PROBLEM SET 4, 18.155
DUE 13 OCTOBER, 2017

Q4.1

Recall from L9 that the space of symbols of order \( m \in \mathbb{R} \) is
\[
S^m(\mathbb{R}^n) = \left\{ a \in C^\infty(\mathbb{R}^n); \|a\|_{m,k} = \sup_{\xi \in \mathbb{R}^n, |\alpha| \leq k} \langle \xi \rangle^{-m+|\alpha|} |\partial_\alpha a(\xi)| < \infty \right\}.
\]

Show that
1. \( S^m(\mathbb{R}^n) \subset S^{m'}(\mathbb{R}^n), m' \geq m. \)
2. \( \langle \xi \rangle^s \in S^s(\mathbb{R}^n). \)
3. The \( S^s \) form a filtered algebra under multiplication
\[ a \in S^m(\mathbb{R}^n), a' \in S^{m'}(\mathbb{R}^n) \implies aa' \in S^{m+m'}(\mathbb{R}^n). \]

Q4.2

Suppose \( \chi \in C^\infty_c(\mathbb{R}^n), \chi(\xi) = 1 \) in \( |\xi| < 1 \). Show that if \( a \in S^m(\mathbb{R}^n) \) then \( a_n(\xi) = a(\xi)\chi(\xi/n) \) is bounded with respect to each of the semi-norms in (1) and that \( a_n \to a \) with respect to each of the norms \( \|\cdot\|_{m',k} \) if \( m' > m. \)

Q4.3

Using the density shown in Q4.2, or otherwise, show that the operators (discussed in L10 but you should be able to see this before then)
\[
(2) \quad b* : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n), \quad b = G a, \quad a \in S^m(\mathbb{R}^n)
\]
form a graded algebra, \( a * (b * \phi) = c * \phi, a \in S^m(\mathbb{R}^n), b \in S^{m'}(\mathbb{R}^n) \) implies \( c \in S^{m+m'}(\mathbb{R}^n). \) [This is the algebra of pseudodifferential operators with constant coefficients on \( \mathbb{R}^n. \)]

Q4.4

Assuming that \( \phi, \psi \in C^\infty_c(\mathbb{R}^n) \) have disjoint supports and that \( E \in C^{-\infty}_c(\mathbb{R}^n) \) has \( \text{singsupp}(E) \subset \{0\} \) show that
\[
C^{-\infty}(\mathbb{R}^n) \ni u \longmapsto \phi(E * (\psi u)) \in C^\infty_c(\mathbb{R}^n).
\]
Q4.5

Explain the precise meaning of the formula
\[
\Delta |x|^{-n+2} = c_n \delta_0, \quad n > 2,
\]
and compute the constant.

Q4.6 (Optional)

Given any (relatively of course) closed subset of an open set \( \Omega \subset \mathbb{R}^n \) show that there is a distribution \( u \in C^{-\infty}(\Omega) \) with this as singular support. The usual argument is called ‘condensation of singularities’ if you want to look it up.

Hint: I described this very quickly in Lecture 9.

Q4.7 (Optional)

A differential operator \( P(D) \) with constant coefficients is said to be hypoelliptic if for every (equivalently any one non-empty) open set \( \Omega \)
\[
\text{singsupp}(P(D)u) = \text{singsupp}(u) \quad \forall \ u \in C^{-\infty}(\Omega).
\]

Show that this condition is equivalent to the existence of a function of slow growth \( v \) such that \( P(\xi)v(\xi) = 1 + e(\xi), \ e \in \mathcal{S}(\mathbb{R}^n). \) [It is actually equivalent to the existence of any smooth function with this property but that involves more work.]