NOTES FOR 18.155 LECTURE 4
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Last time we found some useful elements of $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, smooth functions 'of compact support'. We need to talk about this more later, but let me define the support of a continuous function precisely. In fact here is a definition-lemma:-

$$
\begin{aligned}
(1) \qquad \text{if } u \in \mathcal{C}(\mathbb{R}^n),\ \operatorname{supp}(u) &= \overline{\{u(x) \neq 0\}} \\
&= \Big(\bigcup\{O \subset \mathbb{R}^n; O \text{ is open and } u = 0 \text{ on } O\}\Big)^{\complement} \\
&= \mathbb{R}^n \setminus U,\ U = \text{ the largest open set on which } u = 0.
\end{aligned}
$$

It is perhaps not immediately clear that the last 'definition' makes sense, since it really asserts that there *is* a largest open set on which $u = 0$. Well, there is because if U_1 and U_2 are open and u vanishes on them both then it vanishes on the union $U_1 \cup U_2$. It follows that U is the union of all open sets on which u vanishes, which (noticing the complement) gives the middle equality. The first equality then follows from continuity. In fact the support is a 'sheafy' object as we shall see and I assert that the last definition, or if you like the second, is the best way to go.

You might like to modify the definition to take care of the case that $u \in \mathcal{C}^0(\Omega)$ is only defined on an open set Ω in the first place.

So $\mathcal{C}^0_{\mathrm{c}}(\mathbb{R}^n)$ and $\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$ are the spaces of continuous and smooth functions of compact support.

Proposition 1. *If $u,\ v \in \mathcal{C}^0_c(\mathbb{R}^n)$ then*

$$
\begin{gathered}
u * v(x) = \int u(x-y)v(y)dy = \int u(y)v(x-y)dy = v * u \in \mathcal{C}^0_c(\mathbb{R}^n), \\
\operatorname{supp}(u * v) \subset \operatorname{supp}(u) + \operatorname{supp}(v) \ \textit{and} \\
\sup|(u+v)| \le C_R \sup|u| \sup|v| \ \textit{if} \ \operatorname{supp}(u) + \operatorname{supp}(v) \subset B(0,R).
\end{gathered} \tag{2}
$$

Note that the sum of two sets is the sets of sums of elements; check that it is compact if they are both compact.

These follow respectively by change of variable, the fact that $\operatorname{supp}(u(x-\cdot)) = x - \operatorname{supp}(u)$ so $y \notin \operatorname{supp}(u(x-\cdot)v(y))$ unless $x \in \operatorname{supp}(u) + \operatorname{supp}(v)$ and finally

$$
|\int u(x-y)v(y)dy| \le \operatorname{Vol}(\operatorname{supp}(u) + \operatorname{supp}(v)) \sup|u| \sup|v|. \tag{3}
$$

Now, convolution – which is what this product is called – has smoothing properties

$$
\mathcal{C}^0_{\mathrm{c}}(\mathbb{R}^n) * \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n) \subset \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n).
$$

To see this take the second form of $u * v$ and differentiate under the integral sign – at least briefly you should think of taking difference quotients on the left and hence insider the integral and using the fact (from Taylor's formula with remainder) that the difference quotient converges to the derivative *uniformly* for an element of $\mathcal{C}^\infty_c(\mathbb{R}^n)$. Thus

$$\partial_j(u*v) = u*(\partial_j v) \in \mathcal{C}^0_c(\mathbb{R}^n) \Longrightarrow \partial^\alpha(u*v) = u*(\partial^\alpha v) \tag{4}$$

from which higher differentiability follows by induction.

Now, last time we constructed bump functions. One was a $\chi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\chi \geq 0$, $\operatorname{supp}(\chi) \subset B(0,1)$ which is not identically zero. We can scale it by a positive constant so $\int \chi = 1$. Now consider for a given $u \in \mathcal{C}^0_c(\mathbb{R}^n)$ the sequence

$$u_k = u * \chi_k \in \mathcal{C}^\infty_c(\mathbb{R}^n),\ \chi_k(x) = k^n\chi(kx) \Longrightarrow \int \chi_k = 1. \tag{5}$$

Since $\operatorname{supp}(\chi_k) \subset B(0,k^{-1})$ for all k, the support of u_k is contained in a fixed compact set such as $\operatorname{supp}(u) + B(0,1)$.

