CHAPTER 6

Invertibility of elliptic operators

Next we will use the local elliptic estimates obtained earlier on open sets in $\mathbb{R}^n$ to analyse the global invertibility properties of elliptic operators on compact manifolds. This includes at least a brief discussion of spectral theory in the self-adjoint case.

1. Global elliptic estimates

For a single differential operator acting on functions on a compact manifold we now have a relatively simple argument to prove global elliptic estimates.

PROPOSITION 1.1. *If M is a compact manifold and $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is a differential operator with $\mathcal{C}^\infty$ coefficients which is* elliptic *(in the sense that $\sigma_m(P) \neq 0$) on $T^*M\backslash 0$) then for any s, $M \in \mathbb{R}$ there exist constants C_s, C'_M such that*

$$\begin{gathered} u \in H^M(M),\ Pu \in H^s(M) \Longrightarrow u \in H^{s+m}(M) \\ \|u\|_{s+m} \leq C_s\|Pu\|_s + C'_M\|u\|_M, \end{gathered} \tag{1.1}$$

where m is the order of P.

PROOF. The regularity result in (1.1) follows directly from our earlier local regularity results. Namely, if $M = \bigcup_a \Omega_a$ is a (finite) covering of M by coordinate patches,

$$F_a : \Omega_a \longrightarrow \Omega'_a \subset \mathbb{R}^n$$

then

$$P_a v = (F_a^{-1})^* P F_a^* v,\ v \in \mathcal{C}_c^\infty(\Omega'_a) \tag{1.2}$$

defines $P_a \in \operatorname{Diff}^m(\Omega'_a)$ which is a differential operator in local coordinates with smooth coefficients; the invariant definition of ellipticity above shows that it is elliptic for each a. Thus if φ_a is a partition of unity subordinate to the open cover and $\psi_a \in \mathcal{C}_c^\infty(\Omega_a)$ are chosen with $\psi_a = 1$ in a neighbourhood of $\operatorname{supp}(\varphi_a)$ then

$$\|\varphi'_a v\|_{s+m} \leq C_{a,s}\|\psi'_a P_a v\|_s + C'_{a,m}\|\psi'_a v\|_M \tag{1.3}$$

where $\varphi'_a = (F_a^{-1})^*\varphi_a$ and similarly for $\psi'_a(F_a^{-1})^*\varphi_a \in \mathcal{C}_c^\infty(\Omega'_a)$, are the local coordinate representations. We know that (1.3) holds for every $v \in \mathcal{C}^{-\infty}(\Omega'_a)$ such that $P_a v \in H^M_{\text{loc}}(\Omega'_a)$. Applying (1.3) to $(F_a^{-1})^*u = v_a$, for $u \in H^M(M)$, it follows that $Pu \in H^s(M)$ implies $P_a v_a \in H^M_{\text{loc}}(\Omega'_a)$, by coordinate-invariance of the Sobolev spaces and then conversely

$$v_a \in H^{s+m}_{\text{loc}}(\Omega'_a) \ \forall\ a \Longrightarrow u \in H^{s+m}(M).$$

The norm on $H^s(M)$ can be taken to be

$$\|u\|_s = \left(\sum_a \|(F_a^{-1})^*(\varphi_a u)\|_s^2\right)^{1/2}$$

so the estimates in (1.1) also follow from the local estimates:

$$\begin{aligned}\|u\|^2_{s+m} &= \sum_a \|(F_a^{-1})^*(\varphi_a u)\|^2_{s+m} \\ &\le \sum_a C_{a,s}\|\psi'_a P_a (F_a^{-1})^* u\|_s^2 \\ &\le C_s\|Pu\|_s^2 + C'_M\|u\|^2_M.\end{aligned}$$

$\square$

Thus the elliptic regularity, and estimates, in (1.1) just follow by patching from the local estimates. The same argument applies to elliptic operators on vector bundles, once we prove the corresponding local results. This means going back to the beginning!

As discussed in Section 3, a differential operator between sections of the bundles E_1 and E_2 is represented in terms of local coordinates and local trivializations of the bundles, by a matrix of differential operators

$$P = \begin{bmatrix} P_{11}(z, D_z) & \cdots & P_{1\ell}(z, D_z) \\ \vdots & & \vdots \\ P_{n1}(z, D_z) & \cdots & P_{n\ell}(z, D_z)\end{bmatrix}.$$

The (usual) order of P is the maximum of the orders of the $P_{ij}(z, D_3)$ and the symbol is just the corresponding matrix of symbols

$$\sigma_m(P)(z,\zeta) = \begin{bmatrix} \sigma_m(P_{11})(z,\zeta) & \cdots & \sigma_m(P_{1\ell})(z,\zeta) \\ \vdots & & \vdots \\ \sigma_m(P_{n1})(z,\zeta) & \cdots & \sigma_m(P_{n\ell})(z,\zeta)\end{bmatrix}. \tag{1.4}$$

Such a P is said to be *elliptic* at z if this matrix is invariable for all $\zeta \neq 0$, $\zeta \in \mathbb{R}^n$. Of course this implies that the matrix is square, so the two vector bundles have the same rank, ℓ. As a differential operator, $P \in \text{Diff}^m(M, \mathbb{E})$, $\mathbb{E} = E_1, E_2$, is *elliptic* if it is elliptic at each point.

Proposition 1.2. *If $P \in \mathrm{Diff}^m(M,\mathbb{E})$ is a differential operator between sections of vector bundles $(E_1,E_2) = \mathbb{E}$ which is* elliptic *of order m at every point of M then*

$$u \in \mathcal{C}^{-\infty}(M;E_1),\ Pu \in H^s(M,E_) \Longrightarrow u \in H^{s+m}(M;E_1) \tag{1.5}$$

and for all $s,\ t \in \mathbb{R}$ there exist constants $C = C_s$, $C' = C'_{s,t}$ such that

$$\|u\|_{s+m} \le C\|Pu\|_s + C'\|u\|_t. \tag{1.6}$$

Furthermore, there is an operator

$$Q : \mathcal{C}^\infty(M;E_2) \longrightarrow \mathcal{C}^\infty M;E_1) \tag{1.7}$$

such that

$$PQ - \mathrm{Id}_2 = R_2,\ QP - \mathrm{Id}_1 = R_1 \tag{1.8}$$

are smoothing operators.

