_{n \ge k} f_n(x)$. Thus $F_k$ is an increasing sequence of non-negative functions with limiting function $\liminf_{n\to\infty} f_n$ and $F_k(x) \le f_n(x) \, \forall \; n \ge k$. By the monotone convergence theorem

$$\int \liminf_{n\to\infty} f_n \, d\mu = \lim_{k\to\infty} \int F_k(x) \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

□

We further extend the integral to complex-valued functions, just saying that

$$f : X \to \mathbb{C}$$

is integrable if its real and imaginary parts are both integrable. Then, by definition,

$$\int_E f d\mu = \int_E \operatorname{Re} f d\mu + i \int_E \operatorname{Im} f d\mu$$

for any $E \subset X$ measurable. It follows that if $f$ is integrable then so is $|f|$. Furthermore

$$\left| \int_E f \, d\mu \right| \le \int_E |f| \; d\mu \,.$$

This is obvious if $\int_E f \, d\mu = 0$, and if not then

$$\int_E f \, d\mu = Re^{i\theta} \;\; R > 0 \, , \; \theta \subset [0, 2\pi) \, .$$

Then

$$\begin{aligned}\left|\int_E f\,d\mu\right| &= e^{-i\theta}\int_E f\,d\mu \\ &= \int_E e^{-i\theta} f\,d\mu \\ &= \int_E \mathbb{R}e(e^{-i\theta} f)\,d\mu \\ &\le \int_E \left|\mathbb{R}e(e^{-i\theta} f)\right|\,d\mu \\ &\le \int_E \left|e^{-i\theta} f\right|\,d\mu = \int_E |f|\,d\mu\,.\end{aligned}$$

The other important convergence result for integrals is Lebesgue's *Dominated convergence theorem.*

THEOREM 4.6. *If $f_n$ is a sequence of integrable functions, $f_k \to f$ a.e.*[5] *and $|f_n| \le g$ for some integrable $g$ then $f$ is integrable and*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu\,.$$

PROOF. First we can make the sequence $f_n(x)$ converge by changing all the $f_n(x)$'s to zero on a set of measure zero outside which they converge. This does not change the conclusions. Moreover, it suffices to suppose that the $f_n$ are real-valued. Then consider

$$h_k = g - f_k \ge 0\,.$$

Now, $\liminf_{k\to\infty} h_k = g - f$ by the convergence of $f_n$; in particular $f$ is integrable. By monotone convergence and Fatou's lemma

$$\begin{aligned}\int (g-f)d\mu = \int \liminf_{k\to\infty} h_k\,d\mu &\le \liminf_{k\to\infty} \int (g - f_k)\,d\mu \\ &= \int g\,d\mu - \limsup_{k\to\infty} \int f_k\,d\mu\,.\end{aligned}$$

Similarly, if $H_k = g + f_k$ then

$$\int (g+f)d\mu = \int \liminf_{k\to\infty} H_k\,d\mu \le \int g\,d\mu + \liminf_{k\to\infty} \int f_k\,d\mu.$$

It follows that

$$\limsup_{k\to\infty} \int f_k\,d\mu \le \int f\,d\mu \le \liminf_{k\to\infty} \int f_k\,d\mu.$$

[5]Means on the complement of a set of measure zero.

Thus in fact

$$\int f_k \, d\mu \to \int f \, d\mu \, .$$

□

Having proved Lebesgue's theorem of dominated convergence, let me use it to show something important. As before, let $\mu$ be a positive measure on $X$. We have defined $L^1(X, \mu)$; let me consider the more general space $L^p(X, \mu)$. A measurable function

$$f : X \to \mathbb{C}$$

is said to be '$L^p$', for $1 \leq p < \infty$, if $|f|^p$ is integrable[6], i.e.,

$$\int_X |f|^p \, d\mu < \infty \, .$$

As before we consider equivalence classes of such functions under the equivalence relation

$$f \sim g \Leftrightarrow \mu \{x; (f - g)(x) \neq 0\} = 0 \, . \tag{4.9}$$

We denote by $L^p(X, \mu)$ the space of such equivalence classes. It is a linear space and the function

$$\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p} \tag{4.10}$$

is a norm (we always assume $1 \leq p < \infty$, sometimes $p = 1$ is excluded but later $p = \infty$ is allowed). It is straightforward to check everything except the triangle inequality. For this we start with

