1. Problem 2.1

Show that if $K \in C([0, 1]^2)$ is a continuous function of two variables, then the integral operator

$$Au(x) = \int_0^1 K(x, y)u(y)dy$$

(given by a Riemann integral) is a bounded operator, i.e. a continuous linear map, from $C([0, 1])$ to itself with respect to the supremum norm.

Solution: A continuous function on a compact set, such as $[0, 1]^2$, is uniformly continuous, so given $\epsilon$ there exists $\delta > 0$ such that

$$|x - x'| + |y - y'| < \delta \implies |K(x, y) - K(x', y')| < \epsilon.$$ 

If $u \in C([0, 1])$ is fixed then the integrand in (1) is continuous for each fixed $x \in [0, 1]$, so $Au : [0, 1] \rightarrow C$ is well-defined as a Riemann integral. Moreover

$$|Au(x) - Au(x')| = |\int_0^1 (K(x, y) - K(x', y))u(y)dy| \leq \sup_y |K(x, y) - K(x', y)| \sup |u|$$

by standard properties of the Riemann integral. Using (2) it follows that

$$|x - x'| < \delta \implies |Au(x) - Au(x')| \leq \sup |u| \epsilon$$

so $Au$ is continuous on $[0, 1]$ and (1) defines a map

$$A : C([0, 1]) \rightarrow C([0, 1]).$$

The linearity of this map follows from the linearity of the Riemann integral and

$$|u(x)| \leq \sup |K| \sup |u| \forall x \in [0, 1]$$

shows that it is bounded, i.e. continuous.

2. Problem 2.2

(1) Show that the ‘Dirac delta function at $y \in [0, 1]$’ is well-defined as a continuous linear map

$$\delta_y : C([0, 1]) \ni u \mapsto u(y) \in C$$

with respect to the supremum norm on $C([0, 1])$.

(2) Show that $\delta_y$ is not continuous with respect to the $L^1$ norm $\int_0^1 |u|$.

Solution
(1) The map (1) is clearly linear since
\[
\delta_y(c_1 u_1 + c_2 u_2) = (c_1 u_1 + c_2 u_2)(y) = c_1 \delta_y(u_1) + c_2 \delta_y(u_2)
\]
and it is bounded
\[
|\delta_y(u)| \leq \sup |u|
\]
so continuous.

(2) It suffices to show that there is a sequence \( u_n \) in \( C([0, 1]) \) such that \( \delta_y(u_n) = 1 \) but \( \|u_n\|_{L^1} \to 0 \) since then a bound \( |\delta_y(u)| \leq C\|u\|_{L^1} \) is impossible. Such a sequence is given by the ‘triangle functions’
\[
u_n(x) = \begin{cases} 
0 & x \leq y - 1/n \\
1 - n|y - x| & y - 1/n \leq x \leq y + 1/n \\
0 & x \geq y + 1/n
\end{cases}
\]
restricted to \([0, 1]\). Indeed \( u_n \) is continuous at each point and
\[
u_n(y) = 1, \quad \int_0^1 u_n(y) \leq 1/n.
\]

3. Problem 2.3

A subset \( E \subset \mathbb{R} \) is said to be of measure zero if there exists an absolutely summable sequence \( f_n \in C_c(\mathbb{R}) \) (so \( \sum_n \int |f_n| < \infty \)) such that
\[
E \subset \{ x \in \mathbb{R}; \sum_n |f_n(x)| = +\infty \}.
\]
Show that if \( E \) is of measure zero and \( \epsilon > 0 \) is given then there exists \( f_n \in C_c(\mathbb{R}) \) satisfying (1) and in addition
\[
\sum_n \int |f_n| < \epsilon.
\]

Solution: Take such a series \( f_n \) with \( \sum_n \int |f_n(x)| = C \) and replace it by \( \frac{\epsilon}{C+\epsilon} f_n \) or choose \( N \) so large that
\[
\sum_{n \leq N} \int |f_n(x)| > C - \epsilon
\]
and consider the new series \( u_n = f_{n+N} \) which has
\[
\sum_n \int |u_n(x)| < \epsilon
\]
and for which \( \sum_n |u_n(x)|C \) diverges wherever \( \sum_n |f_n(x)| \) diverges, so in particular on \( E \).
4. Problem 2.4

Using the previous problem (or otherwise ...) show that a countable union of sets of measure zero is a set of measure zero.

