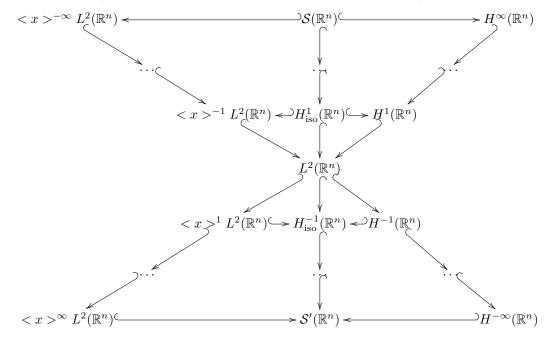
BRIEF NOTES FOR 18.155 LECTURE 1 9 SEPTEMBER 2017

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Abstract.

- Basic aim of the course
- Outline of contents
 - Distributions and function spaces, Fourier transform Constant coefficient differential operators Operators on Hilbert space Elliptic regularity (variable coefficients)
- Prerequisites, including $L^2(\mathbb{R}^n)$.
- Riesz' theorem as the 'weak' definition of $f \in L^2(\mathbb{R}^n)$.
- A biggish diagram of spaces to indicate where we are going in the immediate future (here is a smaller version which almost fits on the page):-



There is a lot to this diagram – and a lot missing. Flipping around the central vertical axis is the Fourier transform (so $S(\mathbb{R}^n)$ etc are invariant under it). Flipping around the central horizontal axis is duality, so $L^2(\mathbb{R}^n)$ is self-dual. Going up the three lines there is a general 'order k' line missing above (and corresponding order -k below) where the dots are. The top spaces are all the intersections of the lines down to $L^2(\mathbb{R}^n)$ and the bottom

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spaces are all the unions of the lines from L^2 to them. The 'isotropic spaces' in the middle are the intersections of the sides above L^2 and the sums below.

- So we start with the space $\mathcal{S}(\mathbb{R}^n)$ which consists of the smooth functions with all derivatives decaying rapidly.
- Differentiability recalled, symmetry of higher derivatives, the notation ∂^αu for α = (α₁,..., α_n) ∈ N₀ⁿ for derivatives and similary x^β for powers.
 So explicitly

(1)
$$\mathcal{S}(\mathbb{R}^n) = \{ u : \mathbb{R}^n \longrightarrow \mathbb{C}; \partial_x^{\alpha} u(x) \text{ exists } \forall \alpha \text{ and }$$

$$\|u\|_N = \sup_{|\alpha| + |\beta| \le N} |x^{\beta} \partial^{\alpha} u| < \infty \ \forall \ N \in \mathbb{N}_0^n \}$$

Here $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ for multiindices, despite the possible confusion.

- Each of the $||u||_N$ is a norm, so $\mathcal{S}(\mathbb{R}^n)$ is a countably normed space.
- A countably normed space has a metric topology given by

(2)
$$d(\phi, \psi) = \sum_{N} 2^{-N} \frac{\|\phi - \psi\|_{N}}{1 + \|\phi - \psi\|_{N}}.$$

You should check that this is a distance – each of the quotients is a distance and the sum is finite.

• In fact $S(\mathbb{R}^n)$ is a complete metric space with respect to this distance, which is to say it is a Fréchet space. You should check that you follow the proof of this; it is a standard sort of completeness argument but needs to be done carefully (by you) at least once. In brief:-

If ϕ_n is Cauchy with respect to the distance then it is Cauchy with respect to each $\|\dot{\|}_N$ (and conversely).

Each $\|\|_N$ is the supremum norm on the $x^{\beta}\partial^{\alpha}\phi_n$ for $|\alpha| + |\beta| \leq N$. From the completeness of the bounded continuous functions it follows that each sequence $x^{\beta}\partial^{\alpha}\phi_n \to u_{\alpha,\beta}$ converges in supremum norm to a bounded continuous limit.

Finally by using standard theorems (in Rudin for instance, at least in 1-D) or better by integrating the derivatives and looking at convergence it follows that the limit of the sequence ϕ_n in supremum norm is $u \in \mathcal{S}(\mathbb{R}^n)$ since $x^{\beta}\partial^{\alpha}u = u_{\alpha,\beta}$.

The uniform convergence of each $x^{\beta}\partial^{\alpha}\phi_n$ to $x^{\beta}\partial^{\alpha}u$ now implies that $\phi_n \to u$ in the metric.

• Finally the space of tempered (also 'temperate') distributions is by definition

(3)
$$\mathcal{S}'(\mathbb{R}^n) = \{ U : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}; U \text{ is linear and continuous.} \}$$

On Tuesday I will talk about the meaning of continuity here (in terms of the norms), the embedding of $L^2(\mathbb{R}^n)$ into $\mathcal{S}'(\mathbb{R}^n)$, various operations on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ and start to talk about the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$:

(4)
$$\hat{\phi}(\xi) = \int e^{-ix \cdot \xi} \phi(x) dx.$$

References

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