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Up-coming:

(1) Harmonic oscillator
(2) Diffeomorphisms of open sets
(3) Densities and duality
(4) Distributions on manifolds
(5) Sobolev spaces

(1) Harmonic oscillator
(2) Diffeomorphisms of open sets

A smooth map between open sets of $\mathbb{R}^n$ is a map $F : U_1 \longrightarrow U_2$ with components in $\mathcal{C}^\infty(U_1)$.

Pull-back is the continuous map given by composition:

$$F^* : \mathcal{C}^\infty(U_2) \longrightarrow \mathcal{C}^\infty(U_1),\ F^*\phi = \phi \circ F. \tag{1}$$

Notice this is 'contravariant'. If $G : U_2 \longrightarrow U_3$ is smooth then $(G \circ F)^* = F^* \circ G^*$.

The continuity of F^* means we get a transpose map 'push-forward' on compactly supported distributions, $\mathcal{C}_{\mathrm{c}}^{-\infty}(U) = \mathcal{C}^\infty(U)'$:

$$F_* : \mathcal{C}_{\mathrm{c}}^{-\infty}(U_1) \longrightarrow \mathcal{C}_{\mathrm{c}}^{-\infty}(U_2),\ (F_*u)(\psi) = u(F^*\psi),\ \psi \in \mathcal{C}^\infty(U_2). \tag{2}$$

Even if $u \in \mathcal{C}_{\mathrm{c}}^\infty(U_1) \subset \mathcal{C}_{\mathrm{c}}^{-\infty}(U_1)$, F_*u will not usually be smooth. For instance if $F(x) = y_0$ is a constant map then you will see that

$$F_*(u) = c\delta_{y_0},\ c = \int u(x)dx. \tag{3}$$

(3) Now a smooth map $F : U_1 \longrightarrow U_2$ is a diffeomorphism if it has a smooth inverse – it is a bijection and the inverse map $G : U_2 \longrightarrow U_1$ is also smooth (it is easy to find smooth bijections which do not have smooth inverses, such as $x \longmapsto x^3$ on $\mathbb{R}$).

For a diffeomorphism, $F^* : \mathcal{C}^\infty(U_2) \longrightarrow \mathcal{C}^\infty(U_1)$ is an isomorphism, with continuous inverse G^*.

In the case of a diffeomorphism several things happen but let's concentrate on the *problem.* Namely in this case if $u \in \mathcal{C}_{\mathrm{c}}^\infty(U_1)$ then $F_*u \in \mathcal{C}_{\mathrm{c}}^\infty(U_2)$ but it is not always equal to $G^*u \in \mathcal{C}_{\mathrm{c}}^\infty(U_2)$! So we have at the very least a notational issue. What happens:-

$$F_*u(\psi) = u(F^*\psi) = \int u(x)\psi(F(x))dx = \int u(G(y))\psi(y)|J_G(y)|dy, \tag{4}$$
$$J_G(y) = \det \frac{\partial G_i(y)}{\partial y_j}.$$

So the issue is the change of variable formula for integration, which involves the absolute value of the determinant of the Jacobian matrix, giving us

$$F_* u = (G^* u)|J_G(y)|. \tag{5}$$

So there is a problem with coordinate transformations, but it is clearly a mild one. The problem can be traced back to our identification of $\mathcal{C}^\infty_{\mathrm{c}}(U)$ with a subspace of $\mathcal{C}^{-\infty}_{\mathrm{c}}(U) = (\mathcal{C}^\infty(U))'$. Something has to give.

The solution is *not* to break this identification, at least not really. It is simply to 'carry along the density factor with us'. So, we want to give the part of the integration formula $f(x)dx = f(x)|dx|$ (where dx or, better $|dx|$ is the Lebesgue density; the second notation is better than the first but not usually adhered to) an independent meaning, so that when use integration to get our pairing we think of it as

$$(f, g|dx|) \longrightarrow \int f(x)g(x)|dx|. \tag{6}$$

This is no longer symmetric. Of course on Euclidean space we always have Lebesgue measure at our disposal so we can think of $g|dx|$ as just being g. When we change coordinates this does not work so well!

(4) Densities and duality: Let's go about this repair mission carefully. We are still working with open subsets of $\mathbb{R}^n$ but now we think of them as manifolds and try keep more careful track of things.

On an open subset $U \subset \mathbb{R}^n$ the tangent bundle of U is just

$$TU = U \times \mathbb{R}^n \tag{7}$$

What exactly is the tangent bundle? It is another manifold, constructed tradionally from curves in the given manifold, U. So consider for each point $p \in U$ the smooth curves say $\chi : (-1, 1) \longrightarrow U$ such that $\chi(0) = 0$. Then the tangent space should be

$$\begin{gathered} T_pU = \{\chi \in \mathcal{C}^\infty((-1, 1); U); \chi(0) = p\}/ \simeq, \\ \chi_1 \simeq \chi_2 \text{ if they are equal to first order at } 0. \end{gathered} \tag{8}$$