Solution: Let $E_j$ be the countable collection of sets of measure zero. Choose a summable series $f_{j,n}$ for each $j$ which satisfies

$$\sum_n \int |f_{j,n}| < 2^{-j}, \quad \sum_n |f_{j,n}(x)| = \infty \text{ for } x \in E_j.$$  

Now, rearrange the countably many terms $f_{j,n}$ into a sequence $g_k \in C_c(\mathbb{R})$ – using for instance a bijection from $\mathbb{N}^2$ to $\mathbb{N}$ applied to the indices. Then, standard rearrangement properties of absolutely summable series (look at Rudin if you need to, we will use this next week) show that

$$\sum_k \int |g_k| = \sum_j \sum_n \int |f_{j,n}| < \sum_j 2^{-j} = 2,$$

$$\sum_k |g_k(x)| \geq \sum_n |f_{j,n}(x)| = \infty \quad \forall \ x \in E_j, \forall j.$$  

Thus $E = \sum_j E_j$ has measure zero.

Problem 2.5

Suppose $E \subset \mathbb{R}$ has the following (well-known) property:-

$$\forall \epsilon > 0 \exists \text{ a countable collection of intervals } (a_i, b_i) \text{ s.t.}$$

$$\sum_i (b_i - a_i) < \epsilon, \quad E \subset \bigcup_i (a_i, b_i).$$

Show that $E$ is a set of measure zero in the sense used in lectures and above.

Solution: for $\epsilon_n = 1/n^2$, we have a countable collection of intervals $(a_i^{(n)}, b_i^{(n)})$ as in the question. Now define $f_i^{(n)}$ to be 1 on $[a_i^{(n)}, b_i^{(n)}]$ and 0 outside $[a_i^{(n)} - \frac{b_i^{(n)} - a_i^{(n)}}{2}, b_i^{(n)} + \frac{b_i^{(n)} - a_i^{(n)}}{2}]$, and define the values elsewhere using linear segment. Then it’s easy to verify $\sum_i f_i^{(n)} = 2(b_i^{(n)} - a_i^{(n)})$, so $\sum_i \sum_j |f_i^{(n)}| < +\infty$, but for any $x \in E$, $\sum_i \sum_j |f_i^{(n)}(x)| = +\infty$ as $\sum_i \int |f_i^{(n)}(x)| \geq 1$ by definition. So $E$ is of measure zero.

5. Problem 2.6 – Extra

Let’s generalize the theorem about $B(V, W)$ given last week to bilinear maps – this may seem hard but just take it step by step!

(1) Check that if $U$ and $V$ are normed spaces then $U \times V$ (the linear space of all pairs $(u, v)$ where $u \in U$ and $v \in V$) is a normed space where addition and scalar multiplication is ‘componentwise’ and the norm is the sum

$$\|(u, v)\|_{U \times V} = \|u\|_U + \|v\|_V.$$

(2) Show that $U \times V$ is a Banach space if both $U$ and $V$ are Banach spaces.

(3) Consider three normed spaces $U, V$ and $W$. Let

$$B : U \times V \rightarrow W$$
be a bilinear map. This means that
\[ B(\lambda \lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 B(u_1, v) + \lambda_2 B(u_2, v), \]
\[ B(u, \lambda \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 B(u, v_1) + \lambda_2 B(u, v_2) \]
for all \( u, u_1, u_2 \in U, v, v_1, v_2 \in V \) and \( \lambda, \lambda_1, \lambda_2 \in \mathbb{C} \). Show that \( B \) is continuous if and only if it satisfies
\[ \|B(u, v)\|_W \leq C\|u\|_V \|v\|_V \quad \forall u \in U, v \in V. \]  

(3) Consider the space \( B \) using the bilinearity. If either vanishes then continuity.