Proof. As already remarked, we need to go back and carry the discussion through from the beginning for systems. Fortunately this requires little more than notational change.

Starting in the constant coefficient case, we first need to observe that ellipticity of a (square) matrix system is equivalent to the ellipticity of the determinant polynomial

$$D_p(\zeta) = \det \begin{bmatrix} P_{11}(\zeta) & \cdots & P_{1k}(\zeta) \\ \vdots & & \vdots \\ P_{k1}(\zeta) & \cdots & P_{kk}(\zeta) \end{bmatrix} \tag{1.9}$$

which is a polynomial degree km. If the P_i's are replaced by their leading parts, of homogeneity m, then D_p is replaced by its leading part of degree km. From this it is clear that the ellipticity at P is equivalent to the ellipticity at D_p. Furthermore the invertibility of matrix in (1.9), under the assumption of ellipticity, follows for $|\zeta| > C$. The inverse can be written

$$P(\zeta)^{-1} = \mathrm{cof}(P(\zeta))/D_p(\zeta).$$

Since the cofactor matrix represents the Fourier transform of a differential operator, applying the earlier discussion to D_p and then composing with this differential operator gives a generalized inverse etc.

For example, if $\Omega \subset \mathbb{R}^n$ is an open set and D_Ω is the parameterix constructed above for D_p on Ω then

$$Q_\Omega = \mathrm{cof}(P(D)) \circ D_\Omega$$

is a 2-sided parameterix for the matrix of operators P:

$$\begin{aligned} PQ_\Omega - \mathrm{Id}_{k\times k} &= R_R \\ Q_\Omega - \mathrm{Id}_{k\times k} &= R_L \end{aligned} \tag{1.10}$$

where R_L, R_R are $k \times k$ matrices of smoothing operators. Similar considerations apply to the variable coefficient case. To construct the global parameterix for an elliptic operator P we proceed as before to piece together the local parameterices Q_a for P with respect to a coordinate patch over which the bundles E_1, E_2 are trivial. Then

$$Qf = \sum_a F_a^* \psi_a' Q_a \varphi_a' (F_a)^{-1} f$$

is a global 1-sided parameterix for P; here φ_a is a partition of unity and $\psi_a \in \mathcal{C}_c^\infty(\Omega_a)$ is equal to 1 in a neighborhood of its support. $\square$

(Probably should be a little more detail.)

2. Compact inclusion of Sobolev spaces

For any $R > 0$ consider the Sobolev spaces of elements with compact support in a ball:

$$\dot{H}^s(B) = \{u \in H^s(\mathbb{R}^n); u) = 0 \text{ in } |x| > 1\}. \tag{2.1}$$

Lemma 2.1. *Tthe inclusion map*

$$\dot{H}^s(B) \hookrightarrow \dot{H}^t(B) \text{ is compact if } s > t. \tag{2.2}$$

Proof. Recall that compactness of a linear map between (separable) Hilbert (or Banach) spaces is the condition that the image of any bounded sequence has a convergent subsequence (since we are in separable spaces this is the same as the condition that the image of the unit ball have compact closure). So, consider a bounded sequence $u_n \in \dot{H}^s(B)$. Now $u \in \dot{H}^s(B)$ implies that $u \in H^s(\mathbb{R}^n)$ and that $\phi u = u$ where $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is equal to 1 in a neighbourhood of the unit ball. Thus the Fourier transform satifies

$$\hat{u} = \hat{\phi} * \hat{u} \Longrightarrow \hat{u} \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{2.3}$$

In fact this is true with uniformity. That is, one can bound any derivative of $\hat{u}$ on a compact set by the norm

$$\sup_{|z| \le R} |D_j \hat{u}| + \max_j \sup_{|z| \le R} |D_j \hat{u}| \le C(R) \|u\|_{H^s} \tag{2.4}$$

where the constant does not depend on u. By the Ascoli-Arzela theorem, this implies that for each R the sequence $\hat{u}_n$ has a convergent subsequence in $\mathcal{C}(\{|\zeta| \le R\})$. Now, by diagonalization we can extract a subsequence which converges in $\mathcal{V}_c(\{|\zeta| \le R\})$ for every R. This implies that the restriction to $\{|\zeta| \le R\}$ converges in the weighted L^2 norm corresponding to H^t, i.e. that $(1 + |\zeta|^2)^{t/2} \chi_R \hat{u}_{n_j} \to (1 + |\zeta|^2)^{t/2} \chi_R \hat{v}$

in L^2 where χ_R is the characteristic function of the ball of radius R. However the boundedness of u_n in H^s strengthens this to

$$(1+|\zeta|^2)^{t/2}\hat{u}_{n_j} \to (1+|\zeta|^2)^{t/2}\hat{v} \text{ in } L^2(\mathbb{R}^n).$$

Namely, the sequence is Cauchy in $L^(\mathbb{R}^n)$ and hence convergnet. To see this, just note that for $\epsilon > 0$ one can first choose R so large that the norm outside the ball is

$$\int_{|\zeta|\geq R}(1+|\zeta|^2)^t|u_n|^2 d\zeta \leq (1+R^2)^{\frac{s-t}{2}}\int_{|\zeta|\geq R}(1+|\zeta|^2)^s|u_n|^2 d\zeta \leq C(1+R^2)^{\frac{s-t}{2}} < \epsilon/2 \tag{2.5}$$

where C is the bound on the norm in H^s. Now, having chosen R, the subsequence converges in $|\zeta| \leq R$. This proves the compactness. $\square$

Once we have this local result we easily deduce the global result.

