One thing that I have not been able to describe is the wavefront set of a distribution, so I ask you to assimilate the definition and deduce some basic properties. This notion involves cones in $\mathbb{R}^n \setminus \{0\}$ so let me define ‘the open cone of aperture $\epsilon > 0$ around a point’ to be

\[
\Gamma(\bar{\xi}, \epsilon) = \left\{ \xi \in \mathbb{R}^n \setminus \{0\} : \frac{|\xi|}{|\bar{\xi}|} - \frac{\bar{\xi}}{|\bar{\xi}|} < \epsilon \right\}.
\]

Make sure you see that this is just a ball around the point in the sphere $\bar{\xi}/|\bar{\xi}| \in S^{n-1}$ extended radially.

If $u \in C^{-\infty}(\Omega), \Omega \subset \mathbb{R}^n$ open, the wave front set of $u$ is the subset

\[
\text{WF}(u) \subset \Omega \times (\mathbb{R}^n \setminus \{0\})
\]

defined in terms of its complement

\[
\Omega \times (\mathbb{R}^n \setminus \{0\}) \ni (\bar{x}, \bar{\xi}) \notin \text{WF}(u) \iff \exists \phi \in C_c^\infty(\Omega), \phi(\bar{x}) \neq 0 \text{ and } \epsilon > 0 \text{ such that } \sup_{\Gamma} |\xi|^N |\mathcal{F}(\phi u)(\xi)| < \infty \forall N, \Gamma = \Gamma(\bar{\xi}, \epsilon).
\]

The idea is that the wavefront set gives information about the (co-) direction of singularities, not just their position.

**Q8.1**

For $u \in C^{-\infty}(\Omega)$ show that

1. $\text{WF}(u) \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$ is closed (as a subset of course)
2. $\text{WF}(u)$ is ‘conic’ i.e.
3. $(x, \xi) \in \text{WF}(u) \implies (x, t\xi) \in \text{WF}(u), (x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}), t > 0.$
4. $\text{WF}(u) \subset \text{singsupp}(u) \times (\mathbb{R}^n \setminus \{0\}).$

**Q8.2**

Given $\bar{\xi} \in \mathbb{R}^n \setminus \{0\}$ and $\epsilon_1 > \epsilon_2 > 0$ small construct a(n almost) conic cut-off $0 \leq \psi \in S^0(\mathbb{R}^n)$ (the symbol space) such that

\[
\text{supp } \psi \subset \Gamma(\bar{\xi}, \epsilon_1), \psi = 1 \text{ on } \Gamma(\bar{\xi}, \epsilon_2) \cap \{|\xi| > 2\}.
\]
Show that \((\bar{x}, \bar{\xi}) \notin \text{WF}(u)\) is equivalent to
\[
\psi \mathcal{F}(\phi u) \in \mathcal{S}(\mathbb{R}^n) \iff b_\psi * (\phi u) \in \mathcal{S}(\mathbb{R}^n), \quad \hat{b}_\psi = \psi,
\]
for some \(\phi \in \mathcal{C}^\infty_c(\Omega), \phi(\bar{x}) \neq 0, \epsilon_1 > \epsilon_2 > 0\).

**Hint:** One way is easy here. The other way the issue is that the definition of \(\text{WF}(u)\) only gives directly the condition that \(b_\psi * \phi u \in H^\infty(\mathbb{R}^n)\) (the intersection of the Sobolev spaces). You should recall that \(b_\psi\) is the sum of a compactly supported distribution and an element of \(\mathcal{S}(\mathbb{R}^n)\).

**Q8.3**

1. Now show that \((\bar{x}, \bar{\xi}) \notin \text{WF}(u)\) implies that for some \(\phi \in \mathcal{C}^\infty_c(\Omega), \phi(\bar{x}) \neq 0\) and some cone \(\Gamma(\bar{x}, \epsilon)\), \(\epsilon > 0\)
\[
b * (\phi u) \in \mathcal{S}(\mathbb{R}^n) \quad \forall \hat{b} \in S^m(\mathbb{R}^n), \quad \text{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon).
\]
2. Recall (you do not have to prove this, I will do it, and more, in class in L20) that if \(b \in S^m(\mathbb{R}^n)\) and \(\phi \in \mathcal{S}(\mathbb{R}^n)\) then there exist \(\phi_\alpha \in \mathcal{S}(\mathbb{R}^n)\) and \(b_\alpha \in S^{m-j}\) such that given \(k\) there exists \(N = N_k\) such that the operator
\[
E_N : u \mapsto b * (\phi u) - \sum_{|\alpha| \leq N} \phi_j(b_j * u)
\]
has Schwartz kernel in \(C^k(\mathbb{R}^{2n})\).
3. Conclude that if \(\text{[8]}\) holds then for any \(\mu \in \mathcal{C}^\infty_c(\Omega)\)
\[
b * (\mu \phi u) \in \mathcal{S}(\mathbb{R}^n) \quad \forall \hat{b} \in S^m(\mathbb{R}^n), \quad \text{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon).
\]
**Hint:** A kernel in \(C^k(\mathbb{R}^{2n})\) defines a map from \(H^{-k}_c(\mathbb{R}^n)\) to \(H^k_{\text{loc}}(\mathbb{R}^n)\) so as \(k\) increases this becomes ‘increasingly a smoothing operator’. If you know something about the support properties as well (from its definition) you get more.
4. Hence deduce that \((\bar{x}, \bar{\xi}) \notin \text{WF}(u)\) is equivalent to the apparently stronger statement that for some \(\epsilon > 0\)
\[
b * (\phi u) \in \mathcal{S}(\mathbb{R}^n) \quad \forall \phi \in \mathcal{C}^\infty_c(\Omega), \quad \text{supp} \phi \subset B(\bar{x}, \epsilon),
\]
\[
\hat{b} \in S^m(\mathbb{R}^n), \quad \text{supp}(\hat{b}) \subset \Gamma(\bar{\xi}, \epsilon).
\]