Lemma 4.7. *If $a \geq 0$, $b \geq 0$ and $0 < \gamma < 1$ then*

$$a^\gamma b^{1-\gamma} \leq \gamma a + (1 - \gamma) b \tag{4.11}$$

*with equality only when $a = b$.*

Proof. If $b = 0$ this is easy. So assume $b > 0$ and divide by $b$. Taking $t = a/b$ we must show

$$t^\gamma \leq \gamma t + 1 - \gamma \, , \ 0 \leq t \, , \ 0 < \gamma < 1 \, . \tag{4.12}$$

The function $f(t) = t^\gamma - \gamma t$ is differentiable for $t > 0$ with derivative $\gamma t^{\gamma - 1} - \gamma$, which is positive for $t < 1$ and negative for $t > 1$. Thus $f(t) \leq f(1)$ with equality only for $t = 1$. Since $f(1) = 1 - \gamma$, this is (5.17), proving the lemma. □

We use this to prove Hölder's inequality

[6]Check that $|f|^p$ is automatically measurable.

LEMMA 4.8. *If $f$ and $g$ are measurable then*

$$\left|\int fgd\mu\right| \leq \|f\|_p \|g\|_q \tag{4.13}$$

*for any $1 < p < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$.*

PROOF. If $\|f\|_p = 0$ or $\|g\|_q = 0$ the result is trivial, as it is if either is infinite. Thus consider

$$a = \left|\frac{f(x)}{\|f\|_p}\right|^p, \; b = \left|\frac{g(x)}{\|g\|_q}\right|^q$$

and apply (5.16) with $\gamma = \frac{1}{p}$. This gives

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{|f(x)|^p}{p\|f\|_p^p} + \frac{|g(x)|^q}{q\|g\|_q^q}.$$

Integrating over $X$ we find

$$\begin{aligned}\frac{1}{\|f\|_p \|g\|_q} &\int_X |f(x)g(x)| \; d\mu \\ &\leq \frac{1}{p} + \frac{1}{q} = 1\,.\end{aligned}$$

Since $\left|\int_X fg\,d\mu\right| \leq \int_X |fg| \; d\mu$ this implies (5.18). $\square$

The final inequality we need is *Minkowski's* inequality.

PROPOSITION 4.9. *If $1 < p < \infty$ and $f, g \in L^p(X, \mu)$ then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p\,. \tag{4.14}$$

PROOF. The case $p = 1$ you have already done. It is also obvious if $f + g = 0$ a.e.. If not we can write

$$|f + g|^p \leq (|f| + |g|)\,|f + g|^{p-1}$$

and apply Hölder's inequality, to the right side, expanded out,

$$\int |f + g|^p \; d\mu \leq (\|f\|_p + \|g\|_p)\;, \left(\int |f + g|^{q(p-1)} \; d\mu\right)^{1/q}.$$

Since $q(p - 1) = p$ and $1 - \frac{1}{q} = 1/p$ this is just (5.20). $\square$

So, now we know that $L^p(X, \mu)$ is a normed space for $1 \leq p < \infty$. In particular it is a metric space. One important additional property that a metric space may have is *completeness*, meaning that every Cauchy sequence is convergent.

Definition 4.10. *A normed space in which the underlying metric space is complete is called a* Banach *space.*

Theorem 4.11. *For any measure space $(X, M, \mu)$ the spaces $L^p(X, \mu)$, $1 \le p < \infty$, are Banach spaces.*

Proof. We need to show that a given Cauchy sequence $\{f_n\}$ converges in $L^p(X, \mu)$. It suffices to show that it has a convergent subsequence. By the Cauchy property, for each $k$ $\exists$ $n = n(k)$ s.t.

$$\|f_n - f_\ell\|_p \le 2^{-k} \ \forall \ \ell \ge n\,. \tag{4.15}$$

Consider the sequence

$$g_1 = f_1\,,\ g_k = f_{n(k)} - f_{n(k-1)}\,,\ k > 1\,.$$

By (5.3), $\|g_k\|_p \le 2^{-k}$, for $k > 1$, so the series $\sum_k \|g_k\|_p$ converges, say to $B < \infty$. Now set

$$h_n(x) = \sum_{k=1}^{n} |g_k(x)|\ ,\ n \ge 1\,,\ h(x) = \sum_{k=1}^{\infty} g_k(x).$$

