No lecture on 24th October.

Read: Notes Chapter 2.

- Compact operators as closure of finite rank operators.
- \( \operatorname{Ran} A^\perp = \operatorname{Nul}(A^*) \).
- Since there is no lecture Tuesday you can (have) use(d) it to check that Baire’s Theorem can be used to show:
  
  Uniform Boundedness=Banach-Steinhaus (straightforward)
  
  Open Mapping Theorem (trickier)

  Closed Graph Theorem (follows easily from OMT) – look at the commutative diagram

\[
\begin{align*}
\operatorname{Gr}(A) & \subset H \times H \\
\pi_1 & \downarrow \quad \downarrow \pi_2 \\
H & \xrightarrow{(I_d,A)} H
\end{align*}
\]

If \( \operatorname{Gr}(A) \) is closed the continuity of \( \pi_1 \), and hence \( A \) follows from the OMT.

Note that if \( A : H \longrightarrow H \) is bounded and a bijection, the continuity of \( A^{-1} \) follows from either the OMT or the CGT.

- Spectrum and resolvent of a bounded operator.
- Spectrum of a compact operator is discrete outside \( \{0\} \).
- Spectrum of a self-adjoint operator is contained in \( \mathbb{R} \subset \mathbb{C} \).
- If \( A = A^* \in \mathcal{B}(H) \) then \( \|A\| = \sup_{\|u\|=1} |\langle Au, u \rangle| \).
- If \( A = A^* \) then
  \[
  \{\alpha\} \cup \{\beta\} \subset \operatorname{spec}(A) \subset [\alpha, \beta], \quad \alpha = \inf_{\|u\|=1} \langle Au, u \rangle, \quad \beta = \sup_{\|u\|=1} \langle Au, u \rangle.
  \]
- If \( A = A^* \) and \( p \) is a polynomial with real coefficients then

\[
\|p(A)\| \leq \sup_{[\alpha, \beta]} |p(z)|.
\]

- (Functional Calculus) If \( A = A^* \in \mathcal{B}(H) \) there is a continuous linear map of norm one

\[
\mathcal{C}([\alpha, \beta]) \ni f \mapsto f(A) \in \mathcal{B}(H) \text{ s.t. } f(A)g(A) = (fg)(A).
\]

where \( f(A) = A^j \) if \( f(x) = x^j \).

- Existence of spectral projections.

\textit{Department of Mathematics, Massachusetts Institute of Technology}  
\textit{E-mail address: rbm@math.mit.edu}