**Q8.4**

Prove a complement to the last part of Q8.1 in the sense that for any \(u \in \mathcal{C}^{-\infty}(\Omega)\) the wavefront set is a refinement of the singular support:
\[
\pi(\text{WF}(u)) = \text{singsupp}(u), \quad \pi(x, \xi) = x
\]

**Q8.5**

This is a somewhat unfair, open-ended, question but I could not resist! Make of it what you will.
We now have three ‘support’ sets for distributions in an open set \( U \) – support itself, singular support and this notion of wavefront set – the first two are subsets of \( U \) but the third is a subset of \( U \times (\mathbb{R}^n \setminus \{0\}) \). Since it is conic we can also think of \( WF(v) \) as a (closed) subset of \( U \times \mathbb{S}^{n-1} \). Try to explain how these three correspond to sheaves, in the three cases

(1) The support corresponds to the sheaf of linear spaces \( \mathcal{C}^{-\infty}(U) \) over \( \mathbb{R}^n \).
(2) The second corresponds to the sheaf of linear spaces \( \mathcal{C}^{-\infty}(U)/\mathcal{C}^\infty(U) \) over \( \mathbb{R}^n \).
(3) The new notion corresponds to a sheaf over \( \mathbb{R}^n \times \mathbb{S}^{n-1} \) where the linear space over an open set \( V \subset \mathbb{R}^n \times \mathbb{S}^{n-1} \) is the quotient

\[
\mathcal{C}^{-\infty}(U)/\{v \in \mathcal{C}^{-\infty}(U) \text{ s.t. } WF(v) \cap V = \emptyset\}, \quad U = \pi_1(V)
\]

being projection onto the first factor.

Hint-Questions
(a) If \( U_1, U_2 \subset \mathbb{R}^n \) are open and \( u \in \mathcal{C}^\infty(U_1 \cap U_2) \) do there exists functions \( u_i \in \mathcal{C}^\infty(U_i) \) such that \( u_1 - u_2 = u \)? It is enough to do this for the constant function 1 on \( U_1 \cap U_2 \).
(b) If \( U_1, U_2 \subset \mathbb{R}^n \) are open, \( V_i \subset U_i \times \mathbb{S}^{n-1} \) open and \( u \in \mathcal{C}^{-\infty}(U_1 \cap U_2) \) is such that \( WF(u) \cap (V_1 \cap V_2) = \emptyset \) do there exists \( u_i \in \mathcal{C}^{-\infty}(U_i) \) with \( WF(u_i) \cap V_i = \emptyset \) and \( u_1 - u_2 = u \)? You could try first writing \( u \) as the difference of two distributions on \( U_1 \cap U_2 \) where one has WF not meeting \( V_1 \) and the other has WF not meeting \( V_2 \).

Q8.6-Opt.

Show the ‘microellipticity of elliptic operators’: If

\[
P(x, D) = \sum_{|\alpha| \leq m} p_{\alpha}(x) D^\alpha
\]

has coefficients \( p_{\alpha} \in \mathcal{C}^\infty(\Omega) \) and is elliptic in \( \Omega \) then

\[
WF(P(x, D)u) = WF(u) \quad \forall \quad u \in \mathcal{C}^{-\infty}(\Omega).
\]

Pseudodifferential operators are also microlocal! You can use the properties of these operators in the notes to show this and deduce \textit{microlocal}
elliptic regularity of differential operators:

(14) if \( P(x, D_x) = \sum_{|\alpha| \leq m} p_{\alpha}(x)D_x^{\alpha}, \ p_{\alpha} \in C^\infty(U), \)

\[
p_{m}(x, \xi) = \sum_{|\alpha| = m} p_{\alpha}(x)\xi^{\alpha}, \text{ then } \forall \ u \in C^{-\infty}(U)
\]

\[
p_{m}(\bar{x}, \bar{\xi}) \neq 0 \implies (\bar{x}, \bar{\xi}) \in WF(P(x, D)u) \iff (\bar{x}, \bar{\xi}) \in WF(u).
\]

Q8.7-opt.

Show that if \( u, v \in C^{-\infty}(\Omega) \) and there is no point \((x, \xi) \in WF(u)\) such that \((x, -\xi) \in WF(v)\) then it is possible to define the product \(uv \in C^{-\infty}(\Omega)\) consistently with multiplication when one element is smooth.

Hint: First think about the corresponding result for singular supports, which is just that \(\text{singsupp}(u) \cap \text{singsupp}(v)\) allows you to define \(uv\) and try to do something similar.