18.100B, FALL 2002
FINAL EXAM: DECEMBER 16

Try each of the questions; the first seven are worht 12 points each, the last one is harder and is worth 16. You may use theorems from class, or the book, provided you can recall them correctly! This includes standard properties of the exponential and trigonometric functions. No books or papers are permitted.

Overall comment: There were many problems arising simply from a failure to express ideas clearly and directly. I think this was a pretty easy exam, except that I asked for lots of explanations and this bothered quite a few people.

Problem 1

Show that the set $\{z \in \mathbb{C}; z = \exp(it^3) \text{ for some } t \in \mathbb{R}\}$ is connected.

Solution. As was shown in class, the function $\exp : \mathbb{C} \longrightarrow \mathbb{C}$ is continuous. As a polynomial, $\mathbb{R} \ni t \longmapsto it^3 \in \mathbb{C}$ is continuous, so the composite $\mathbb{R} \ni t \longmapsto \exp(it^3) \in \mathbb{C}$ is continuous. Thus, as the image of the connected set $\mathbb{R}$ under a continuous map, the given set is connected. □

Comment: Several round-about proofs that were acceptable.

Problem 2

Let $\{x_n\}$ be a sequence in a metric space X and suppose that there is a point $p \in X$ with the property that every subsequence of $\{x_n\}$ has a subsequence which converges to p. Show that $\{x_n\}$ converges to p.

Solution. Suppose that $\{x_n\}$ does not converge to p. Then, for some $\epsilon > 0$ there exists no N with the property that $d(x_n, p) < \epsilon$ if $n > N$. This means that the set $\{n \in \mathbb{N}; d(x_n, p) \geq \epsilon\}$ is infinite. Hence there is a strictly increasing map $\mathbb{N} \ni i \longmapsto n_i$ with values in this set. This means that the subsequence x_{n_i} satisfies $d(x_{n_i}, p) \geq \epsilon$ for all i. But such a sequence can have no subsequence converging to p. Hence the only remaining possibility is that x_n does converge to p. □

Comment: Many tried to prove this by elimating convergent subsequences repeatedly. This cannot work since there might always be an infinite sequence remaining.

Problem 3

(1) Why is the function $f(x) = \exp(\frac{x^3-15}{x^2+x+1})$ continuously differentiable on $[0, 1]$?
(2) Why does f have a minimum value on this interval?

Solution. On $[0, 1]$ the polynomial $x^2 + x + 1 \geq 1$ so the rational function $(x^3 - 15)/(x^2 + x + 1)$ is continuously differentiable, with values in $\mathbb{R}$, by the quotient rule. Since exp is continuously differentiable on $\mathbb{R}$ the composite, f, is continuously differentiable (by the chain rule).

By a theorem in Rudin, a real-valued continuous function on a compact metric space, such as $[0, 1]$ has a minimum. □

Comment: Generally well done, as it should have been.

Problem 4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable and suppose that 0 is a local maximum of f, i.e. for some $\epsilon > 0$ $f(x) \leq f(0)$ for all $x \in (-\epsilon, \epsilon)$. Show that $f''(0) \leq 0$.

Solution. We know from class that $f'(0) = 0$ since 0 is a local maximum – in any case this follows that the difference quotient $(f(x) - f(0))/x$ is ≥ 0 for $x \in (-\epsilon, 0)$ and ≤ 0 for $x \in (0, \epsilon)$. By the mean value theorem for each n, large, there is a point $c_n \in (-1/n, 0)$ such that $f(x) - f(0) = xf'(c_n)$, so $f'(c_n) \geq 0$. Since f' is differentiable we must have $f'(c_n)/c_n \to f''(0)$ as $n \to \infty$ which shows that $f''(0) \leq 0$. □

Comment: Alternatively one can assume that $f''(0) > 0$ and arrive at a contradiction. However, one cannot assume that $f''(x) > 0$ for x near 0, since I did not say that f'' is continuous – this lead many people into error. Similarly you cannot just assume that $f'(x) \geq 0$ for $x \in (-\delta, 0)$ and small enough δ. It is simply not true in general.

Problem 5

If $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable function and g' is bounded on $\mathbb{R}$, show that g is uniformly continuous.

Solution. By assumption, there exists $M \in \mathbb{R}$ such that $|g'(x)| < M$ for all $x \in \mathbb{R}$. By the Mean Value Theorem, if $x < y \in \mathbb{R}$ there exists $c \in (x, y)$ such that $g(y) - g(x) = g'(c)(y - x)$. It follows that

$$|g(y) - g(x)| \leq M|x - y|$$

since this is true whe $x = y$, and for $x \neq y$, exchanging x and y as necessary. Thus, given $\epsilon > 0$ choosing $\delta < \epsilon/M$, $|g(y) - g(x)| < \epsilon$ whenever $|y - x| < \delta$ and this is the definition of uniform continuity. □

Comment: Arguments involving the integration of g' received few marks, since it is not assumed that g' is integrable. Claims that the difference quotients were uniformly bounded without appeal to the MVT were not successful.