Here of course the equivalence condition is vague. On $\mathbb{R}^n$ we have several ways to interpret this. The easiest is to replace the general χ by the linear (affine) maps through p. Obviously this depends on the linear structure. The second is to use the derivatives at 0, this uses the linear structure as well. A third, more general method is to look at the ideal of smooth functions which vanish at p, $\mathcal{I}_p \subset \mathcal{C}^\infty(U)$. If $f \in \mathcal{C}^\infty(U)$ the composite, $\chi^* f$ is a smooth function on $(-1, 1)$ so we can say

$$\chi_1 \simeq \chi_2 \Longrightarrow \chi_1^*(fg) - \chi_2^*(fg) = O(t^2) \ \forall \ f, g \in \mathcal{I}_p. \tag{9}$$

This only uses the linear structure on $\mathcal{C}^\infty((-1, 1))$ and so makes sense much more generally. The finite linear span of the products of pairs of elements of $\mathcal{I}_p$ is the ideal $\mathcal{I}_p^2$ (by definition) so

$$\chi_1 \simeq \chi_2 \Longrightarrow \chi_1^*(u) - \chi_2^*(u) = O(t^2) \ \forall \ u \in \mathcal{I}_p^2. \tag{10}$$

Now the derivative of the curve gives the standard identification

$$T_pU \ni [\chi] \longrightarrow \frac{d\chi}{dt}(0) \in \mathbb{R}^n \tag{11}$$

which is (7). This is all pedantic stuff, but it pays to get it clear. It means that the Jacobian, also called the differential, of a smooth map is well-defined

(12) $F_* : T_pU_1 \longrightarrow T_{F(p)}U_2,\ F_*[\chi] = [\chi \circ F]$

$$\text{since } F^*\mathcal{I}_{F(p)} \subset \mathcal{I}_p \Longrightarrow F^*\mathcal{I}^2_{F(p)} \subset \mathcal{I}^2_p$$

You should check carefully that according to the identification above, using the chain rule,

$$F_* : T_pU_1 = \mathbb{R}^n \longrightarrow T_{F(p)}U_2 = \mathbb{R}^n \text{ is the Jacobian matrix } F_* = \frac{\partial F_i}{\partial x_j}. \tag{13}$$

Now we recall some linear alebra. If V, W are finite dimensional vector spaces over the reals then the duals, V' W' are well-defined and

$$V \otimes W = \{B : V' \times W' \longrightarrow \mathbb{R} \text{ bilinear}\} \tag{14}$$

is one possible definition of the tensor product. So $v \otimes w \in V \otimes W$, (the 'dyadic product') for $v \in V$ and $w \in W$ is the element such that

$$(v \otimes w)(v', w') = v(v')w(w'). \tag{15}$$

There is a natural isomorphism $V \times W \longrightarrow W \otimes V$ given by switching the order. If $V = W$ then the higher tensor powers $V^{\otimes k}$ have an action of the permutation group Σ_k by order switching and the subspace

$$\lambda^k V \subset V^{\otimes k} \text{ of totally antisymetric elements} \tag{16}$$

is of particular importance (so is the symmetric part, especially for $k = 2$!)

If $L : V_1 \longrightarrow V_2$ is linear then there are induced linear maps $L_k : V_1^{\otimes k} \longrightarrow V_2^{\otimes k}$ and these restrict to $L_k : \lambda^k V \longrightarrow \lambda^k V$. Of particular importance for us for the moment is that

$$\text{if } k = \dim V, L_k = \det L : \lambda^k V \longrightarrow \lambda^k V \tag{17}$$

where the 'maximal degree' $\lambda^k V$ is one-dimensional. This is not so much a theorem as a definition.

Now for the slightly confusing part. On a manifold (of course for the moment I am only talking about open subsets of $\mathbb{R}^n$ but I am doing it in such a way that it generalizes directly) the cotangent bundle is the dual of T_p but we can define it directly

$$T_pU = \mathcal{I}_p/\mathcal{I}_p^2 \tag{18}$$

and *see* that it is naturally identified with the dual of T_p from the definition of the latter. One slightly confusing thing is that the form bundles have fibres at each point

$$\Lambda^k_p U = \lambda^k T^*_p U \tag{19}$$

which is why I was using a 'little λ' above.

(5) Now, if $F : U_1 \longrightarrow U_2$ is a smooth map then $F^*(\mathcal{I}_{F(p)}) \subset \mathcal{I}_p$ for any $p \in U_1$ and so

$$F^* : T^*_{F(p)}U_2 \longrightarrow T^*_p U_1 \tag{20}$$

is also often called the differential – because it is the transpose of $F_* : T_pU_1 \longrightarrow T_{F(p)}U_2$.

Now you should do the little computation to see that the induced map on maximal forms

$$F^* : \Lambda^n_{F(p)} U_2 \longrightarrow \Lambda^n_p U_1 \text{ is } \det \frac{\partial F_i}{\partial x_j} \tag{21}$$

the determinant of the Jacobian.

