Was due in 2-251, by Noon, Tuesday November 5.
This was a bit of a stinker.
Rudin:

(1) Chapter 5, Problem 12

Solution. In $x > 0$, $|x|^3 = x^3$ so is infinitely differentiable, being a polynomial, and has derivative $3x^2$. Similarly in $x < 0$, $|x|^3 = -x^3$ is again a polynomial and has derivative $-3x^2$. The limit

$$\lim_{0\neq t\to 0} \frac{f(0)-f(t)}{0-t} = \lim_{0\neq t\to 0} |t|^3/t = 0$$

so f is differentiable at 0 and $f'(x) = 3x|x|$ everywhere. As already noted this is differentiable in $x \neq 0$ and has derivative $6|x|$. The limit

$$\lim_{0\neq t\to 0} \frac{f'(0)-f'(t)}{0-t} = \lim_{0\neq t\to 0} 3|t| = 0 \tag{1}$$

again exists, so $f''(x) = 6|x|$ exists everywhere. Finally the third derivative exists for $x \neq 0$ and is $f^{(3)}(x) = 6\operatorname{sgn} x$, $\operatorname{sgn} x = \pm 1$ as $x > 0$ or $x < 0$. The limit of

$$\frac{f(0)-f''(t)}{0-t} = \frac{f(0)-f(t)}{0-t} 6\operatorname{sgn} t$$

does not exist as $0 \neq t \to 0$, so $f^{(3)}(0)$ does not exist. □

(2) Chapter 5, Problem 14

Solution. By assumption, $f(x)$ is convex and differentiable on (a, b). Thus

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \ \forall\ t \in [0,1],\ x \leq y \in (a,b).$$

For any three points $x < y < z \in (a, b)$ the difference quotient satisfies

$$\frac{f(x)-f(y)}{x-y} \leq \frac{f(x)-f(z)}{x-z} \leq \frac{f(y)-f(z)}{y-z}$$

as shown last week. Letting $y \downarrow x$ in the first inequality, and using the differentiability of f shows that

$$f'(x) \leq \frac{f(x)-f(z)}{x-z} \leq \frac{f(y)-f(z)}{y-z}$$

where x, y, z are again any points satisfying $x < y < z$. Now letting $y \uparrow z$ we conclude that $f'(x) \leq f'(z)$ if $x < z$.

Conversely, suppose $f'(x)$ is monotonically increasing on (a, b). Using the mean value theorem, if $x < z$ then $f(z) - f(x) = (z - x)f'(q)$ for some $q \in (x, z)$ so $f'(x) \leq \frac{f(z)-f(x)}{z-x} \leq f'(z)$. For three points $x < z < y$ this gives

$$\frac{f(z)-f(x)}{z-x} \leq \frac{f(z)-f(y)}{z-y}$$

and setting $t = \frac{y-z}{y-x}$ so $z = tx + (1-t)y$ this is precisely

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \ \forall\ t \in (0,1)$$

which is convexity.

If $f''(x)$ exists for all $x \in (a,b)$ and $f'' \geq 0$ then $f'(x)$ is increasing and so f is convex. Conversely if f is convex then f' is increasing and hence $f'' \geq 0$. □

(3) Chapter 5, Problem 15

I should have said not to do the last part, since I have not talked much about differentiation of vector-valued functions.

Solution. The question is quite as clear as should be, you are supposed to assume that M_0 and M_2 are finite.

