No problem set due Feb 28, but I will post a list of questions from which the test on Feb 27 will be drawn.

**Problem 3.1**
Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function with Riemann integral satisfying
\[
\sup_R \int_{-R}^{R} |f(x)| \, dx < \infty.
\]
Show that \( f \in L^1(\mathbb{R}) \).

**Problem 3.2**
Show that the function \( \frac{\sin x}{(1+|x|)} \) is not an element of \( L^1(\mathbb{R}) \).

**Problem 3.3**
We say that a function \( f : \mathbb{R} \to \mathbb{C} \) is in \( L^2(\mathbb{R}) \) if there exists a sequence \( f_n \in C(\mathbb{R}) \) such that \( f_n(x) \to f(x) \) a.e. and there exists \( F \in L^1(\mathbb{R}) \) such that \(|f_n|^2 \leq F(x)\) a.e. Show that if \( f \in L^2(\mathbb{R}) \) then \( \chi_{[-R,R]} f \in L^1(\mathbb{R}) \) and that
\[
\left( \int \chi_{[-R,R]} |f| \right)^2 \leq (2R) \int |f|^2.
\]

**Problem 3.4**
Show that the function with \( F(0) = 0 \) and
\[
F(x) = \begin{cases} 
0 & x > 1 \\
\exp(i/x) & 0 < |x| \leq 1 \\
0 & x < -1,
\end{cases}
\]
is an element of \( L^1(\mathbb{R}) \).
Problem 3.5
Suppose $f \in L^1(\mathbb{R})$ is real-valued. Show that there is a sequence $f_n \in C_c(\mathbb{R})$ and another element $F \in L^1(\mathbb{R})$ such that

$$f_n(x) \to f(x) \text{ a.e. on } \mathbb{R}, \quad |f_n(x)| \leq F(x) \text{ a.e.}$$

Problem 3.6 – extra
(1) Suppose that $O \subset \mathbb{R}$ is a bounded open subset, so $O \subset (-R,R)$ for some $R$. Show that the characteristic function of $O$

$$\chi_O(x) = \begin{cases} 1 & x \in O \\ 0 & x \notin O \end{cases}$$

is an element of $L^1(\mathbb{R})$.

(2) If $O$ is bounded and open define the length (or Lebesgue measure) of $O$ to be $l(O) = \int \chi_O$. Show that if $U = \bigcup_j O_j$ is a (at most) countable union of bounded open sets such that $\sum_j l(O_j) < \infty$ then $\chi_U \in L^1(\mathbb{R})$; again we set $l(U) = \int \chi_U$.

(3) Conversely show that if $U$ is open and $\chi_U \in L^1(\mathbb{R})$ then $U = \bigcup_j O_j$ is the union of a countable collection of bounded open sets with $\sum_j l(O_j) < \infty$.

(4) Show that if $K \subset \mathbb{R}$ is compact then its characteristic function is an element of $L^1(\mathbb{R})$.

Problem 3.7 – extra
Prove that for any $\epsilon > 0$ any set of measure zero is covered by a countable collection of open intervals the sum of whose lengths is less than $\epsilon$. 

Department of Mathematics, Massachusetts Institute of Technology
Email address: rbm@math.mit.edu