(4) Let \( \mathcal{M}(U; V; W) \) be the space of all such continuous bilinear maps. Show that this is a linear space and that
\[ \|B\| = \sup_{\|u\|=1,\|v\|=1} \|B(u, v)\|_W \]
is a norm.

(5) Show that \( \mathcal{M}(U; V; W) \) is a Banach space if \( W \) is a Banach space.

Solution: Third last part only and brief. An estimate (3) implies continuity, since if \( u_n \to u \) and \( v_n \to v \) then
\[ \|B(u_n, v_n) - B(u, v)\|_W \leq \|B(u_n, v_n) - B(u_n, v)\|_W + \|B(u_n, v) - B(u, v)\|_W \leq C(\|u_n\| \|v_n - v\| + \|u_n - u||\|v||) \to 0. \]
Conversely, if \( B \) is continuous then \( B^{-1}(\{\|w\| < 1\}) \ni 0 \) is open, so
\[ \|u\| + \|v\| < \epsilon \implies \|B(u, v)\| \leq 1 \]
for some \( \epsilon > 0 \). If \( u \) and \( v \) are non-zero then
\[ \|\epsilon/A(\|u\|/\|v\|)\| < \epsilon \implies \|B(u, v)\| \leq 4\|u\|\|v|| \]
using the bilinearity. If either vanishes then \( B(u, v) \) vanishes so (3) is equivalent to continuity.

Everything else is very similar to the linear case.

6. Problem 2.7 – Extra

Consider the space \( C_c(\mathbb{R}^n) \) of continuous functions \( u : \mathbb{R}^n \to \mathbb{C} \) which vanish outside a compact set, i.e. in \( |x| > R \) for some \( R \) (depending on \( u \)). Check (quickly) that this is a linear space.

Show that if \( y \in \mathbb{R}^{n-1} \) and \( u \in C_c(\mathbb{R}^n) \) then
\[ U_y : \mathbb{R} \ni t \to u(y, t) \in \mathbb{C} \]
defines an element \( U_y \in C_c(\mathbb{R}) \). Fix an overall ‘rectangle’ \([-R, R]^n\) and only consider functions \( C_{c,R}(\mathbb{R}) \) vanishing outside this rectangle. With this restriction on supports show for each \( R \) that \( \mathbb{R}^{n-1} \ni y \mapsto U_y \) is a continuous map into \( C_{c,R}(\mathbb{R}) \) with respect to the supremum norm which vanishes for \( |y| > R \), i.e. has compact support.

Conclude that ‘integration in the last variable’ gives a continuous linear map (with respect to supremum norms)
\[ C_{c,R}(\mathbb{R}^n) \ni u \to v \in C_{c,R}(\mathbb{R}^{n-1}), \quad v(y) = \int U_y. \]
By iterating this statement show that the iterated Riemann integral is well defined
\[ \int : C_c^\infty(\mathbb{R}^n) \to \mathbb{C} \]
and that \( \int |u| \) is a norm which is independent of \( R \) – so defined on the whole of \( C_c^\infty(\mathbb{R}^n) \).

Solution: \( y \mapsto U_y \) is continuous as \([-R, R]^n\) is compact so \( u \) is uniformly continuous then one easily gets the bound. The iterated Riemann integral is a norm: non-negative, absolute homogeneity, triangle inequality follows immediately, if \( u \neq 0 \), then \( |u| > 0 \) in an open neighborhood of some points, hence the integral is positive. The independence on \( R \) is because \( u \) vanishes outside the rectangle.