Problem 6

Using standard properties of the cosine function explain why the formula

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3} \cos(nx)$$

defines a continuously differentiable function on the real line.

Solution. Since $|\cos(nx)| \leq 1$, and the series $\sum\limits_{n=11}^{\infty} \frac{1}{n^3}$ converges, $\sum\limits_{n=1}^{\infty} \frac{1}{n^3} \cos(nx)$ converges uniformly on $\mathbb{R}$, by Weierstrass Theorem. Since the terms are continuous, the function f exists and is continuous. The term-by-term differentiated series,

$\sum\limits_{n=1}^{\infty} -\frac{1}{n^2}\sin(nx)$ has terms bounded by the convergent series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ so it too converges uniformly on $\mathbb{R}$ by the same argument. By a Theorem in Rudin the uniform convergence of the term-by-term differentiated series and the convergence of the series (even at one point) implies that the sum is differentiable and that f' is the sum of the series of derivatives. Since the terms in the latter are continuous, f' is also continuous by its uniform convergence. □

Comment: Sorry about the typo here. Not mentioning uniform convergence meant few marks, not proving it meant losing quite a few.

Problem 7

Explain carefully why the Riemann-Stieltjes integral

$$\int_0^1 \exp(3x^2)d\alpha$$

exists for any increasing function $\alpha : [0,1] \longrightarrow \mathbb{R}$.

Solution. By a theorem in Class/Rudin any continuous function on a bounded interval is Riemann-Stieltjes integrable with respect to any increasing α. In this case $\exp(3x^2)$ is continuous, as the composite continuous functions. □

Comment: At least everyone got this right. Mind you there were some close calls.

Problem 8

Let $A : [0,1]^2 \longrightarrow \mathbb{R}$ be a continuous function satisfying

$$\sup_{[0,1]^2} |A(x,y)| \le \frac{1}{2}.$$

(1) Show that if $f \in \mathcal{C}([0,1])$ then

$$g(x) = \int_0^1 A(x,y)f(y)dy \in \mathcal{C}([0,1]).$$

(2) Estimate $\|g\| = \sup_{[0,1]} |g(x)|$ in terms of $\|f\|$.

(3) If $h \in \mathcal{C}([0,1])$ is a fixed function show that

$$(Gf)(x) = h(x) + \int_0^1 A(x,y)f(y)dy$$

defines a contraction G on $\mathcal{C}([0,1])$ sending f to Gf.

(4) Show that there exists a unique $f \in \mathcal{C}([0,1])$ such that

$$f(x) = h(x) + \int_0^1 A(x,y)f(y)dy \ \forall\ x \in [0,1].$$

Solution. (1) Since A is continuous, if $f \in \mathcal{C}([0,1])$ then for each $x \in [0,1]$, $A(x,y)f(y)$ is continuous and hence integrable on $[0,1]$. Thus $g(x)$ exists for each $x \in [0,1]$. To see that it is continuous, note that A is uniformly continuous, since $[0,1]^2$ is compact. Thus given $\epsilon > 0$ there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|A(x,y) - A(x',y)| < \epsilon$ for all $y \in [0,1]$. Thus

$$|g(x) - g(x')| = |\int_0^1 (A(x,y) - A(x',y))f(y)dy| \leq \epsilon \sup|f|$$

shows that g is continuous, so $g \in \mathcal{C}([0,1])$.

(2) Estimate the integral

$$|g(x)| \leq \int_0^1 |A(x,y)||f(y)|dy \leq \frac{1}{2}\sup|f|$$

shows that $\|g\| \leq \frac{1}{2}\|f\|$.

(3) If $f_1,\ f_2 \in \mathcal{C}([0,1])$ then

$$G(f_1)(x) - G(f_2)(x) = \int_0^1 A(x,y)(f_1(y) - f_2(y))dy.$$

By the estimate above, $d(G(f_1), G(f_2)) = \|G(f_1) - G(f_2)| \leq \frac{1}{2}\|f_1 - f_2\| = \frac{1}{2}d(f_1, f_2)$ in terms of the distance on $\mathcal{C}([0,1])$. Thus G is a contraction.

(4) Since $\mathcal{C}([0,1])$ is complete, the Contraction Mapping Principle implies that there is a unique solution of $G(f) = f$ which is the desired equation.

$\square$

Comment: Many people did not see there was anything to prove as regards the continuity of g in the first part, or tried to use the FTC (with not good effect). In the third part there was often confusion on what the distance being estimated was (and why the $h(x)$ drops out.) In the last part you should have remembered to say that $\mathcal{C}([0,1])$ is complete, since it is one of the conditions in the Contraction Mapping Principle.