Proposition 2.2. *On a compact manifold the inclusion $H^s(M) \hookrightarrow H^t(M)$, for any $s > t$, is compact.*

Proof. If $\phi_i \in \mathcal{C}_c^\infty(U_i)$ is a partition of unity subordinate to an open cover of M by coordinate patches $g_i : U_i \longrightarrow U_i' \subset \mathbb{R}^n$, then

$$u \in H^s(M) \Longrightarrow (g_i^{-1})^*\phi_i u \in H^s(\mathbb{R}^n),\ \operatorname{supp}((g_i^{-1})^*\phi_i u) \Subset U_i'. \tag{2.6}$$

Thus if u_n is a bounded sequence in $H^s(M)$ then the $(g_i^{-1})^*\phi_i u_n$ form a bounded sequence in $H^s(\mathbb{R}^n)$ with fixed compact supports. It follows from Lemma 2.1 that we may choose a subsequence so that each $\phi_i u_{n_j}$ converges in $H^t(\mathbb{R}^n)$. Hence the subsequence u_{n_j} converges in $H^t(M)$. $\square$

3. Elliptic operators are Fredholm

If V_1, V_2 are two vector spaces then a linear operator $P : V_1 \to V_2$ is said to be *Fredholm* if these are finite-dimensional subspaces $N_1 \subset V_1$, $N_2 \subset V_2$ such that

$$\begin{gathered}\{v \in V_1;\ Pv = 0\} \subset N_1 \\ \{w \in V_2;\ \exists\ v \in V_1,\ Pv = w\} + N_2 = V_2.\end{gathered} \tag{3.1}$$

The first condition just says that the null space is finite-dimensional and the second that the range has a finite-dimensional complement – by shrinking N_1 and N_2 if necessary we may arrange that the inclusion in (3.1) is an equality and that the sum is direct.

THEOREM 3.1. *For any elliptic operator, $P \in \mathrm{Diff}^m(M;\mathbb{E})$, acting between sections of vector bundles over a compact manifold,*

$$P : H^{s+m}(M;E_1) \longrightarrow H^s(M;E_2)$$
$$\text{and } P : \mathcal{C}^\infty(M;E_1) \longrightarrow \mathcal{C}^\infty(M;E_2)$$

are Fredholm for all $s \in \mathbb{R}$.

The result for the $\mathcal{C}^\infty$ spaces follows from the result for Sobolev spaces. To prove this, consider the notion of a Fredholm operator between Hilbert spaces,

$$P : H_1 \longrightarrow H_2. \tag{3.2}$$

In this case we can unwind the conditions (3.1) which are then equivalent to the three conditions

$$\begin{gathered}\mathrm{Nul}(P) \subset H_1 \text{ is finite-dimensional.}\\ \mathrm{Ran}(P) \subset H_2 \text{ is closed.}\\ \mathrm{Ran}(P))^\perp \subset H_2 \text{ is finite-dimensional.}\end{gathered} \tag{3.3}$$

Note that *any* subspace of a Hilbert space with a finite-dimensional complement is closed so (3.3) does follow from (3.1). On the other hand the ortho-complement of a subspace is the same as the ortho-complement of its closure so the first and the third conditions in (3.3) do *not* suffice to prove (3.1), in general. For instance the range of an operator can be dense but not closed.

The main lemma we need, given the global elliptic estimates, is a standard one:-

LEMMA 3.2. *If $R : H \longrightarrow H$ is a compact operator on a Hilbert space then $\mathrm{Id} - R$ is Fredholm.*

PROOF. A compact operator is one which maps the unit ball (and hence any bounded subset) of H into a precompact set, a set with compact closure. The unit ball in the null space of $\mathrm{Id} - R$ is

$$\{u \in H;\ \|u\| = 1,\ u = Ru\} \subset R\{u \in H;\ \|u\| = 1\}$$

and is therefore precompact. Since is it closed, it is compact and any Hilbert space with a compact unit ball is finite-dimensional. Thus the null space of $\mathrm{Id} - R$ is finite-dimensional.

Consider a sequence $u_n = v_n - Rv_n$ in the range of $\mathrm{Id} - R$ and suppose $u_n \to u$ in H; we need to show that u is in the range of $\mathrm{Id} - R$. We may assume $u \neq 0$, since 0 is in the range, and by passing to a subsequence suppose that $\|u_n\| \neq 0$; $\|u_n\| \to \|u\| \neq 0$ by assumption. Now consider $w_n = v_n/\|v_n\|$. Since $\|u_n\| \neq 0$, $\inf_n \|v_n\| \neq 0$, since other wise there is a subsequence converging to 0, and so w_n is well-defined and of norm 1. Since $w_n = Rw_n + u_n/\|v_n\|$ and $\|v_n\|$ is bounded below, w_n must have a convergence subsequence, by the compactness of R. Passing to such a subsequence, and relabelling, $w_n \to w$, $u_n \to u$, $u_n/\|v_n\| \to cu$, $c \in \mathbb{C}$. If $c = 0$ then $(\operatorname{Id} - R)w = 0$. However, we can assume in the first place that $u_n \perp \operatorname{Nul}(\operatorname{Id} - R)$, so the same is true of w_n. As $\|w\| = 1$ this is a contradiction, so $\|v_n\|$ is bounded above, $c \neq 0$, and hence there is a solution to $(\operatorname{Id} - R)w = u$. Thus the range of $\operatorname{Id} - R$ is closed.

