BRIEF NOTES FOR AND AFTER 18.155 LECTURE 2
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- The topology of $\mathcal{S}(\mathbb{R}^n)$ continued.
  Continuity in terms of the norms
  Definition of $\mathcal{S}'(\mathbb{R}^n)$
  Embedding of $L^2(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$.
  Operations on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$
  $\mathcal{S}(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.
- The Fourier transform is an isomorphism on Schwartz space of test functions – will be completed on Thursday.
  For $\phi \in L^1(\mathbb{R}^n)$

$$\mathcal{F}\phi = \hat{\phi}(\xi) = \int e^{-ix\cdot\xi}\phi(x)dx. \tag{1}$$

$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{C}^0_\infty(\mathbb{R}^n)$ (bounded continuous functions)
Continuity of FT as a map on $\mathcal{S}(\mathbb{R}^n)$ and

$$\partial_{\xi_j}\hat{\phi} = -i\widehat{x_j\phi},\ \xi_j\hat{\phi} = -i\widehat{\partial_{x_j}\phi} \tag{2}$$

I messed up the proof of this in class – it is in the lecture notes but here is the argument a little more cleanly:- By the FTC

$$e^{-ix_is} - 1 = \int_0^{x_i}(\frac{d}{dt}e^{-its})dt = -is\int_0^{x_i} e^{-its}dt \tag{3}$$

(even for $x_i < 0$) so

$$|\frac{e^{-ix_is} - 1}{s}| \le |\int_0^{x_i}| \le |x_i| \tag{4}$$

This means that that if $u \in \mathcal{S}(\mathbb{R}^n)$ (in fact if $(1 + |x|)u \in L^1$)

$$\frac{e^{-ix\cdot(\xi+se_i} - e^{-ix\cdot\xi}}{s}u(x) \to -ix_iu(x) \tag{5}$$

converges points wise and is bounded by $(1 + |x_i|)|u(x)|$ so by Lebesgue Dominated Convergence

$$\frac{\hat{u}(\xi + se_i) - \hat{u}(\xi)}{s} \to -i\widehat{x_iu(x)} \tag{6}$$

shows that $\hat{u}$, for $u \in \mathcal{S}(\mathbb{R}^n)$ is differentiable, and in fact that the partial derivatives are bounded and continuous. Repeating the argmument it follows that if $u \in \mathcal{S}(\mathbb{R}^n)$ then all derivatives of $\hat{u}$ exist and are bounded.

I gave the similar argument by integration by parts that $\xi_i\hat{u} = -i\widehat{\partial_i u}$ if $u \in \mathcal{S}(\mathbb{R}^n)$. Combining the two it follows that

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{7}$$

Schwartz functions vanishing at zero
Translations
Inversion formula

$$\mathcal{F}\mathcal{G} = \mathcal{G}\mathcal{F} = \mathrm{Id},\ \mathcal{G}\phi(\xi) = (2\pi)^{-n}(\hat{\phi})(-\xi) \tag{8}$$

Coming up:

- Fourier transform of tempered distributions
- Density of test functions in square-integrable functions
- Fourier transform of square-integrable functions
- Sobolev spaces

References

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