Abstract. Notes before and after lecture – if you have questions, ask!

Singular support and symbols

- Singular support: Since $\mathcal{C}^\infty(\Omega) \subset \mathcal{C}^{-\infty}(\Omega)$ is a well-defined subset the definition

$$u \in \mathcal{C}^\infty(U)$$

is smooth on $\Omega \subset U$ both open if $u|_\Omega \in \mathcal{C}^\infty(\omega)$ makes sense.

Lemma 1. If $u \in \mathcal{C}^{-\infty}(\Omega)$ and

$$(1) \quad \mathcal{O} = \bigcup \{U \subset \Omega; U \text{ is open and } u|_U \in \mathcal{C}^\infty(U)\}$$

then $u|_\mathcal{O} \in \mathcal{C}^\infty(\mathcal{O})$.

Thus there is a largest open set to which $u \in \mathcal{C}^{-\infty}(\Omega)$ restricts to be smooth and we may unambiguously define the (relatively) closed set

$$(3) \quad \text{singsupp}(u) = \Omega \setminus \mathcal{O}.$$ 

Obviously $\text{singsupp}(u) \subset \text{supp}(u)$.

Proof. By assumption there is an open covering $U_a$ of $\mathcal{O}$ such that for each $a$, $u|_{U_a} = v_a \in \mathcal{C}^\infty(U_a)$. The fact that $\mathcal{C}^\infty(U) \subset \mathcal{C}^{-\infty}(U)$ is well-defined and the (obvious) pre-sheaf property of the $\mathcal{C}^{-\infty}(U)$ means that for any $a, b$

$$(4) \quad v_a|_{U_a \cap U_b} = v_b|_{U_a \cap U_b} = u|_{U_a \cap U_b}$$

so by the (again obvious) sheaf property (‘locality’) of the $\mathcal{C}^\infty(U)$s there exists one function $v \in \mathcal{C}^\infty(\mathcal{O})$ such that $v|_{U_a} = v_a$ for all $a$. Now, we just have to show that $u|_{\mathcal{O}} = v$. This just means that $u(\psi) = v(\psi) = \int v\psi$ for all $\psi \in \mathcal{C}^\infty(\mathcal{O})$. Since supp($\psi$) is compact it has a finite cover by the $U_a = U_i$ and we know that we can then decompose using a partition of unity to get $\psi = \sum \psi_i$, $\psi_i \in \mathcal{C}^\infty(U_i)$. So

$$(5) \quad u(\psi) = \sum_i u(\psi_i) = \sum_i v(\psi_i) = v(\psi).$$

□

Check some of the basic properties of singular support, in particular that if $u \in \mathcal{C}^{-\infty}(\Omega)$ and $\psi \in \mathcal{C}^\infty(\Omega)$ then

$$(6) \quad \text{singsupp}(\psi u) \subset \text{singsupp}(u).$$
What is more important for us is that

**Proposition 1.** If \( u \in C^ {-\infty} (\mathbb{R}^ n) \) and \( v \in C^ {-\infty} (\mathbb{R}^ n) \) then

\[
\text{singsupp}(u * v) \subset \text{singsupp}(u) + \text{singsupp}(v).
\]

This follows from the smoothness of the convolution if either factor is smooth and the same inclusion, (7), for support.

Exercise: If you are so inclined you might like to check that the quotient spaces \( C^{-\infty}(U)/CI(U) \) for open sets \( U \) form a sheaf over \( \mathbb{R}^ n \). This is called the sheaf of microfunctions. Note that this is not a general fact, the quotient of sheaf by a subsheaf is always a presheaf but not in general a sheaf; here no ‘sheafification’ is required.

**Symbols:-** Ellipticity of \( P(D) \). Last time I showed that ellipticity of \( P(D) \) is equivalent to the fact that there is a smooth function of compact support \( \psi \) such that

\[
a(\xi) = \frac{1 - \psi(\xi)}{P(\xi)} \in \mathcal{C}^\infty(\mathbb{R}^ n) \quad \text{and} \quad |a(\xi)| \leq C|\xi|^{-m}.
\]

In fact \( a \) has the special properties of a symbol of order \(-m\).

A symbol of order \( s \) (for any real \( s \)) is a function \( a \in \mathcal{C}^\infty(\mathbb{R}^ n) \) satisfying the estimates

\[
|\partial^\alpha a(\xi)| \leq C_\alpha |\xi|^{s - |\alpha|} \iff \sup_{\xi \in \mathbb{R}^ n} \langle \xi \rangle^{-s + |\alpha|}|\partial^\alpha a(\xi)| \leq \infty \quad \forall \ \alpha \in \mathbb{N}_0^ n.
\]

We write \( S^s(\mathbb{R}^ n) \) for the linear space of such symbols. It is a Fréchet space with topology given by the seminorms implicit in (9). That, for an elliptic \( P(\xi), a \in S^{-m}(\mathbb{R}^ n) \) follows by differentiating (8) – by induction we find

\[
\partial^\alpha a(\xi) = \frac{Q_\alpha}{P(\xi)^{1 + |\alpha|}}
\]

where \( Q_\alpha \) is a polynomial of degree \((m-1)|\alpha|\) – proof as usual by differentiating again. This gives (9).

**Symbols and the Fourier transform:-** The (inverse) Fourier transform of a symbol is a distribution \( \mathcal{G} a \in \mathcal{S}'(\mathbb{R}^ n) \) satisfying

\[
singsupp(\mathcal{G} a) \subset \{0\}, \quad \mathcal{G} a \in C^{-\infty}_c(\mathbb{R}^ n) + \mathcal{S}(\mathbb{R}^ n), \quad P(D)\mathcal{G} a = \delta + R, \quad R \in \mathcal{S}(\mathbb{R}^ n).
\]

- Parametrices for constant coefficient elliptic operators.
- Local Sobolev spaces
- Local elliptic regularity.