The ortho-complement of the range $\operatorname{Ran}(\operatorname{Id} - R)^\perp$ is the null space at $\operatorname{Id} - R^*$ which is also finite-dimensional since R^* is compact. Thus $\operatorname{Id} - R$ is Fredholm. □

PROPOSITION 3.3. *Any smoothing operator on a compact manifold is compact as an operator between (any) Sobolev spaces.*

PROOF. By definition a smoothing operator is one with a smooth kernel. For vector bundles this can be expressed in terms of local coordinates and a partition of unity with trivialization of the bundles over the supports as follows.

$$
\begin{gathered}
Ru = \sum_{a,b} \varphi_b R \varphi_a u \\
\varphi_b R \varphi_a u = F_b^* \varphi_b' R_{ab} \varphi_a' (F_a^{-1})^* u \qquad (3.4) \\
R_{ab} v(z) = \int_{\Omega_a'} R_{ab}(z, z') v(z'),\ z \in \Omega_b',\ v \in \mathcal{C}_c^\infty(\Omega_a'; E_1)
\end{gathered}
$$

where R_{ab} is a matrix of smooth sections of the localized (hence trivial by refinement) bundle on $\Omega_b' \times \Omega_a$. In fact, by inserting extra cutoffs in (3.4), we may assume that R_{ab} has compact support in $\Omega_b' \times \Omega_a'$. Thus, by the compactness of sums of compact operators, it suffices to show that a single smoothing operator of compact support compact support is compact on the standard Sobolev spaces. Thus if $R \in \mathcal{C}_c^\infty(\mathbb{R}^{2n}$

$$
H^{L'}(\mathbb{R}^n) \ni u \mapsto \int_{\mathbb{R}^n} R(z) \in H^L(\mathbb{R}^n) \qquad (3.5)
$$

is compact for any L, L'. By the continuous inclusion of Sobolev spaces it suffices to take $L' = -L$ with L a large even integer. Then $(\Delta + 1)^{L/2}$ is an isomorphism from $(L^2(\mathbb{R}^n))$ to $H^{-L}(\mathbb{R}^2)$ and from $H^L(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Thus the compactness of (3.5) is equivalent to the compactness of

$$
(\Delta + 1)^{L/2} R (\Delta + 1)^{L/2} \text{ on } L^2(\mathbb{R}^n). \qquad (3.6)
$$

This is still a smoothing operator with compactly supported kernel, then we are reduced to the special case of (3.5) for $L = L' = 0$. Finally then it suffices to use Sturm's theorem, that R is uniformly approximated by polynomials on a large ball. Cutting off on left and right then shows that

$$\rho(z)R_i(z,z')\rho(z') \to Rz,z') \text{ uniformly on } \mathbb{R}^{2n}$$

the R_i is a polynomial (and $\rho(z)\rho(z') = 1$ on $\operatorname{supp}(R)$) with $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. The uniform convergence of the kernels implies the convergence of the operators on $L^2(\mathbb{R}^n)$ in the norm topology, so R is in the norm closure of the finite rank operators on $L^2(\mathbb{R}^n)$, hence is compact. □

Proof of Theorem 3.1. We know that P has a 2-sided parameterix $Q : H^s(M;E_2) \longrightarrow H^{s+m}(M;E_1)$ (for any s) such that

$$PQ - \mathrm{Id}_2 = R_2,\ QP - \mathrm{Id}_2 = R_1,$$

are both smoothing (or at *least* C^N for arbitrarily large N) operators. Then we can apply Proposition 3.3 and Lemma 3.2. First

$$QP = \mathrm{Id} - R_1 : H^{s+m}(M;E_1) \longrightarrow H^{s+m}(M;E_2)$$

have finite-dimensional null spaces. However, the null space of P is certainly contained in the null space of $\mathrm{Id} - R$, so it too is finite-dimensional. Similarly,

$$PQ = \mathrm{Id} - R_1 : H^s(M;E_2) \longrightarrow H^s(M;E_2)$$

has closed range of finite codimension. But the range of P certainly *contains* the range of $\mathrm{Id} - R$ so it too must be closed and of finite codimension. Thus P is Fredholm as an operator from $H^{s+m}(M;E_2)$ to $H^s(M;E_2)$ for any $s \in \mathbb{R}$.

So consider P as an operator on the $\mathcal{C}^\infty$ spaces. The null space of $P : H^m(M;E_1) \longrightarrow H^0(M;E_2)$ consists of $\mathcal{C}^\infty$ sections, by elliptic regularity, so must be equal to the null space on $\mathcal{C}^\infty(M;E_1)$ — which is therefore finite-dimensional. Similarly consider the range of $P : H^m(M;E_1) \longrightarrow H^0(M;E_2)$. We know this to have a finite-dimensional complement, with basis $v_1, \ldots, v_n \in H^0(M;E_2)$. By the density of $\mathcal{C}^\infty(M;E_2)$ in $L^2(M;E_2)$ we can approximate the v_i's closely by $w_i \in \mathcal{C}^\infty(M;E_2)$. On close enough approximation, the w_i must span the complement. Thus $PH^m(M;E_1)$ has a complement in $L^2(M;E_2)$ which is a finite-dimensional subspace of $\mathcal{C}^\infty(M;E_2)$; call this N_2. If $f \in \mathcal{C}^\infty(M;E_2) \subset L^2(M;E_2)$ then there are constants c_i such that

$$f - \sum_{i=1}^N c_i w_i = Pu,\ u \in H^m(M;E_1).$$

Again by elliptic regularity, $u \in \mathcal{C}^\infty(M; E_1)$ thus N_2 is a complement to $P\mathcal{C}^\infty(M; E_1)$ in $\mathcal{C}^\infty(M; E_2)$ and P is Fredholm. $\square$

The point of Fredholm operators is that they are 'almost invertible' — in the sense that they are invertible up to finite-dimensional obstructions. However, a Fredholm operator may not itself be *close* to an invertible operator. This defect is measured by the index

$$\begin{gathered}\operatorname{ind}(P) = \dim \operatorname{Nul}(P) - \dim(\operatorname{Ran}(P)^\perp) \\ P : H^m(M; E_1) \longrightarrow L^2(M; E_2).\end{gathered}$$

4. Generalized inverses

Written, at least in part, by Chris Kottke.