(6) So the problem is that we need to get the *absolute value* of this determinant into the picture to handle the way our integrals transform. To do this we make the following observation. For any vector space V of dimension n there is a natural isomorphism

$$\lambda^n V = (\lambda^n V')'. \tag{22}$$

That is, the elements of $\lambda^n V$ are just the linear maps

$$\upsilon : \lambda^n V' \longrightarrow \mathbb{R},\ n = \dim V. \tag{23}$$

Now, this is just a one-dimensional vector space but there is another one-dimensional vector space which is very similar but not the same. Namely we can consider

$$\omega V = \{\mu : \lambda^n V' \longrightarrow \mathbb{R}; \mu(sw) = |s|\mu(w)\ \forall\ s \in \mathbb{R},\ w \in \lambda^n V'\}. \tag{24}$$

In higher dimensions this would not be a vector space, but in dimension one it is – the space of absolutely homogeneous functions of degree 1. Check it carefully! Notice that

$$\upsilon \in \lambda^n V \Longrightarrow |\upsilon| \in \omega V. \tag{25}$$

This of course is not a linear map

On a manifold (open subset $U \subset \mathbb{R}^n$) we define

$$\Omega_p U = \omega(T^*_p U). \tag{26}$$

These fit together to form a smooth manifold $\Omega U = U \times \mathbb{R}$.

A very important point is that the Lebesgue measure gives a smooth section of this one dimensional (trivial) bundle.

Note that this 'density bundle' is trivial on any manifold, which is not the case for its close relative $\Lambda^n M$ – which is trivial only when the manifold is orientable. Still ΩM is not *canonically trivial.* Over $\mathbb{R}^n$ it is trivialized by the Lebesgue measure.

You can also write

$$\Omega M = |\Lambda^n M|,\ n = \dim M. \tag{27}$$

A smooth n-form ν on M does define a section $|\nu|$ of ΩM but it is not smooth unless the n-form is non-vanishing, in which case the manifold is orientable. Still, ΩM *always* has a global positive (this makes sense) smooth section – it just does not have a natural one.

(7)

Proposition 1. *If $F : U_1 \longrightarrow U_2$ is a diffeomorphism (it doesn't work for general smooth maps) between open subsets of $\mathbb{R}^n$, then there is a natural pull-back map*

$$F^* : \Omega_{F(p)} U_2 \longrightarrow \Omega_p U_1 \tag{28}$$

given by multiplcation by $|\det\frac{\partial F}{\partial x}|$ *so taking smooth sections to smooth sections. If* $u\in\mathcal{C}^\infty_{\mathrm{c}}(U;\Omega U)$ *is a smooth (even continuous) section of compact support then the integral*

$$\int u\in\mathbb{R} \text{ is well-defined and } \int_{U_1}F^*u=\int_{U_2}u\ \forall\ u\in\mathcal{C}^\infty(U_2;\Omega U_2). \tag{29}$$

(8) Distributions on manifolds:-

Let me write down formally what happens, all this can be deduced from the discussion above. On any smooth manifold, M, there is a well-defined, naturally oriented real line bundle, the density bundle ΩM, such that there is an invariant integral

$$\int:\mathcal{C}^\infty_{\mathrm{c}}(M;\Omega M)\longrightarrow\mathbb{C} \tag{30}$$

(if we allow complex sections). This induces a pairing

$$\mathcal{C}^\infty(M)\times\mathcal{C}^\infty_{\mathrm{c}}(M;\Omega M)\ni(u,\phi)\longrightarrow u\phi\longrightarrow\int u\phi. \tag{31}$$

We use this map $\mathcal{C}^\infty(M)$ into the dual $\mathcal{C}^{-\infty}(M)=(\mathcal{C}^\infty_{\mathrm{c}}(M;\Omega))'$ (this is a definition of $\mathcal{C}^{-\infty}(M)$. We also define $\mathcal{C}^{-\infty}_{\mathrm{c}}(M;\Omega)=(\mathcal{C}^\infty(M))'$ and the same pairing gives us an injection $\mathcal{C}^\infty_{\mathrm{c}}(M;\Omega M)\longrightarrow\mathcal{C}^{-\infty}_{\mathrm{c}}(M;\Omega M)$.

Now, if $F:M_1\longrightarrow M_2$ is a smooth map there are induced linear maps

$$\begin{gathered}F^*:\mathcal{C}^\infty(M_2)\longrightarrow\mathcal{C}^\infty(M_1),\\ F_*:\mathcal{C}^{-\infty}_{\mathrm{c}}(M_1;\Omega M_1)\longrightarrow\mathcal{C}^{-\infty}_{\mathrm{c}}(M_2;\Omega M_2)\\ \text{s.t. } F_*((F^*u)v)=uF_*v\ \forall\ u\in\mathcal{C}^\infty(M_2),\ v\in\mathcal{C}^{-\infty}_{\mathrm{c}}(M_1;\Omega M_1).\end{gathered} \tag{32}$$

If we choose a global smooth positive section $0<\nu\in\mathcal{C}^\infty(M;\Omega M)$ then multiplcation extends to isomorphisms

$$\mathcal{C}^\infty(M)\dot\nu=\mathcal{C}^\infty(M;\Omega M),\ \mathcal{C}^{-\infty}(M)\dot\nu=\mathcal{C}^{-\infty}(M;\Omega M). \tag{33}$$

Saying more might be confusing!

(9) Sobolev spaces

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