PROBLEM SET 6, 18.155
DUE 4 NOVEMBER, 2017

A bounded operator, $P \in \mathcal{B}(H)$, on a (separable, infinite dimensional) Hilbert space is semi-Fredholm if it has closed range and one (at least) of $\operatorname{Nul}(P)$ and $\operatorname{Nul}(P^*)$ is finite dimensional.

Q1 If $R \subset H$ is a closed subspace with a finite dimensional orthocomplement, and $R \subset R_1 \subset H$ is another subspace, show that R_1 is closed and has finite dimensional complement.

Q2 Suppose P is semi-Fredholm and has range with finite dimensional complement, show that

(a) There exists a right inverse, an operator $Q \in \mathcal{B}(H)$ such that

$$PQ = \operatorname{Id} - \Pi_{\operatorname{Nul}(P^*)},\ \Pi_{\operatorname{Nul}(P^*)} \text{ a projection of finite rank.}$$

(b) Conversely, show that if $P \in \mathcal{B}(H)$ and there exist $Q' \in \mathcal{B}(H)$ which is a right parametrix modulo compact operators in the sense that

$$PQ' = \operatorname{Id} - K,\ K \in \mathcal{K}(H), \tag{1}$$

then P is semi-Fredholm with range of finite codimension.

Q3 Using such a parameterix (or otherwise ...) show that the set of semi-Fredholm operators is open.

Hint: For the ones with closed range of finite codimension, expand the composite $(P + A)Q$ and observe that $(\operatorname{Id} + AQ)$ is invertible if $\|A\|$ is small enough; apply the inverse on the right (to the identity you found) and deduce the result from the previous question. For the others, think adjoints.

Q4 Show that P is Fredholm iff P and P^* are operators as in Q2 and conclude from Q3 that the set of Fredholm operators is open.

Q5 Show that if $A \in \mathcal{B}(H)$ is self-adjoint, Fredholm but not invertible, then its index vanishes and 0 is an isolated point in its spectrum.

Q6-optional Show that $\operatorname{GL}(H)$ is a (path) connected metric space (it is actually contractible.)

Hint: Use the polar decomposition to reduce the question to $\operatorname{U}(H)$, the group of unitary operators. To connect a unitary operator to the identity through unitary (or just invertible) operators use the spectral theorem to write the self-adjoint part as the difference of two non-negative self-adjoint operators which commute with each other and with the antisymmetric part of the unitary operator. Use these to 'pry it apart'.

Q7-optional Show that an operator is Hilbert-Schmidt (resp of trace class) iff the eigenvaluses (repeated with multiplicty) of its self-adjoint and anti-self-adjoint parts each from a sequence in l^2 (resp l^1); define general Schatten ideals by replacing this by l^p and show that these are Banach spaces with respect to an appropriately chosen norm.