PROBLEM SET 8, 18.155
DUE 1 DECEMBER, 2017

One thing that I have not been able to describe is the *wavefront set* of a distribution, so I ask you to assimilate the definition and deduce some basic properties. This notion involves cones in $\mathbb{R}^n \setminus \{0\}$ so let me define 'the open cone of aperture $\epsilon > 0$ around a point' to be

$$\Gamma(\bar{\xi}, \epsilon) = \left\{ \xi \in \mathbb{R}^n \setminus \{0\}; |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}| < \epsilon \right\}. \tag{1}$$

Make sure you see that this is just a ball around the point in the sphere $\bar{\xi}/|\bar{\xi}| \in \mathbb{S}^{n-1}$ extended radially.

If $u \in \mathcal{C}^{-\infty}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open, the wave front set of u is the subset

$$\mathrm{WF}(u) \subset \Omega \times (\mathbb{R}^n \setminus \{0\}) \tag{2}$$

defined in terms of its complement

$$\begin{aligned} &\Omega \times (\mathbb{R}^n \setminus \{0\}) \ni (\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u) \Longleftrightarrow \\ &\quad \exists\, \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \phi(\bar{x}) \neq 0 \text{ and } \epsilon > 0 \text{ such that} \\ &\qquad \sup_\Gamma |\xi|^N |\mathcal{F}(\phi u)(\xi)| < \infty\ \forall\ N,\ \Gamma = \Gamma(\bar{\xi}, \epsilon). \end{aligned} \tag{3}$$

The idea is that the wavefront set gives information about the (co-)direction of singularities, not just their position.

Q8.1

For $u \in \mathcal{C}^{-\infty}(\Omega)$ show that

(1) $\mathrm{WF}(u) \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$ is closed (as a subset of course)

(2) $\mathrm{WF}(u)$ is 'conic' i.e.

$$(x, \xi) \in \mathrm{WF}(u) \Longrightarrow (x, t\xi) \in \mathrm{WF}(u),\ (x, \xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\}),\ t > 0. \tag{4}$$

(3)

$$\mathrm{WF}(u) \subset \mathrm{singsupp}(u) \times (\mathbb{R}^n \setminus \{0\}). \tag{5}$$

Q8.2

Given $\bar{\xi} \in \mathbb{R}^n \setminus \{0\}$ and $\epsilon_1 > \epsilon_2 > 0$ small construct a(n almost) conic cut-off $0 \leq \psi \in S^0(\mathbb{R}^n)$ (the symbol space) such that

$$\mathrm{supp}\, \psi \subset \Gamma(\bar{\xi}, \epsilon_1),\ \psi = 1 \text{ on } \Gamma(\bar{\xi}, \epsilon_2) \cap \{|\xi| > 2\}. \tag{6}$$

Show that $(\bar{x},\bar{\xi}) \notin \mathrm{WF}(u)$ is equivalent to

$$\psi\mathcal{F}(\phi u) \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow b_\psi * (\phi u) \in \mathcal{S}(\mathbb{R}^n),\ \hat{b}_\psi = \psi, \tag{7}$$

for some $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, $\phi(\bar{x}) \neq 0$, $\epsilon_1 > \epsilon_2 > 0$.

Hint:- One way is easy here. The other way the issue is that the definition of $\mathrm{WF}(u)$ only gives directly the condition that $b_\psi * \phi u \in H^\infty(\mathbb{R}^n)$ (the intersection of the Sobolev spaces). You should recall that b_ψ is the sum of a compactly supported distribution and an element of $\mathcal{S}(\mathbb{R}^n)$.

Q8.3

(1) Now show that $(\bar{x},\bar{\xi}) \notin \mathrm{WF}(u)$ implies that for some $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$, $\phi(\bar{x}) \neq 0$ and some cone $\Gamma(\bar{x},\epsilon)$, $\epsilon > 0$

$$b * (\phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \hat{b} \in S^m(\mathbb{R}^n),\ \mathrm{supp}(\hat{b}) \subset \Gamma(\bar{\xi},\epsilon). \tag{8}$$

(2) Recall (you do not have to prove this, I will do it, and more, in class in L20) that if $b \in S^m(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ then there exist $\phi_\alpha \in \mathcal{S}(\mathbb{R}^n)$ and $b_\alpha \in S^{m-j}$ such that given k there exists $N = N_k$ such that the operator

$$E_N : u \longmapsto b * (\phi u) - \sum_{|\alpha| \leq N} \phi_j (b_j * u) \tag{9}$$

has Schwartz kernel in $\mathcal{C}^k(\mathbb{R}^{2n})$.

(3) Conclude that if (8) holds then for any $\mu \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$

$$b * (\mu\phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \hat{b} \in S^m(\mathbb{R}^n),\ \mathrm{supp}(\hat{b}) \subset \Gamma(\bar{\xi},\epsilon).$$

Hint: A kernel in $\mathcal{C}^k(\mathbb{R}^{2n})$ defines a map from $H^{-k}_{\mathrm{c}}(\mathbb{R}^n)$ to $H^k_{\mathrm{loc}}(\mathbb{R}^n)$ so as k increases this becomes 'increasingly a smoothing operator'. If you know something about the support properties as well (from its definition) you get more.