As discussed above, a bounded operator between Hilbert spaces,

$$T : H_1 \longrightarrow H_2$$

is Fredholm if and only if it has a parametrix up to compact errors, that is, there exists an operator

$$S : H_2 \longrightarrow H_1$$

such that

$$TS - \mathrm{Id}_2 = R_2,\ ST - \mathrm{Id}_1 = R_1$$

are both compact on the respective Hilbert spaces H_1 and H_2. In this case of Hilbert spaces there is a "preferred" parametrix or generalized inverse.

Recall that the adjoint

$$T^* : H_2 \longrightarrow H_1$$

of any bounded operator is defined using the Riesz Representation Theorem. Thus, by the continuity of T, for any $u \in H_2$,

$$H_1 \ni \phi \longrightarrow \langle T\phi, u\rangle \in \mathbb{C}$$

is continuous and so there exists a unique $v \in H_1$ such that

$$\langle T\phi, u\rangle_2 = \langle \phi, v\rangle_1,\ \forall\ \phi \in H_1.$$

Thus v is determined by u and the resulting map

$$H_2 \ni u \mapsto v = T^*u \in H_1$$

is easily seen to be continuous giving the adjoint identity

$$\langle T\phi, u\rangle = \langle \phi, T^*u\rangle,\ \forall\ \phi \in H_1,\ u \in H_2 \tag{4.1}$$

In particular it is always the case that

$$\operatorname{Nul}(T^*) = (\operatorname{Ran}(T))^\perp \tag{4.2}$$

as follows directly from (4.1). As a useful consequence, if $\operatorname{Ran}(T)$ is closed, then $H_2 = \operatorname{Ran}(T) \oplus \operatorname{Nul}(T^*)$ is an orthogonal direct sum.

PROPOSITION 4.1. *If $T : H_1 \longrightarrow H_2$ is a Fredholm operator between Hilbert spaces then T^* is also Fredholm,* $\operatorname{ind}(T^*) = -\operatorname{ind}(T)$, *and T has a unique generalized inverse $S : H_2 \longrightarrow H_1$ satisfying*

$$TS = \operatorname{Id}_2 - \Pi_{\operatorname{Nul}(P^*)}, \quad ST = \operatorname{Id}_1 - \Pi_{\operatorname{Nul}(P)} \tag{4.3}$$

PROOF. A straightforward exercise, but it should probably be written out! □

Notice that $\operatorname{ind}(T)$ is the difference of the two non-negative integers $\dim \operatorname{Nul}(T)$ and $\dim \operatorname{Nul}(T^*)$. Thus

$$\dim \operatorname{Nul}(T) \;\geq\; \operatorname{ind}(T) \tag{4.4}$$

$$\dim \operatorname{Nul}(T^*) \;\geq\; -\operatorname{ind}(T) \tag{4.5}$$

so if $\operatorname{ind}(T) \neq 0$ then T is definitely *not* invertible. In fact it cannot then be made invertible by small bounded perturbations.

PROPOSITION 4.2. *If H_1 and H_2 are two seperable, infinite-dimensional Hilbert spaces then for all $k \in \mathbb{Z}$,*

$$\operatorname{Fr}_k = \{T : H_1 \longrightarrow H_2;\ T \text{ is Fredholm and } \operatorname{ind}(T) = k\}$$

is a non-empty subset of $B(H_1, H_2)$, the Banach space of bounded operators from H_1 to H_2.

PROOF. All separable Hilbert spaces of infinite dimension are isomorphic, so Fr_0 is non-empty. More generally if $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of H_1, then the shift operator, determined by

$$S_k e_i = \begin{cases} e_{i+k}, & i \geq 1,\ k \geq 0 \\ e_{i+k}, & i \geq -k,\ k \leq 0 \\ 0, & i < -k \end{cases}$$

is easily seen to be Fredholm of index k in H_1. Composing with an isomorphism to H_2 shows that $\operatorname{Fr}_k \neq \emptyset$ for all $k \in \mathbb{Z}$. □

One important property of the spaces $\operatorname{Fr}_k(H_1, H_2)$ is that they are stable under compact perturbations; that is, if $K : H_1 \longrightarrow H_2$ is a compact operator and $T \in \operatorname{Fr}_k$ then $(T + K) \in \operatorname{Fr}_k$. That $(T + K)$ is Fredholm is clear, sinces a parametrix for T is a parametrix for $T + K$, but it remains to show that the index itself is stable and we do this in steps. In what follows, take $T \in \operatorname{Fr}_k(H_1, H_2)$ with kernel $N_1 \subset H_1$. Define $\tilde{T}$ by the factorization

$$T : H_1 \longrightarrow \tilde{H}_1 = H_1/N_1 \xrightarrow{\tilde{T}} \operatorname{Ran} T \hookrightarrow H_2, \tag{4.6}$$

so that $\tilde{T}$ is invertible.

LEMMA 4.3. *Suppose* $T \in \mathrm{Fr}_k(H_1, H_2)$ *has kernel* $N_1 \subset H_1$ *and* $M_1 \supset N_1$ *is a finite dimensional subspace of* H_1 *then defining* $T' = T$ *on* $M_1^\perp$ *and* $T' = 0$ *on* M_1 *gives an element* $T' \in \mathrm{Fr}_k$.

PROOF. Since $N_1 \subset M_1$, T' is obtained from (4.6) by replacing $\tilde{T}$ by $\tilde{T}'$ which is defined in essentially the same way as T', that is $\tilde{T}' = 0$ on M_1/N_1, and $\tilde{T}' = \tilde{T}$ on the orthocomplement. Thus the range of $\tilde{T}'$ in $\mathrm{Ran}(T)$ has complement $\tilde{T}(M_1/N_1)$ which has the same dimension as M_1/N_1. Thus T' has null space M_1 and has range in H_2 with complement of dimension that of $M_1/N_1 + N_2$, and hence has index k. $\square$

LEMMA 4.4. *If* A *is a finite rank operator* $A : H_1 \longrightarrow H_2$ *such that* $\mathrm{Ran}\, A \cap \mathrm{Ran}\, T = \{0\}$*, then* $T + A \in \mathrm{Fr}_k$.