We now claim that

$$u_k \to u \text{ uniformly }.$$

Well, the way to see this is the estimate the supremum norm of the difference and the trick is to write the difference as

$$u(x)-u_k(x) = u(x)-\int u(x-y)\chi_k(y)dy = \int (u(x)-u(x-y)\chi_k(y)dy \text{ since } \int \chi_k = 1. \tag{6}$$

Then from the usual supremum estimate of integrals as used to get (3) ,

$$\sup|u-u_k| \leq C\int |u((x)-u(x-y)|\chi_k(y) \leq C\sup|u-u_k|\int \chi_k = C\sup|u-u_k| \tag{7}$$

where we also use the fact that $\chi_k \geq 0$ and that it has integral 1.

So we have shown that if $u \in \mathcal{C}^0_c(\mathbb{R}^n)$ there exists a sequence $u_k \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with support in a fixed compact set (very close to $\operatorname{supp}(u)$ if we want) such that $u_k \to u$ uniformly. However this implies

$$\mathcal{C}^\infty_c(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \text{ is dense w.r.t. } \|\cdot\|_{L^2} \tag{8}$$

(because $\mathcal{C}^0_c(\mathbb{R}^n)$ is dense and the convergence here implies convergence in L^2 (by LDC if you don't know otherwise). Same argument works for L^1 but I ask you to do this more directly in the homework this week.

So now we know that

$$\mathcal{C}^\infty_c(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \text{ are dense.} \tag{9}$$

The density of the first in the second (with respect to the topology of $\mathcal{S}(\mathbb{R}^n)$) I mentioned at the end last time. Take a different bump function $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with support in $B(0,2)$, with $\psi \geq 0$ (which does not matter here) and $\psi = 1$ on $B(0,1)$. Then if $\phi \in \mathcal{S}(\mathbb{R}^n)$ define

$$\mathcal{C}^\infty_c(\mathbb{R}^n) \ni \phi_k(x) = \phi(x)\psi(\frac{x}{k}) = \phi(x) \text{ on } B(0,k) \tag{10}$$

since $\psi(\frac{x}{k}) = 1$ if $|x| \leq k$. To estimate the norms of the difference $(\phi - \phi_k)(x) = \phi(x)(1-\psi)(\frac{x}{k})$consider

$$x^\beta\partial^\alpha_x(\phi_k)(x) = \sum_{\gamma\leq\alpha} c_{\gamma,\alpha}x^\beta\partial^{\beta-\gamma}\phi(x)k^{|\gamma|}(\partial^\gamma(1-\psi)(\frac{x}{k}) \tag{11}$$

where the powers of k are a bit worrying in terms of estimates! However, the point is that $(1-\psi)(\frac{x}{k}) = 0$ if $|x| \leq k$ so all terms vanish there. The supremum is then over $|x| \geq k$, all the derivatives of $1-\psi$ are bounded so, using the rapid decay of derivatives of ϕ we see that all terms have a similar bound giving

$$|x^\beta \partial_x^\alpha(\phi_k)(x)| \leq C(1+k)^{-N-10}k^N, \text{ if } |\alpha|+|\beta| \leq N. \tag{12}$$

This implies $\phi_k \to \phi$ in $\mathcal{S}(\mathbb{R}^n)$.

This in turn implies that the map

$$L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n),\ f \longmapsto U_f,\ U_f(\phi) = \int f\phi \tag{13}$$

is injective – since if $U_f = 0$ then $U_f(g_k) = \int f g_k = 0$ for a sequence $\mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n) \ni g_k \to \overline{f}$ in $L^2(\mathbb{R}^n)$ so $\int |f|^2 = 0$ and $f = 0$ in $L^2(\mathbb{R}^n)$.

Note that this argument works for all the weighted L^2 space

$$\langle x\rangle^s L^2(\mathbb{R}^n) \ni f \longrightarrow U_f \in \mathcal{S}'(\mathbb{R}^n),\ U_f(\phi) = \int f\phi \tag{14}$$

where the map to $\mathcal{S}'(\mathbb{R}^n)$ is consistent with the inclusions $\langle x\rangle^t L^2(\mathbb{R}^n) \subset \langle x\rangle^s L^2(\mathbb{R}^n)$ if $t \leq s$ (because the weight is not involved in the defininition of U_f). To get injectivity for the weighted space, if $f \in \langle x\rangle^s L^2(\mathbb{R}^n)$ we take $\mathcal{C}_{\mathrm{c}}^\infty(\mathbb{R}^n) \ni g_k \to \langle x\rangle^{2s}\overline{f}$ with respect to norm on $\langle x\rangle^s L^2(\mathbb{R}^n)$ – since this is the same as saying

$$\langle x\rangle^{-s} g_k \longrightarrow \langle x\rangle^s \overline{f} \tag{15}$$

with respect to the L^2 norm – which we can arrange.

