18.155 LECTURE 10
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Abstract. Notes before and after lecture – if you have questions, ask!

Local elliptic regularity and maybe start on the Schwartz kernel theorem. Next week the SKT and then start operators on Hilbert space.

(1) More on the Fourier transform.

$\mathcal{S}(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$.
$\mathcal{S}'(\mathbb{R}^n) * \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n)$.
$\widehat{u * \psi} = \hat{u}\hat{\psi}$, $u \in \mathcal{S}'(\mathbb{R}^n)$, $\psi \in \mathcal{S}(\mathbb{R}^n)$.
$L^1 * L^2 \subset L^2$.
$\mathcal{S}(\mathbb{R}^n) * \left(\langle x\rangle^m L^2\right) \subset \langle x\rangle^m L^2$.
$b* : H^s(\mathbb{R}^n) \longrightarrow H^{s-m}(\mathbb{R}^n)$, $\hat{b} \in S^m(\mathbb{R}^n)$.

(2) Localization, $\mathcal{S}(\mathbb{R}^n) \cdot H^m(\mathbb{R}^n) \subset H^m(\mathbb{R}^n)$.

Definition: For any open set $\Omega \subset \mathbb{R}^n$

$$H^m_{\text{loc}}(\Omega) = \{u \in \mathcal{C}^{-\infty}(\Omega); \phi u \in H^m(\mathbb{R}^n),\ \forall\ \phi \in \mathcal{C}^\infty_{\text{c}}(\Omega)\}. \tag{1}$$

$U \subset \Omega$ open sets, $\big|_U : H^m_{\text{loc}}(\Omega) \longrightarrow H^m_{\text{loc}}(U)$.

(3) Local elliptic regularity:

$$P \text{ elliptic of order } m,\ u \in \mathcal{C}^{-\infty}(\Omega),\ f \in H^s_{\text{loc}}(\Omega) \Longrightarrow u \in H^{m+s}(\Omega). \tag{2}$$

(4) If $K \in \mathcal{S}'(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ then

$$A_K\phi(\psi) = K(\psi(x)\phi(y)) \Longrightarrow A_K : \mathcal{S}(\mathbb{R}^m) \longrightarrow \mathcal{S}(\mathbb{R}^n)$$

is a continuous linear operator. Converse is the Schwartz kernel theorem.

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