(4) Hence deduce that $(\bar{x},\bar{\xi}) \notin \mathrm{WF}(u)$ is equivalent to the apparently stronger statement that for some $\epsilon > 0$

$$\begin{aligned} b * (\phi u) \in \mathcal{S}(\mathbb{R}^n)\ \forall\ \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega),\ \mathrm{supp}\,\phi \subset B(\bar{x},\epsilon),& \\ \hat{b} \in S^m(\mathbb{R}^n),\ \mathrm{supp}(\hat{b}) \subset \Gamma(\bar{\xi},\epsilon).& \end{aligned} \tag{10}$$

Q8.4

Prove a complement to the last part of Q8.1 in the sense that for any $u \in \mathcal{C}^{-\infty}(\Omega)$ the wavefront set is a refinement of the singular support:-

$$\pi(\mathrm{WF}(u)) = \mathrm{singsupp}(u),\ \pi(x,\xi) = x \tag{11}$$

Q8.5

This is a somwhat unfair, open-ended, question but I could not resist! Make of it what you will.

We now have three 'support' sets for distributions in an open set U – support itself, singular support and this notion of wavefront set – the first two are subsets of U but the third is a subset of $U \times (\mathbb{R}^n \setminus \{0\})$. Since it is conic we can also think of $\mathrm{WF}(v)$ as a (closed) subset of $U \times \mathbb{S}^{n-1}$. Try to explain how these three correspond to sheaves, in the three cases

(1) The support corresponds to the sheaf of linear spaces $\mathcal{C}^{-\infty}(U)$ over $\mathbb{R}^n$.
(2) The second corresponds to the sheaf of linear spaces

$$\mathcal{C}^{-\infty}(U)/\mathcal{C}^{\infty}(U)$$

over $\mathbb{R}^n$.
(3) The new notion corresponds to a sheaf over $\mathbb{R}^n \times \mathbb{S}^{n-1}$ where the linear space over an open set $V \subset \mathbb{R}^n \times \mathbb{S}^{n-1}$ is the quotient

$$\mathcal{C}^{-\infty}(U)/\{v \in \mathcal{C}^{-\infty}(U) \text{ s.t. } \mathrm{WF}(v) \cap V = \emptyset\},\ U = \pi_1(V) \tag{12}$$

being projection onto the first factor.
Hint-Questions
(a) If U_1, $U_2 \subset \mathbb{R}^n$ are open and $u \in \mathcal{C}^{\infty}(U_1 \cap U_2)$ do there exists functions $u_i \in \mathcal{C}^{\infty}(U_i)$ such that $u_1 - u_2 = u$? It is enough to do this for the constant function 1 on $U_1 \cap U_2$.
(b) If U_1, $U_2 \subset \mathbb{R}^n$ are open, $V_i \subset U_i \times \mathbb{S}^{n-1}$ open and $u \in \mathcal{C}^{-\infty}(U_1 \cap U_2)$ is such that $\mathrm{WF}(u) \cap (V_1 \cap V_2) = \emptyset$ do there exists $u_i \in \mathcal{C}^{-\infty}(U_i)$ with $\mathrm{WF}(u_i) \cap V_i = \emptyset$ and $u_1 - u_2 = u$? You could try first writing u as the difference of two distributions on $U_1 \cap U_2$ where one has WF not meeting V_1 and the other has WF not meeting V_2.

Q8.6-Opt.

Show the 'microellipticity of elliptic operators': If

$$P(x, D) = \sum_{|\alpha| \le m} p_\alpha(x) D^\alpha$$

has coefficients $p_\alpha \in \mathcal{C}^{\infty}(\Omega)$ and is elliptic in Ω then

$$\mathrm{WF}(P(x, D)u) = \mathrm{WF}(u)\ \forall\ u \in \mathcal{C}^{-\infty}(\Omega). \tag{13}$$

Pseudodifferential operators are also microlocal! You can use the properties of these operators in the notes to show this and deduce *microlocal* *elliptic regularity* of differential operators:

$$\text{(14)} \quad \text{if } P(x, D_x) = \sum_{|\alpha| \le m} p_\alpha(x) D_x^\alpha,\ p_\alpha \in \mathcal{C}^\infty(U),$$
$$p_m(x, \xi) = \sum_{|\alpha| = m} p_\alpha(x) \xi^\alpha, \quad \text{then } \ \forall\ u \in \mathcal{C}^{-\infty}(U)$$
$$p_m(\bar{x}, \bar{\xi}) \neq 0 \Longrightarrow (\bar{x}, \bar{\xi}) \in \mathrm{WF}(P(x, D)u) \text{ iff } (\bar{x}, \bar{\xi}) \in \mathrm{WF}(u).$$

Q8.7-opt.

Show that if $u,\ v \in \mathcal{C}^{-\infty}(\Omega)$ and there is no point $(x, \xi) \in \mathrm{WF}(u)$ such that $(x, -\xi) \in \mathrm{WF}(v)$ then it is possible to define the product $uv \in \mathcal{C}^{-\infty}(\Omega)$ consistently with multiplication when one element is smooth.

Hint: First think about the corresponding result for singular supports, which is just that $\operatorname{singsupp}(u) \cap \operatorname{singsupp}(v)$ allows you to define uv and try to do something similar.