- I will finish the discussion of the trace ideal.

  An operator is of trace class, $A \in (T)(H)$ if it is a finite sum of products of Hilbert-Schmidt operators, $A = \sum_{i=1}^{N} B_i D_i,\ B_i,\ D_i \in \mathrm{HS}(H)$.

  If $A \in (T)(H)$ then the trace norm

$$\|A\|_{\mathrm{Tr}} = \sup \sum_i |\langle Ae_i, f_i\rangle| < \infty. \tag{1}$$

  If $A \in \mathcal{B}(H)$ and $\|A\|_{\mathrm{Tr}} < \infty$ then $P = (AA^*)^{\frac{1}{4}} \in \mathrm{HS}(H)$ and hence $A$ is the product of two Hilbert-Schmidt operators.

  The trace functional is defined by

$$\mathrm{Tr}(A) = \sum_i \langle Ae_i, e_i\rangle_H,\ A \in \mathrm{Tr}(H),\ \mathrm{Tr} : (T)(H) \longrightarrow \mathbb{C} \tag{2}$$

  is linear, continuous and independent of the orthonormal basis used to define it.

  If $A \in (T)(H)$ and $B \in \mathcal{B}(H)$ then

$$\mathrm{Tr}([A, B]) = 0. \tag{3}$$

  Conversely, if $A \in (T)(H)$ and $\mathrm{Tr}(A) = 0$ then $A$ is a sum of such commutators.

  If $A = A^* \in (T)(H)$ then

$$\mathrm{Tr}(A) = \sum_i \lambda_i \tag{4}$$

  is the sum of the eigenvalues repeated with multiplicity. This is true even if $A$ is not self-adjoint (Lidskii's theorem) but harder to prove.
- Fredholm operators.

  If $L \in \mathcal{B}(H)$ has closed range then it has a unique 'generalized inverse' $Q \in \mathcal{B}(H)$ satisfying

$$QL = \mathrm{Id} - P_{\mathrm{null}(L)},\ LQ = \mathrm{Id} - P_{\mathrm{null}(L^*)}. \tag{5}$$

- By definition $L$ is *Fredholm* if it has closed range and both the null space and orthocomplement to the range are finite dimensional.

  The index is the difference of the dimensions, defined for $L$ Fredholm:-

$$\mathrm{ind}(L) = \dim \mathrm{null}(L) - \dim \mathrm{null}(L^*). \tag{6}$$

  If $L \in \mathcal{B}(H)$ then $L$ is Fredholm if and only if it has an inverse $Q'$ modulo compact operators

$$Q'L - \mathrm{Id}, LQ' - \mathrm{Id} \in \mathcal{K}(H). \tag{7}$$

  Hence iff it has an inverse modulo errors in $(T)(H)$.

  The Fredholm operators are open in $\mathcal{B}(H)$ and stable under the addition of compact operators.

  If $L$ is Fredholm and $Q$ is an inverse modulo errors in $(T)(H)$ then (Calderón's formula)

$$\mathrm{ind}(L) = \mathrm{Tr}([L, Q]). \tag{8}$$

  The index is constant on components of $\mathcal{F}(H)$ and labels them.

- The harmonic oscillator on $\mathbb{R}^n$,

$$H = \Delta + |x|^2 = \sum_{i=1}^{n} (-\partial_1^2 - \cdots - \partial_n^2) + |x|^2 \tag{9}$$

  has eigenfunctions (the Hermite functions) $h_\alpha \in \mathcal{S}(\mathbb{R}^n)$, $\alpha \in \mathbb{N}_0^n$, forming an orthonormal basis of $L^2(\mathbb{R}^n)$.
- If $u \in \mathcal{S}'(\mathbb{R}^n)$ the Fourier-Bessel series

$$u = \sum_\alpha u(h_\alpha) h_\alpha \tag{10}$$

  converges in $\mathcal{S}'(\mathbb{R}^n)$ and

$$u \in \mathcal{S}(\mathbb{R}^n) \Longleftrightarrow \text{(10) converges in } \mathcal{S}(\mathbb{R}^n). \tag{11}$$

Department of Mathematics, Massachusetts Institute of Technology
*E-mail address*: `rbm@math.mit.edu`