PROOF. First note that $\mathrm{Nul}(T + A) = \mathrm{Nul}\, T \cap \mathrm{Nul}\, A$ since

$$x \in \mathrm{Nul}(T+A) \Leftrightarrow Tx = -Ax \in \mathrm{Ran}\, T \cap \mathrm{Ran}\, A = \{0\} \Leftrightarrow x \in \mathrm{Nul}\, T \cap \mathrm{Nul}\, A.$$

Similarly the range of $T+A$ restricted to $\mathrm{Nul}\, T$ meets the range of $T+A$ restricted to $(\mathrm{null}\, T)^\perp$ only in 0 so the codimension of the $\mathrm{Ran}(T + A)$ is the codimension of $\mathrm{Ran}\, A_N$ where A_N is A as a map from $\mathrm{Nul}\, T$ to $H_2/\,\mathrm{Ran}\, T$. So, the equality of row and column rank for matrices,

$$\mathrm{codim}\, \mathrm{Ran}(T+A) = \mathrm{codim}\, \mathrm{Ran}\, T - \dim \mathrm{Nul}(A_N) = \dim \mathrm{Nul}(T) - k - \dim \mathrm{Nul}(A_N) = \dim \mathrm{Nul}(T+$$

Thus $T + A \in \mathrm{Fr}_k$. $\square$

PROPOSITION 4.5. *If* $A : H_1 \longrightarrow H_2$ *is any finite rank operator, then* $T + A \in \mathrm{Fr}_k$.

PROOF. Let $E_2 = \mathrm{Ran}\, A \cap \mathrm{Ran}\, T$, which is finite dimensional, then $E_1 = \tilde{T}^{-1}(E_2)$ has the same dimension. Put $M_1 = E_1 \oplus N_1$ and apply Lemma 4.3 to get $T' \in \mathrm{Fr}_k$ with kernel M_1. Then

$$T + A = T' + A' + A$$

where $A' = T$ on E_1 and $A' = 0$ on $E_1^\perp$. Then $A' + A$ is a finite rank operator and $\mathrm{Ran}(A' + A) \cap \mathrm{Ran}\, T' = \{0\}$ and Lemma 4.4 applies. Thus

$$T + A = T' + (A' + A) \in \mathrm{Fr}_k(H_1, H_2).$$

$\square$

PROPOSITION 4.6. *If* $B : H_1 \longrightarrow H_2$ *is compact then* $T + B \in \mathrm{Fr}_k$.

PROOF. A compact operator is the sum of a finite rank operator and an operator of arbitrarily small norm so it suffices to show that $T + C \in \mathrm{Fr}_k$ where $\|C\| < \epsilon$ for ϵ small enough and then apply Proposition 4.5. Let $P : H_1 \longrightarrow \tilde{H}_1 = H_1/N_1$ and $Q : H_2 \longrightarrow \operatorname{Ran} T$ be projection operators. Then

$$C = QCP + QC(\mathrm{Id} - P) + (\mathrm{Id} - Q)CP + (\mathrm{Id} - Q)C(\mathrm{Id} - P)$$

the last three of which are finite rank operators. Thus it suffices to show that

$$\tilde{T} + QC : \tilde{H}_1 \longrightarrow \operatorname{Ran} T$$

is invertible. The set of invertible operators is open, by the convergence of the Neumann series so the result follows. □

REMARK 1. In fact the Fr_k are all *connected* although I will not use this below. In fact this follows from the multiplicativity of the index:-

$$\mathrm{Fr}_k \circ \mathrm{Fr}_l = \mathrm{Fr}_{k+l} \tag{4.7}$$

and the connectedness of the group of invertible operators on a Hilbert space. The topological type of the Fr_k is actually a point of some importance. A fact, which you should know but I am not going to prove here is:-

THEOREM 4.7. *The open set* $\mathrm{Fr} = \bigcup_k \mathrm{Fr}_k$ *in the Banach space of bounded operators on a separable Hilbert space is a* classifying space *for even K-theory.*

That is, if X is a reasonable space – for instance a compact manifold – then the space of homotopy classes of continuous maps into Fr may be canonically identified as an Abelian group with the (complex) K-theory of X :

$$\mathrm{K}^0(X) = [X; \mathrm{Fr}]. \tag{4.8}$$

5. Self-adjoint elliptic operators

Last time I showed that elliptic differential operators, acting on functions on a compact manifold, are Fredholm on Sobolev spaces. Today I will first quickly discuss the rudiments of spectral theory for self-adjoint elliptic operators and then pass over to the general case of operators between sections of vector bundles (which is really only notationally different from the case of operators on functions).

To define self-adjointness of an operator we need to define the adjoint! To do so requires invariant integration. I have already talked about this a little, but recall from 18.155 (I hope) Riesz' theorem identifying (appropriately behaved, i.e. Borel outer continuous and inner regular) measures on a locally compact space with continuous linear functionals on $\mathcal{C}^0_0(M)$ (the space of continuous functions 'vanishing at infinity'). In the case of a manifold we define a smooth positive measure, also called a positive density, as one given in local coordinates by a smooth positive multiple of the Lebesgue measure. The existence of such a density is guaranteed by the existence of a partition of unity subordinate to a coordinate cover, since the we can take

$$\nu = \sum_j \phi_j f_j^* |dz| \tag{5.1}$$

where $|dz|$ is Lebesgue measure in the local coordinate patch corresponding to $f_j : U_j \longrightarrow U'_j$. Since we know that a smooth coordinate transforms $|dz|$ to a positive smooth multiple of the new Lebesque measure (namely the absolute value of the Jacobian) and two such positive smooth measures are related by

$$\nu' = \mu\nu,\ 0 < \mu \in \mathcal{C}^\infty(M). \tag{5.2}$$

In the case of a compact manifold this allows one to define integration of functions and hence an inner product on $L^2(M)$,

$$\langle u, v\rangle_\nu = \int_M u(z)\overline{v(z)}\nu. \tag{5.3}$$

It is with respect to such a choice of smooth density that adjoints are defined.

