18.155 LECTURE 18
14 NOVEMBER, 2017

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As we saw last week a linear differential operator with smooth coefficients on an open set $U \subset \mathbb{R}^n$

$$P(x, D_x) = \sum_{|\alpha| \le m} p_\alpha(x) D_x^\alpha,\ p_\alpha \in \mathcal{C}^\infty(U), \tag{1}$$

has a 'principal symbol'

$$P_m(x, \xi) = \sum_{|\alpha| = m} p_\alpha(x) \xi^\alpha \in \mathcal{C}^\infty(T^*U) \tag{2}$$

which is a well-defined function on the cotangent bundle – the symbol 'transforms as a function' on T^*U if you change coordinates.

What we are aiming to prove is elliptic regularity in open sets. The differential operator gives a map

$$P : \mathcal{C}^{-\infty}(U) \longrightarrow \mathcal{C}^{-\infty}(U) \tag{3}$$

Theorem 1. *If P is elliptic in U, i.e. $P_m(x, \xi) \ne 0$ if $0 \ne \xi \in \mathbb{R}^n$ then*

$$\begin{aligned} P(x, D_x) &: \mathcal{C}^{-\infty}(U) \longrightarrow \mathcal{C}^{-\infty}(U) \text{ and} \\ P(x, D_x)u &\in H^s_{\text{loc}}(U) \Longleftrightarrow u \in H^{s+m}_{\text{loc}}(U) \end{aligned} \tag{4}$$

Let's recall the constant coefficient case. Then $U = \mathbb{R}^n$ and $p_m(\xi)$ really is a polynomial. We defined a distribution in $\mathcal{S}'(\mathbb{R}^n)$ by

$$\hat{b} = a(\xi) = \frac{1 - \phi(\xi)}{p(\xi)}. \tag{5}$$

Here $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ is equal to 1 on a large set so that the denominator is non-zero on the complement, as is possible by ellipticity. Then we showed some nice things about b. In fact, what we did was 'encapsulate' some estimates satisfied by a into the definition of a space of 'symbols'

$$a \in S^M(\mathbb{R}^n) \Longrightarrow a \in \mathcal{C}^\infty(\mathbb{R}^n) \text{ and } \|a\|_{M,p} = \sup_{|\alpha| \le p} \langle \xi \rangle^{-M+|\alpha|} |D^\alpha a(\xi)| < \infty. \tag{6}$$

This means any derivative $D^\alpha_\xi a$ has absolute value bounded by a constant multiple of $(1 + |\xi|)^{M-|\alpha|}$. As usual, the $\|a\|_{M,p}$ give a Fréchet topology to $S^M(\mathbb{R}^n)$ – it is a complete metric space.

Now, what we showed about b, as the inverse Fourier transform of an element of $S^M(\mathbb{R}^n)$, in this case for $M = -m$, is that it is singular only at the origin and is the sum of a compactly supported distribution and an element of $\mathcal{S}(\mathbb{R}^n)$. That is,

$$\begin{gathered}\mathcal{G} : S^M(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n), \\ \text{and if } \chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n),\ \chi \equiv 1 \text{ in } B(0,\epsilon),\ \epsilon > 0 \text{ then} \\ (1-\chi)\mathcal{G} : S^M(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)\end{gathered} \tag{7}$$

are continuous maps. Thus each $\mathcal{S}(\mathbb{R}^n)$ norm on $(1-\chi)b$ is bounded by a multiple of some $\|a\|_{M,p}$.

Then we looked at the operator defined by convolution with b. The decay of b means that convolution with any element of $\mathcal{S}'(\mathbb{R}^n)$ is well-defined and

$$P(b * f) = f + E * f,\ b * (Pu) = u + E * u,\ E \in \mathcal{S}(\mathbb{R}^n),\ f \in \mathcal{S}'(\mathbb{R}^n). \tag{8}$$

From this and a little playing with localization, namely showing that $b*$ is a 'pseudolocal operator'

$$\operatorname{singsupp}(b * u) \subset \operatorname{singsupp}(u),\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n), \tag{9}$$

the result (4) follows – this is the constant coefficient case.

So, how to generalize this; it will take us a little while.

First we can see that it is enough to work near a given point in U. We want to escape the problems related to the open set U and get back to that at the end. So, take $p \in U$ and $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(U)$ which is supported very close to p and equal to 1 in a slightly smaller neighborhood of p. Then look at

$$P'(x, D) = \chi P(x, D) + (1-\chi)P(p, D). \tag{10}$$

This has smooth coefficients which are constant outside a compact set and it is equal as an operator to $P(x, D)$ when applied to functions supported sufficiently close to p. Moreover if the support of χ is sufficiently small

$$P'(x, D) = \sum_{|\alpha| \le m} p'_\alpha(x) D^\alpha \text{ is elliptic globally.}$$

So we will proceed to discuss regularity for $P'(x, D)$ and then come back to $P(x, D)$ itself afterwards. I will drop the 'prime' and for the moment consider

$$P(x, D) = P_\infty(D) + \sum_{|\alpha| \le m} q_\alpha(x) D^\alpha,\ q_\alpha \in \mathcal{S}(\mathbb{R}^n) \tag{11}$$

which certainly includes $P'(x, D)$. We will assume that P is elliptic, which implies that the constant coefficient operator at infinity, $P_\infty(D)$, is also elliptic.

This means that we actually have uniform ellipticity, that there is actually a constant C and a positive constant c such that

$$|p(x, \xi)| \ge c|\xi|^m \text{ in } |\xi| \ge C. \tag{12}$$

So, we can just use one cut-off $\chi \in \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}^n)$, to excise the zeros and look at the smooth function

$$a(x, \xi) = \frac{1 - \chi(\xi)}{p(x, \xi)}. \tag{13}$$

You could just use the principal part here in place of the whole 'characteristic polynomial' but let me follow the construction in the constant coefficient case closely. Clearly

$$p(x, \xi) a(x, \xi) = 1 - \chi(\xi) \tag{14}$$

as before.