Then by the monotone convergence theorem

$$\int_X h^p \, d\mu = \lim_{n\to\infty} \int_X |h_n|^p \, d\mu \le B^p\,,$$

where we have also used Minkowski's inequality. Thus $h \in L^p(X, \mu)$, so the series

$$f(x) = \sum_{k=1}^{\infty} g_k(x)$$

converges (absolutely) almost everywhere. Since

$$|f(x)|^p = \lim_{n\to\infty} \left| \sum_{k=1}^{n} g_k \right|^p \le h^p$$

with $h^p \in L'(X, \mu)$, the dominated convergence theorem applies and shows that $f \in L^p(X, \mu)$. Furthermore,

$$\sum_{k=1}^{\ell} g_k(x) = f_{n(\ell)}(x) \text{ and } \left|f(x) - f_{n(\ell)}(x)\right|^p \le (2h(x))^p$$

so again by the dominated convergence theorem,

$$\int_X \left|f(x) - f_{n(\ell)}(x)\right|^p \to 0\,.$$

Thus the subsequence $f_{n(\ell)} \to f$ in $L^p(X, \mu)$, proving its completeness.

$\square$

Next I want to return to our starting point and discuss the Riesz representation theorem. There are two important results in measure theory that I have not covered — I will get you to do most of them in the problems — namely the Hahn decomposition theorem and the Radon-Nikodym theorem. For the moment we can do without the latter, but I will use the former.

So, consider a locally compact metric space, $X$. By a Borel measure on $X$, or a signed Borel measure, we shall mean a function on Borel sets

$$\mu : \mathcal{B}(X) \to \mathbb{R}$$

which is given as the difference of two finite positive Borel measures

$$\mu(E) = \mu_1(E) - \mu_2(E)\,. \tag{4.16}$$

Similarly we shall say that $\mu$ is Radon, or a signed Radon measure, if it *can be written* as such a difference, with both $\mu_1$ and $\mu_2$ finite Radon measures. See the problems below for a discussion of this point.

Let $M_{\mathrm{fin}}(X)$ denote the set of finite Radon measures on $X$. This is a normed space with

$$\|\mu\|_1 = \inf(\mu_1(X) + \mu_2(X)) \tag{4.17}$$

with the infimum over all Radon decompositions (4.16). Each signed Radon measure defines a continuous linear functional on $\mathcal{C}_0(X)$:

$$\int \cdot\, d\mu : \mathcal{C}_0(X) \ni f \longmapsto \int_X f \cdot\, d\mu\,. \tag{4.18}$$

THEOREM 4.12 (Riesz representation.). *If $X$ is a locally compact metric space then every continuous linear functional on $\mathcal{C}_0(X)$ is given by a unique finite Radon measure on $X$ through* (4.18).

Thus the dual space of $\mathcal{C}_0(X)$ *is* $M_{\mathrm{fin}}(X)$ – at least this is how such a result is usually interpreted

$$(\mathcal{C}_0(X))' = M_{\mathrm{fin}}(X), \tag{4.19}$$

see the remarks following the proof.

PROOF. We have done half of this already. Let me remind you of the steps.

We started with $u \in (\mathcal{C}_0(X))'$ and showed that $u = u_+ - u_-$ where $u_\pm$ are *positive* continuous linear functionals; this is Lemma 1.5. Then we showed that $u \geq 0$ defines a finite positive Radon measure $\mu$. Here $\mu$ is defined by (15.4) on open sets and $\mu(E) = \mu^*(E)$ is given by (15.12) on general Borel sets. It is finite because

$$\begin{aligned}(4.20)\qquad \mu(X) &= \sup\{u(f)\,;\, 0\le f\le 1\,,\ \operatorname{supp} f \Subset X\,,\ f\in C(X)\}\\ &\le \|u\|\,.\end{aligned}$$

From Proposition 3.19 we conclude that $\mu$ is a Radon measure. Since this argument applies to $u_\pm$ we get two positive finite Radon measures $\mu_\pm$ and hence a signed Radon measure

$$\mu = \mu_+ - \mu_- \in M_{\text{fin}}(X). \tag{4.21}$$

In the problems you are supposed to prove the Hahn decomposition theorem, in particular in Problem 14 I ask you to show that (4.21) *is* the Hahn decomposition of $\mu$ — this means that there is a Borel set $E\subset X$ such that $\mu_-(E)=0\,,\ \mu_+(X\setminus E)=0$.

What we have defined is a linear map

$$(\mathcal{C}_0(X))' \to M(X),\ u\longmapsto \mu\,. \tag{4.22}$$

We want to show that this is an isomorphism, i.e., it is $1-1$ and onto.