LEMMA 5.1. *If $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is a differential operator with smooth coefficients and ν is a smooth positive measure then there exists a unque differential operator with smooth coefficients $P^* : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ such that*

$$\langle Pu, v\rangle_\nu = \langle u, P^*v\rangle_\nu\ \forall\ u,\ v \in \mathcal{C}^\infty(M). \tag{5.4}$$

PROOF. First existence. If ϕ_i is a partition of unity subordinate to an open cover of M by coordinate patches and $\phi'_i \in \mathcal{C}^\infty(M)$ have supports in the same coordinate patches, with $\phi'_i = 1$ in a neighbourhood of $\operatorname{supp}(\phi_i)$ then we know that

$$Pu = \sum_i \phi'_i P \phi_i u = \sum_i f_i^* P_i (f_i^{-1})^* u \tag{5.5}$$

where $f_i : U)i \longrightarrow U'_i$ are the coordinate charts and P_i is a differential operator on U'_i with smooth coefficients, all compactly supported in U'_i. The existence of P^* follows from the existence of $(\phi'_i P \phi_i)^*$ and hence

P_i^* in each coordinate patch, where the P_i^* should satisfy

$$\int_{U_i'} (P_i)u'\overline{v'}\mu' dz = \int_{U_i'} u'\overline{P_i^* v'}\mu' dz, \ \forall\ u', v' \in \mathcal{C}^\infty(U_i'). \tag{5.6}$$

Here $\nu = \mu'|dz|$ with $0 < \mu' \in \mathcal{C}^\infty(U_i')$ in the local coordinates. So in fact P_i^* is unique and given by

$$P_i^*(z, D)v' = \sum_{|\alpha|\le m} (\mu')^{-1} D^\alpha \overline{p_\alpha(z)}\mu' v' \text{ if } P_i = \sum_{|\alpha|\le m} p_\alpha(z) D^\alpha. \tag{5.7}$$

The uniqueness of P^* follows from (5.4) since the difference of two would be an operator $Q : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ satisfying

$$\langle u, Qv\rangle_\nu = 0 \ \forall\ u,\ v \in \mathcal{C}^\infty(M) \tag{5.8}$$

and this implies that $Q = 0$ as an operator. □

Proposition 5.2. *If $P : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ is an elliptic differential operator of order $m > 0$ which is (formally) self-adjoint with respect to some smooth positive density then*

(5.9)

$$\operatorname{spec}(P) = \{\lambda \in \mathbb{C}; (P-\lambda) : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M) \text{ is not an isomorphism}\}$$

is a discrete subset of $\mathbb{R}$, for each $\lambda \in \operatorname{spec}(P)$

$$E(\lambda) = \{u \in \mathcal{C}^\infty(M); Pu = \lambda u\} \tag{5.10}$$

is finite dimensional and

$$L^2(M) = \sum_{\lambda\in\operatorname{spec}(P)} E(\lambda) \text{ is orthogonal.} \tag{5.11}$$

Formal self-adjointness just means that $P^* = P$ as differential operators acting on $\mathcal{C}^\infty(M)$. Actual self-adjointness means a little more but this follows easily from formal self-adjointness and ellipticity.

Proof. First notice that $\operatorname{spec}(P) \subset \mathbb{R}$ since if $Pu = \lambda u$ with $u \in \mathcal{C}^\infty(M)$ then

$$\lambda\|u\|\nu^2 = \langle Pu, u\rangle = \langle u, Pu\rangle = \bar\lambda\|u\|\nu^2 \tag{5.12}$$

so $\lambda \notin \mathbb{R}$ implies that the null space of $P - \lambda$ is trivial. Since we know that the range is closed and has complement the null space of $(P - \lambda)^* = P - \bar\lambda$ it follows that $P - \lambda$ is an isomorphism on $\mathcal{C}^\infty(M)$ if $\lambda \notin \mathbb{R}$.

If $\lambda \in \mathbb{R}$ then we also know that $E(\lambda)$ is finite dimensional. For any $\lambda \in \mathbb{R}$ suppose that $(P - \lambda)u = 0$ with $u \in \mathcal{C}^\infty(M)$. Then we know that $P - \lambda$ is an isomorphism from $E(\lambda)^\perp$ to itself which extends by continuity to an isomorphism from the closure of $E^\perp(\lambda)$ in $H^m(M)$ to $E^\perp(\lambda) \subset L^2(M)$. It follows that $P - \lambda'$ defines such an isomorphism for

$|\lambda = l'| < \epsilon$ for some $\epsilon > 0$. However acting on $E(\lambda)$, $P - \lambda' = (\lambda - \lambda')$ is also an isomorphism for $\lambda' \neq \lambda$ so $P - \lambda'$ is an isomorphism. This shows that $E(\lambda') = \{0\}$ for $|\lambda' - \lambda| < \epsilon$.

This leaves the completeness statement, (5.11). In fact this really amounts to the existence of a non-zero eigenvalue as we shall see. Consider the generalized inverse of P acting on $L^2(M)$. It maps the orthocomplement of the null space to itself and is a compact operator, as follows from the a priori estimats for P and the compactness of the embedding of $H^m(M)$ in $L^2(M)$ for $m > 0$. Futhermore it is self-adjoint. A standard result shows that a compact self-adjoint operator either has a non-zero eigenvalue or is itself zero. For the completeness it is enough to show that the generalized inverse maps the orthocomplement of the span of the $E(\lambda)$ in $L^2(M)$ into itself and is compact. It is therefore either zero or has a non-zero eigenvalue. Any corresponding eigenfunction would be an eigenfunction of P and hence in one of the $E(\lambda)$ so this operator must be zero, meaning that (5.11) holds. $\square$

For single differential operators we first considered constant coefficient operators, then extended this to variable coefficient operators by a combination of perturbation (to get the a priori estimates) and construction of parametrices (to get approximation) and finally used coordinate invariance to transfer the discussion to a (compact) manifold. If we consider matrices of operators we can follow the same path, so I shall only comment on the changes needed.

