This week I ask you to think about \textit{unbounded self-adjoint operators} which you might well encounter. By definition this means a linear map
\[ A : D(A) \longrightarrow H \]
where \( D(A) \subset H \) is a dense linear subspace of a (separable) Hilbert space \( H \) and in addition we assume/require symmetry:
\[ (Au, v) = (u, Av) \quad \forall \ u, v \in D(A) \]
and a maximality condition on \( D(A) \):
\[ \text{if } v \in H \text{ and } D \ni u \mapsto (Au, v) \text{ extends to be continuous on } H \text{ then } v \in D(A). \]

\textbf{Q7.1}

Show that if \( P(\xi) \) is a real and elliptic polynomial of degree \( m \) in \( n \) variables and \( V \in \mathcal{C}^\infty_c(\mathbb{R}^n) \) is real-valued then
\[ P(D) + V(x) : H^m(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \]
is an unbounded self-adjoint operator (with \( H^m(\mathbb{R}^n) \) as domain).

\textbf{Q7.2}

If \( A \) is unbounded self-adjoint show that
(1) The graph of \( A \) is closed in \( H \times H \).
(2) The spectrum, defined as
\[ \text{Spec}(A) = \{ z \in \mathbb{C}; A - z \text{ Id} : D(A) \longrightarrow H \text{ is not a bijection} \} \]
is a closed subset of \( \mathbb{R} \).
(3) \( R(A, z) = (A - z \text{ Id})^{-1} \in \mathcal{B}(H) \) for \( z \in \mathbb{C} \setminus \text{Spec}(A) \) is holomorphic in \( z \).

\textbf{Q7.3}

If \( A \) is unbounded self-adjoint let \( R(A, i) = BU \) be the polar decomposition of \( R(A, i) \).
(1) Show that \( D(A) = \text{Ran}(B) \).
(2) If $E$ is the spectral subspace for $B$ corresponding to the interval $(-\infty, \frac{1}{2}]$ (the $\frac{1}{2}$ is chosen somewhat randomly in $(0,1)$) then

$$D(A) = E^\perp + \text{Ran}(B\big|_E),$$

$A : E^\perp \longrightarrow E^\perp$ is bounded and self-adjoint

$A : \text{Ran}(B\big|_E) \longrightarrow E$ is a bijection and

$A^{-1} : E \longrightarrow E$ is bounded and self-adjoint.

Q7.4

Show how to define $f(A) \in \mathcal{B}(H)$ where $A$ is unbounded self-adjoint and $f \in C_0(\mathbb{R}; \mathbb{R})$ is real valued and vanishes at infinity in such a way as to give a continuous linear map

$C_0(\mathbb{R}; \mathbb{R}) \ni f \longrightarrow f(A) \in \mathcal{B}(H)$ s.t.

$$f(A)g(A) = (fg)(A), \quad B = (A^2 + 1)^{-\frac{1}{2}}$$

where $B$ is given in (Q3).

Q7.5

For the special case of (Q1), $\Delta + V$, $P(\xi) = |\xi|^2$, show that

$\text{Spec}(\Delta + V) \subset [M, \infty)$, $M = \inf_{x \in \mathbb{R}^n} V(x)$.

Q7.6-Opt

Continue (Q5) to show that the part of the spectrum below 0 (if any) consists of a finite number of eigenvalues of finite multiplicity.

Q7.7-Opt

Under the same conditions as (Q5) show that $\text{Spec}(\Delta + V) \supset [0, \infty)$. 