Following the hint, recall that Taylor's theorem shows that

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2}f''(\xi)$$

for some $\xi \in (x, x+2h)$ which can be written

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi).$$

Thus

$$|f'(x)| \leq \frac{1}{2h}[|f(x+2h)| + |f(x)|] + h|f''(\xi)|$$

and so with M_0 an upper bound for $|f|$ and M_2 and upper bound for $f''|$,

$$|f'(x)| \leq hM_2 + \frac{M_0}{h},\ \forall\ h > 0,\ x \in (a, \infty).$$

Taking the supremum over x for each $h > 0$ we find

$$M_1 \leq hM_2 + \frac{M_0}{h}\ \forall\ h > 0.$$

We can assume M_0, $M_2 > 0$ since if $M_2 = 0$ then f is linear and M_0 is infinite. If $M_0 = 0$ then $f \equiv 0$. The right side is differentiable in h with derivative $M_2 - h^{-2}M_0$. This vanishes when $h = \sqrt{M_0/M_2} > 0$, substituting this gives

$$M_1 \leq 2\sqrt{M_0 M_2} \Longleftrightarrow M_1^2 \leq 4M_0M_2.$$

For the given

$$f(x) = \begin{cases} 2x^2 - 1 & -1 < x < 0 \\ \frac{x^2-1}{x^2+1} & 0 \leq x < \infty \end{cases}$$

we see that

$$f'(x) = \begin{cases} 4x & -1 < x < 0 \\ \frac{4x}{(x^2+1)^2} & 0 < x < \infty \end{cases}$$

also exists at 0 where it has the value 0. Then $f''(x)$ also exists at 0, taking the value 4 and

$$f''(x) = \begin{cases} 4 & -1 < x < 0 \\ \frac{4(1-x^2)}{(x^2+1)^3} & 0 < x < \infty \end{cases}$$

Now, $f' < 0$ in $x < 0$ and $f' > 0$ for $x > 0$ so

$$\sup |f(x)| = M_0 = 1.$$

Similarly $f'' \geq 0$ in $x < 1$ and $f'' < 0$ in $x > 1$ so f' takes its maximum value at $x = 1$ and since it is positive for $x > 0$ its minimum is -4 so

$$M_1 = \sup |f'(x)| = 4.$$

Finally then $M_2 = \sup |f''| = 4$ since in $x > 0$ it decreases to its zero at $x = 1$ and for $x > 1$, $f'' > -4x^2/(x^2+1)^3 \geq -4$. Thus equality can occur.

Yes, the result is true for vector valued functions for the usual Euclidean norms. Let $f = (f_1, f_2, \dots, f_k)$ be a function with values in $\mathbb{R}^k$. Thus the assumption is that each of the components satisfies the assumptions of the question and we set

$$M_i = \sup_{x \in (a,\infty)} |f^{(i)}(x)|$$

with the Euclidean norm. Now, suppose that $a = (a_1, \dots, a_k) \in \mathbb{R}^k$ is a (constant) vector. We can apply the result above to $g(x) = a \cdot f(x) = a_1 f_1(x) + \cdots + a_k f_k(x)$. We see then that for any $x \in (a, \infty)$

$$|g'(x)| \leq 4 \sup |g| \sup |g''| \leq 4|a|^2 M_0 M_2.$$

Now we can set $a = f'(x)$ for a given x and divide by a factor of $|f'(x)|^2$ and so conclude that

$$|f'(x)|^2 \leq 4 M_0 M_2.$$

Taking the supremum over x now gives the vector-valued result. □

(4) Chapter 6, Problem 2

Solution. Since f is continuous it is Riemann integrable and $f \geq 0$, either $f = 0$ or there exists an interval of positive length, $l > 0$, in $[a, b]$ on which $f(x) \geq c > 0$. Then there exists a partition, with the end points of this interval as two of its points, such that

$$L(P, f) \geq lc > 0.$$

Since $\int_a^b f dx \geq L(P, f)$ for any partition, this implies $\int_a^b f dx > 0$ so $\int_a^b f dx = 0$ must imply $f \equiv 0$.

Or, you could use the fundamental theorem of calculus. □

(5) Chapter 6, Problem 4

Solution. For any partition P we have

$$U(P, f) - L(P, f) = \sum_{i=1, x_{i-1} > x_i}^{n} (x_i - x_{i-1})(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f).$$

Now, any interval of non-zero length contains both rational and irrational points, so the difference of $\sup f$ and $\inf f$ is always one. It follows that

$$U(P, f) - L(P, f) = (b - a)$$

so the function cannot be Riemann integrable. □