A $k \times l$ matrix of differential operators (so with k rows and l columns) maps l-vectors of smooth functions to k vectors:

$$P_{ij}(D) = \sum_{|\alpha| \leq m} c_{\alpha,i,j} D^\alpha, \ (P(D)u)_i(z) = \sum_j P_{ij}(D) u_j(z). \tag{5.13}$$

The matrix $P_{ij}(\zeta)$ is invertible if and only if $k = l$ and the polynomial of order mk, $\det P(\zeta) \neq 0$. Such a matrix is said to be elliptic if $\det P(\zeta)$ is elliptic. The cofactor matrix defines a matrix P' of differential operators of order $(k-1)m$ and we may construct a parametrix for P (assuming it to be elliptic) from a parametrix for $\det P$:

$$Q_P = Q_{\det P} P'(D). \tag{5.14}$$

It is then easy to see that it has the same mapping properties as in the case of a single operator (although notice that the product is no longer commutative because of the non-commutativity of matrix multiplication)

$$Q_P P = \operatorname{Id} - R_L, \ P Q_P = \operatorname{Id} - R_R \tag{5.15}$$

where R_L and R_R are given by matrices of convolution operators with all elements being Schwartz functions. For the action on vector-valued functions on an open subset of $\mathbb{R}^n$ we may proceed exactly as before, cutting off the kernel of Q_P with a properly supported function which is 1 near the diagonal

$$Q_\Omega f(z) = \int_\Omega q(z-z')\chi(z,z')f(z')dz'. \tag{5.16}$$

The regularity estimates look exactly the same as before if we define the local Sobolev spaces to be simply the direct sum of k copies of the usual local Sobolev spaces

(5.17)
$$Pu = f \in H^s_{\mathrm{loc}}(\Omega) \Longrightarrow \|\psi u\|_{s+m} \le C\|\psi P(D)u\|_s + C'\|\phi u\|_{m-1} \text{ or } \|\psi u\|_{s+m} \le C\|\phi P(D)u\|_s + C''\|$$

where ψ, $\phi \in \mathcal{C}^\infty_c(\Omega)$ and $\phi = 1$ in a neighbourhood of ψ (and in the second case C'' depends on M.

Now, the variable case proceed again as before, where now we are considering a $k \times k$ matrix of differential operators of order m. I will not go into the details. A priori estimates in the first form in (5.17), for functions ψ with small support near a point, follow by perturbation from the constant coefficient case and then in the second form by use of a partition of unity. The existence of a parametrix for the variable coefficient matrix of operators also goes through without problems – the commutativity which disappears in the matrix case was not used anyway.

As regards coordinate transformations, we get the same results as before. It is also notural to allow transformations by variable coefficient matrices. Thus if $G_i(z) \in \mathcal{C}^\infty(\Omega; \mathrm{GL}(k,\mathbb{C})$ $i = 1, 2$, are smooth family of invertible matrices we may consider the composites PG_2 or $G_1^{-1}P$, or more usually the 'conjugate' operator

$$G_1^{-1}P(z,D)G)2 = P'(z,D). \tag{5.18}$$

This is also a variable coefficient differential operator, elliptic if and only if $P(z,D)$ is elliptic. The Sobolev spaces $H^s_{\mathrm{loc}}(\Omega;\mathbb{R}^k)$ are invariant under composition with such matrices, since they are the same in each variable.

Combining coordinate transformations and such matrix conjugation allows us to consider not only manifolds but also vector bundles over manifolds. Let me briefly remind you of what this is about. Over an open subset $\Omega \subset \mathbb{R}^n$ one can introduce a vector bundle as just a subbundle of some trivial N-dimensional bundle. That is, consider a smooth $N \times N$ matrix $\Pi \in \mathcal{C}^\infty(\Omega; M(N,\mathbb{C}))$ on Ω which is valued in the projections (i.e. idempotents) meaning that $\Pi(z)\Pi(z) = \Pi(z)$ for all $z \in \Omega$. Then the range of $\Pi(z)$ defines a linear subspace of $\mathbb{C}^N$ for each $z \in \Omega$ and together these form a vector bundle over Ω. Namely these spaces fit together to define a manifold of dimension $n+k$ where k is the rank of $\Pi(z)$ (constant if Ω is connected, otherwise require it be the same on all components)

$$E_\Omega = \bigcup_{z\in\Omega} E_z,\ E_z = \Pi(z)\mathbb{C}^N. \tag{5.19}$$

If $\bar z \in \Omega$ then we may choose a basis of $E_{\bar z}$ and so identify it with $\mathbb{C}^k$. By the smoothness of $\Pi(z)$ in z it follows that in some small ball $B(\bar z, r)$, so that $\|\Pi(z)(\Pi(z) - \Pi(\bar z))\Pi(z)\| < \frac{1}{2}$) the map

(5.20)

$$E_{B(\bar z,r)} = \bigcup_{z\in B(\bar z,r)} E_z,\ E_z = \Pi(z)\mathbb{C}^N \ni (z,u) \longmapsto (z, E(\bar z)u) \in B(\bar z,r)\times E_{\bar z} \simeq B(\bar z,r)\times\mathbb{C}^k$$

is an isomorphism. Injectivity is just injectivity of each of the maps $E_z \longrightarrow E_{\bar z}$ and this follows from the fact that $\Pi(z)\Pi(\bar z)\Pi(z)$ is invertible on E_z; this also implies surjectivity.

6. Index theorem

Addenda to Chapter 6