We first show that it is $1-1$. That is, suppose $\mu=0$. Given the uniqueness of the Hahn decomposition this implies that $\mu_+=\mu_-=0$. So we can suppose that $u\ge 0$ and $\mu=\mu_+=0$ and we have to show that $u=0$; this is obvious since

$$\begin{aligned}(4.23)\qquad \mu(X) = \sup\{u(f);\ \operatorname{supp} u\Subset X,\ 0\le f\le 1\ f\in C(X)\} = 0\\ \Rightarrow u(f)=0 \text{ for all such } f\,.\end{aligned}$$

If $0\le f\in C(X)$ and $\operatorname{supp} f\Subset X$ then $f'=f/\|f\|_\infty$ is of this type so $u(f)=0$ for every $0\le f\in C(X)$ of compact support. From the decomposition of continuous functions into positive and negative parts it follows that $u(f)=0$ for every $f$ of compact support. Now, if $f\in\mathcal{C}_o(X)$, then given $n\in\mathbb{N}$ there exists $K\Subset X$ such that $|f|<1/n$ on $X\setminus K$. As you showed in the problems, there exists $\chi\in\mathcal{C}(X)$ with $\operatorname{supp}(\chi)\Subset X$ and $\chi=1$ on $K$. Thus if $f_n=\chi f$ then $\operatorname{supp}(f_n)\Subset X$ and $\|f-f_n\|=\sup(|f-f_n|<1/n$. This shows that $\mathcal{C}_0(X)$ is the closure of the subspace of continuous functions of compact support so by the assumed *continuity* of $u$, $u=0$.

So it remains to show that *every* finite Radon measure on $X$ arises from (4.22). We do this by starting from $\mu$ and constructing $u$. Again we use the Hahn decomposition of $\mu$, as in (4.21)[7]. Thus we assume $\mu\ge 0$ and construct $u$. It is obvious what we want, namely

$$u(f) = \int_X f\,d\mu\,,\ f\in\mathcal{C}_c(X)\,. \tag{4.24}$$

[7]Actually we can just take any decomposition (4.21) into a difference of positive Radon measures.

Here we need to recall from Proposition 3.2 that continuous functions on $X$, a locally compact metric space, are (Borel) measurable. Furthermore, we know that there is an increasing sequence of simple functions with limit $f$, so

$$\left|\int_X f\,d\mu\right| \le \mu(X)\cdot\|f\|_\infty\,. \tag{4.25}$$

This shows that $u$ in (4.24) *is* continuous and that its norm $\|u\| \le \mu(X)$. In fact

$$\|u\| = \mu(X)\,. \tag{4.26}$$

Indeed, the inner regularity of $\mu$ implies that there is a compact set $K \Subset X$ with $\mu(K) \ge \mu(X) - \frac{1}{n}$; then there is $f \in \mathcal{C}_c(X)$ with $0 \le f \le 1$ and $f = 1$ on $K$. It follows that $\mu(f) \ge \mu(K) \ge \mu(X) - \frac{1}{n}$, for any $n$. This proves (4.26).

We still have to show that if $u$ is defined by (4.24), with $\mu$ a finite positive Radon measure, then the measure $\tilde{\mu}$ defined from $u$ via (4.24) is precisely $\mu$ itself.

This is easy provided we keep things clear. Starting from $\mu \ge 0$ a finite Radon measure, define $u$ by (4.24) and, for $U \subset X$ open

$$\tilde{\mu}(U) = \sup\left\{\int_X f d\mu,\ 0 \le f \le 1,\ f \in C(X),\ \operatorname{supp}(f) \Subset U\right\}. \tag{4.27}$$

By the properties of the integral, $\tilde{\mu}(U) \le \mu(U)$. Conversely if $K \Subset U$ there exists an element $f \in \mathcal{C}_c(X)$, $0 \le f \le 1$, $f = 1$ on $K$ and $\operatorname{supp}(f) \subset U$. Then we know that

$$\tilde{\mu}(U) \ge \int_X f d\mu \ge \mu(K). \tag{4.28}$$

By the inner regularity of $\mu$, we can choose $K \Subset U$ such that $\mu(K) \ge \mu(U) - \epsilon$, given $\epsilon > 0$. Thus $\tilde{\mu}(U) = \mu(U)$.

This proves the Riesz representation theorem, modulo the decomposition of the measure - which I will do in class if the demand is there! In my view this is quite enough measure theory. $\square$

Notice that we have in fact proved something stronger than the statement of the theorem. Namely we have shown that under the correspondence $u \longleftrightarrow \mu$,

$$\|u\| = |\mu|\,(X) =: \|\mu\|_1\,. \tag{4.29}$$

Thus the map is an *isometry*.