18.155 LECTURE 8
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Read: Chater 3 Section 6.
I have some loose ends to tie up:

Lemma 1. *If* $\phi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\partial^\alpha \phi(0) = 0$ *for all* $|\alpha| \le N$ *then for* $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\|\phi\chi(n\cdot)\|_N \to 0 \text{ as } n \to \infty,\ \|\psi\|_N = \sup_{x, |\alpha|+|\beta| \le N} |x^\beta \partial^\alpha \psi(x)|. \tag{1}$$

Proposition 1. *If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\operatorname{supp}(u) \subset \{0\}$ *then*

$$u = \sum_{|\alpha| \le N} c_\alpha \partial^\alpha \delta_0 \tag{2}$$

for some N *and constants* c_α.

I will sneak in a discussion of linear transformations which prepares the way for later treatment of the coordinate-invariance of distributions and is useful here. Observe that if $L \in \mathrm{GL}(n, \mathbb{R})$ is an invertible $n \times n$ (real) matrix, so $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and for functions $v : \mathbb{R}^n \longrightarrow \mathbb{C}$ we define (as usual) $L^* v = v \circ L$ then $L^* : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ and by change-of-variable

$$\int (L^*\phi)(x)\psi(x)dx = \int \phi(y)(L^{-1})^*\psi(y)|\det L|^{-1}dy. \tag{3}$$

This means that – reverting to our formal embedding $\mathcal{S}(\mathbb{R}^n) \ni \phi \longmapsto U_\phi \in \mathcal{S}'(\mathbb{R}^n)$

$$U_{L^*\phi}(\psi) = \int (L^*\phi)\psi = U_\phi(|\det L|^{-1}(L^{-1})^*\psi) \tag{4}$$

so we are obliged by demands of consistency (and weak density) to *define*

$$(L^*u)(\psi) = u(|\det L|^{-1}(L^{-1})^*\psi)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n) \Longrightarrow \\ L^* : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \text{ is a bijection.} \tag{5}$$

Notice that this definition does not make sense if L is not invertible, however $(L^{-1})^* = (L^*)^{-1}$ as one would want!

Why do I introduce this now? One reason it is very useful to think about *homogeneity* – simple behaviour under scaling of the coordinates. Let $M_t x = tx$ for $t > 0$, be the corresponding linear transformation. Then from the definition above, for $u \in \mathcal{S}'(\mathbb{R}^n)$, becomes

$$(M_t^* u)(\psi) = u(t^{-n}\psi(\cdot/t)) \tag{6}$$

and this is (the only possible definition) consistent with scaling of functions.

Observe that $|x|^{-n+s}$, $s > 0$, is a locally integrable function on $\mathbb{R}^n$ which defines a tempered distribution and which is indeed homogeneous of degree $-n+s$: If

$u = |x|^{-n+s} \in \mathcal{S}'(\mathbb{R}^n)$ then

$$
\begin{aligned}
(7) \quad (M_t^* u)(\psi) &= t^{-n} u(\psi(\cdot/t)) = t^{-n} \int |x|^{-n+s} \psi(\frac{x}{t}) dx \\
&= \int |x|^{-n+s} \psi(\frac{x}{t}) d(x/t) = \int |ty|^{-n+s} \psi(y) dy = t^{-n+s} u(\psi)
\end{aligned}
$$

$$\text{that is } M_t^* u = t^{-n+s} u,\ u = |x|^{-n+s}.$$

So we say an elment $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree z for any $z \in \mathbb{C}$ if

$$M_t^* u = t^z u,\ \text{i.e. } u(\psi(\cdot/t) = t^{z+n} u(\psi)\ \forall\ t > 0,\ \psi \in \mathcal{S}(\mathbb{R}^n). \tag{8}$$

We have shown that $|x|^{-n+s}$ is homogeneous of degree $-n + s$. You might like to check that this remains meaningful, and correct, if $s \in \mathbb{C}$, $\operatorname{Re} s > 0$. It does not make sense directly if $\operatorname{Re} s \le 0$ because $|x|^{-n+s}$ is not then locally integrable across the origin.

Lemma 2. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ is homogeneous of degree $z \in \mathbb{C}$ then $x^\beta \partial^\alpha u$ is homogeneous of degree $s + |\beta| - |\alpha|$.*

Lemma 3. *The distribution $\partial^\alpha \delta_0$ is homogeneous of degree $-n - |\alpha|$.*

Proposition 2. *For appropriate constants c_n, $E_n = c_n |x|^{-n+2}$ $(n > 2)$, $E_2 = c_2 \log |x|$, is a fundamental solution of $\Delta = -\sum_i \partial_i^2$.*

Proof. I leave it to you in the next homework (not due until Oct 13) to compute the constants c_n. □

Another important example of fundamental solutions is of the Cauchy-Riemann operator

$$\overline{\partial} = (\partial_x + i\partial_y) \text{ on } \mathbb{R}^2 = \mathbb{C} \tag{9}$$

where a factor of a 'half' is often included. Most of the elementary aspects of the theory of one complex variable can be deduced from the fact that

Lemma 4. *$E = \frac{1}{2\pi}(x + iy)^{-1}$ is a fundamental solution of $\overline{\partial}$.*

Proof. You can get this in various ways. One, relating to Dirac operators (that I hope to get to a bit later) is to use the fundamental solution for Δ when $n = 2$ discussed above and the identity

$$\Delta_2 = (\partial_x + i\partial_y)(\partial_x - i\partial_y) \Longrightarrow E = (\partial_x - i\partial_y) E_2 \tag{10}$$

must be a fundamental solution for $\overline{\partial}$. □

Next we start to consider *ellipticity* of a differential operator with constant coefficients $P(D)$ (remember $D_i = \frac{1}{i}\partial_i$) or equivalently of its 'characteristic polynomial' $P(\xi)$.

See the notes starting around Definition 6.11.

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