Now, the idea is to 'quantize' a into an operator $a(x, D)$, generalizing the relationship between $p(x, D)$ and $p(x, \xi)$. Before doing that, let's notice that a is a 'variable coefficient symbol' as one might expect. In fact this is just called a symbol anyway. We know what happens when we differentiate with respect to ξ and the same inductive argument really applies to derivatives with respect to x.

Lemma 1. *The function a in* (13) *satisfies*

$$(15)\qquad \begin{gathered} a = a_\infty + \tilde{a},\ a_\infty \in S^M(\mathbb{R}^n), \\ \sup_{\mathbb{R}^n \times \mathbb{R}^n} |\partial_x^\beta \partial_\xi \tilde{a}(x, \xi)| \le C_{N,\alpha,\beta} \langle x \rangle^{-N} \langle \xi \rangle^{M-|\beta|},\ \forall\ \alpha, \beta \in \mathbb{N}_0^n,\ N,\ M = -m. \end{gathered}$$

In fact in the case at hand, $\tilde{a}$ has compact support in x so the decay in x is trivial and all we are saying is that derivatives with respect to x do not affect the decay in ξ.

Another way of describing these estimates is that

$$(16)\qquad a \in S^M(\mathbb{R}^n) + \mathcal{S}(\mathbb{R}^n_x; S^M(\mathbb{R}^n)),\ M = -m.$$

This is how we will think about it in fact, that the variable part is just a 'symbol valued Schwartz function'.

Now, we want to turn $a(x, \xi)$ into an operator in a way which is consistent with how $p(x, \xi)$ is related to $P(x, D)$ and in the constant coefficient case to how $a(\xi)$ is relate to $b* = a(D)$. The way we have written out differential operators is with 'coefficients on the left' – first differentiate and then multipl. For a product of a function and a constant coefficient symbol this clearly means

$$(17)\qquad f(x)a(\xi) \longmapsto f(x)b(y - x),\ \hat{b} = a.$$

We can do this in general:-

Proposition 1. *The partial Fourier transform*

$$(18)\quad \mathcal{F}_{z\to\zeta}\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \ni b(x, z) \longmapsto a(x, \zeta),\ \mathcal{F}_{z\to\zeta} b = a(x, \zeta) = \int e^{-iz\cdot\zeta} b(x, z) dz$$

is an ismomorphism of $\mathcal{S}(\mathbb{R}^{2n})$ to $\mathcal{S}(\mathbb{R}^{2n})$ which extends to an isomorphism of $\mathcal{S}'(\mathbb{R}^{2n})$ to $\mathcal{S}'(\mathbb{R}^{2n})$.

Since the Schwartz Kernel Theorem tells us that operators $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ are in 1-1 correspondence with elements of $\mathcal{S}'(\mathbb{R}^{2n})$ we can certainly get this sort of operator by taking the partial inverse Fourier transform

$$(19)\qquad \begin{gathered} a(x, \xi) \longmapsto b(x, y - x),\ \mathcal{F}_{z\to\zeta} b(x, z) = a(x, \zeta) \\ b(x, x - z) = (2\pi)^{-n} \int e^{i(y-x)\cdot\xi} a(x, \xi) \end{gathered}$$

where the second formulation is more poetic perhaps, but of course it is fine on Schwartz functions.

Definition 1. A pseudodifferential operator $a(x, D_x) \in \Psi^M_{\mathcal{S}}(\mathbb{R}^n)$ (slightly special because of the restrictions on the symbol which is why I hav added the subscript) is an operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ with Schwartz kernel $b(x, y - z)$ where $b \in \mathcal{S}'(\mathbb{R}^{2n})$ is given by (19) with a as in (16)

This would be pretty useless unless we can find some good properties. The first two are:

$$a(x,D_x)\in\Psi^M_{\mathcal{S}}(\mathbb{R}^n)\Longrightarrow a(x,D_x):H^s(\mathbb{R}^n)\longrightarrow H^{s-M}(\mathbb{R}^n)\text{ is bounded }\forall\ s\in\mathbb{R}\tag{20}$$

and

$$\begin{gathered}a(x,D_x)\in\Psi^M_{\mathcal{S}}(\mathbb{R}^n),\ P(x,D_x)\text{ as in (13)}\Longrightarrow\\ P(x,D_x)a(x,D_x)=r(x,D_x)\in\Psi^{m+M}_{\mathcal{S}}(\mathbb{R}^n),\\ r(x,\xi)-P(x,D_x)a(x,D_x)\in S^{m+M-1}(\mathbb{R}^n)+\mathcal{S}(\mathbb{R}^n;S^{m+M-1}(\mathbb{R}^n).\end{gathered}\tag{21}$$

These, and other properties, are not so hard to prove. Before doing that, notice that (21) which holds if $a(x,D_x)$ happens to be a differential operator of the same type as $P(x,D)$ is pretty much what we want. It says that for the $a(x,\xi)$ in (13), constructed from an elliptic $P(x,D_x)$ we get

$$P(x,D_x)a(x,D_x)=\mathrm{Id}-E*+R(x,D_x),\ R(x,D_x)\in\Psi^{-1}_{\mathcal{S}}(\mathbb{R}^n).\tag{22}$$

Combined with (20) this shows that

$$u\in H^{-N}(\mathbb{R}^n),\ P(x,D_x)u\in H^s(\mathbb{R}^n)\Longrightarrow u\in H^{m+s}(\mathbb{R}^n).\tag{23}$$

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