Finally it is time to set $U_f \equiv f$ and regard these injections as inclusions; we also know that $\mathcal{F}$ extends (by continuity from $\mathcal{S}(\mathbb{R}^n)$) to an isomorphism of $L^2(\mathbb{R}^n)$ and that this is consitent with the definition of $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}^n)$.

Now we are in a position to define the 'L^2-based Sobolev spaces' using the Fourier transform

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n); \hat{u} \in \langle \xi\rangle^{-s} L^2(\mathbb{R}^n)\}. \tag{16}$$

Clearly $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$. These spaces decrease as s increases

$$H^{s'}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \text{ if } s' \geq s \tag{17}$$

and each is a Hilbert space with respect to the appropriate norm

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{u}(\xi)|^2 d\xi. \tag{18}$$

The idea is that H^s consists, for $s \geq 0$, of L^2 functions 'with up to s derivatives in L^2'. For non-integral s in particular this requires a bit of explanation! However, for the moment I will continue with the properties of these Sobolev spaces. One thing indicating that there is some sense in the vague statement above is

$$\partial_j : H^s(\mathbb{R}^n) \longrightarrow H^{s-1}(\mathbb{R}^n),\ \forall\ s \in \mathbb{R}. \tag{19}$$

You might ask how this is defined, but the answer is that $\mathcal{S}(\mathbb{R}^n)$ is dense in both domain and range space and ∂_j acts on it, so the claim is that it extends by continuity. The proof is a little diagram that we already know:-

(20)
$$
\begin{tikzcd}
H^s(\mathbb{R}^n) \arrow[rrr, dotted, "\partial_j"] \arrow[ddd, "\mathcal{F}"] & & & H^{s-1}(\mathbb{R}^n) \arrow[ddd, "\mathcal{F}"] \\
& \mathcal{S}(\mathbb{R}^n) \arrow[r, "\partial_j"] \arrow[ul, hook] \arrow[d, leftrightarrow, "\mathcal{F}"] & \mathcal{S}(\mathbb{R}^n) \arrow[ur, hook] \arrow[d, leftrightarrow, "\mathcal{F}"] & \\
& \mathcal{S}(\mathbb{R}^n) \arrow[r, "\times i\xi_j"'] \arrow[dl, hook] & \mathcal{S}(\mathbb{R}^n) \arrow[dr, hook] & \\
\langle\xi\rangle^{-s}L^2(\mathbb{R}^n) \arrow[rrr, "\times i\xi_j"'] & & & \langle\xi\rangle^{-s+1}L^2(\mathbb{R}^n)
\end{tikzcd}
$$

Here the top, dotted, arrow is defined by composition around the outside – consistent with the inside and unique because of the density of the inclusions.

No theorem about $\times x_j$ on Sololev spaces, why?

Now, one very important result

Theorem 1 (Sobolev embedding)**.** *If* $s > k + n/2$, $k \in \mathbb{N}_0$, *then*

(21)
$$H^s(\mathbb{R}^n) \subset \{u : \mathbb{R}^n \longrightarrow \mathbb{C};\ \textit{has } k \textit{ continuous bounded derivatives}\} = \mathcal{C}^k_\infty(\mathbb{R}^n).$$

In fact the range is in $\mathcal{C}^k_0(\mathbb{R}^n)$ – all the derivatives vanish at infinity. The space on the right is a Banach space with respect to

(22)
$$\|u\|_{\mathcal{C}^k} = \sup_{x\in\mathbb{R}^n,\ |\alpha|\le k} |\partial^\alpha u(x)|$$

Why? Start from $\langle\xi\rangle^{-s} \in L^1(\mathbb{R}^n)$ if $s > n/2$ and $\mathcal{G}\mathcal{L}^1(\mathbb{R}^n) \subset \mathcal{C}^0_\infty(\mathbb{R}^n)$.

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