PSEUDODIFFERENTIAL OPERATORS

Abstract. A brief treatment of classical pseudodifferential operators on $\mathbb{R}^n$ intended to be reasonably complete but approachable and enough to get elliptic regularity.

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Introduction

These pages should ultimately be incorporated into a new version of the ‘Graduate Analysis’ notes for 18.155 at MIT. The only ‘innovation’ here, relative to traditional treatments, is the use of expansions in Hermite functions to define the operators and prove $L^2$ (i.e. Sobolev) boundedness.

The material here can be found in many places although it may be difficult initially to see the relationships between the different treatments. First there are the original papers, on pseudodifferential operators.

- Calderón and Zygmund (singular integral operators) [1, 2].
- Seeley in [9] and [10].
- Kohn and Nirenberg [8].

Then there are various newer sources:

- Lectures by Mark Joshi [7]
- Michael Taylor’s book [12]
- My microlocal analysis notes

If $P(x, D)$ is a differential operator with smooth coefficients, say bounded with all derivatives, on $\mathbb{R}^n$,

\begin{equation}
(0.1) \quad P(x, D)u(x) = \sum_{|\alpha| \leq m} P_\alpha(x)D^\alpha u(x), \ u \in \mathcal{S}(\mathbb{R}^n), \ P(x, \xi) = \sum_{|\alpha| \leq m} P_\alpha(x)\xi^\alpha
\end{equation}

then it can be written in terms of the Fourier transform of $u$ as

\begin{equation}
(0.2) \quad P(x, D)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} P(x, \xi) \hat{u}(\xi) d\xi.
\end{equation}

This a rather ‘fake’ formula in the sense that it only works because we can decompose the integral into a finite sum of terms, using the fact that $P(x, \xi)$ is a polynomial in $\xi$, so that it becomes the inverse Fourier transform

\begin{equation}
(0.3) \quad P(x, D)u(x) = \sum_{|\alpha| \leq m} P_\alpha(x)(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \xi^\alpha \hat{u}(\xi) d\xi
\end{equation}

and we recover (0.1). Nevertheless the idea is that we can interpret, and manipulate, ‘improper’ (usually called oscillatory) integrals of the type (0.2) directly leading to
the theory of pseudodifferential operators. In particular such an integral makes sense for the space of symbols of order $m$ which is usually taken to mean the space of smooth functions on $\mathbb{R}^n \times \mathbb{R}^n$ which satisfy estimates (which we have encountered before)

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \leq C_{\alpha, \beta} (1 + |\xi|)^{m-|\beta|}. \tag{0.4}$$

The most important class of operators correspond to classical symbols which are discussed below, but the term means that they have asymptotic expansions in homogeneous functions in $\xi$ as $|\xi| \to \infty$.

Still the theory passes through the more general symbols (0.4) (or even more general ones) and one reason for this is that the oscillatory integrals are defined using density arguments from rapidly decreasing functions and, just as for bounded continuous functions on $\mathbb{R}$, rapidly decreasing functions are not dense in classical symbols in the natural topology but are dense (with loss of $\epsilon$ in the order) in the symbol topology given by the constants in (0.4).

In this brief introduction to pseudodifferential operators I want to avoid this discussion and get to the local theory as quickly as possible. The approach is based on the observation that the formula (0.3) ‘works’ because the symbol $P(x, \xi)$ is in the case of a differential operator a finite sum of products of functions of $x$ (the coefficients) and functions of $\xi$ (giving monomial differential operators); the general case is handled below by expansion in terms of such products.

To do this I first limit attention to symbols with rapidly decreasing coefficients – i.e. which are Schwartz in the $x$ variable; one disadvantage is that the identity is thereby excluded from the ‘algebra’ but this is not an issue for local regularity questions. Then the simple idea is that one can expand these functions in terms of the Hermite functions $e_\gamma \in \mathcal{S}(\mathbb{R}^n)$, the eigenfunctions of the harmonic oscillator,

$$a(x, \xi) = \sum_{\gamma} e_\gamma(x) a_\gamma(\xi), \quad a_\gamma(\xi) = \int_{\mathbb{R}^n} a(x, \xi) e_\gamma(x) dx. \tag{0.5}$$

This series converges rapidly and the coefficients $a_\gamma(\xi)$ are classical symbols. Then the operator can be defined directly as in (0.3)

$$a(x, D) u(x) = \sum_{\gamma} (2\pi)^{-n} e_\gamma(x) \int_{\mathbb{R}^n} e^{ix \cdot \xi} a_\gamma(\xi) \hat{u}(\xi) d\xi = \sum_{\gamma} e_\gamma(x) a_\gamma(D) u, \quad a_\gamma(D) u = A_\gamma * u \tag{0.6}$$

where $A_\gamma = a_\gamma$ is the sort of convolution operator we dealt with earlier. The series (0.6) converges rapidly as a series of bounded operators on Sobolev spaces so these operators are easily defined and bounded. The crucial theorem is that the product of two such operators is of the same form, that the operators form an algebra.

Really for no particular reason I elected to initially define the algebra of pseudodifferential operators with the coefficients on the right. This may be confusing but is a decision easily reversed and of no particular consequence. Indeed one of the basic result is that one gets the same class of operators either way – this is discussed below.

A word about the Schwartz kernel theorem. This is not used here but the relationship between operators and kernels is used. Consider the space of continuous
linear maps (with the weak topology on $\mathcal{S}'(\mathbb{R}^p)$)

(0.7) \[ \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^p) = \mathcal{B}(\mathcal{S}(\mathbb{R}^n); \mathcal{S}'(\mathbb{R}^p)) \]

(0.8) \[ \mathcal{B}(\mathcal{S}(\mathbb{R}^n); \mathcal{S}'(\mathbb{R}^p)) \leftrightarrow \mathcal{S}'(\mathbb{R}^p + n) \]

Schwartz’ kernel Theorem

(0.9) \[ (A\phi)(\psi) = \int \psi(x)A(x,y)\phi(y), \ \phi \in \mathcal{S}(\mathbb{R}^n), \ \psi \in \mathcal{S}(\mathbb{R}^p) \]

where the actual relationship between operator $A$ and kernel $A$ (why waste a letter when we should think of them as the same thing) is given by the pairing

(0.10) \[ (A\phi)(\psi) = (A(x,y), \psi(x)\phi(y)). \]

Here I have added the variables formally because it is usual to ‘reverse’ them in this way so that this looks like the integral operator (0.9). The harder part of Schwartz kernel theorem is to show that each continuous operator corresponds to a kernel. The other direction is easier and what we really use is the fact that the kernel, once it is shown to exist, is unique. This follows from the formula (0.10) which determines the operator from the kernel and conversely shows that the pairing of the kernel with finite sums of products of Schwartz functions in the two variables is fixed by the operator. These finite sums are dense in $\mathcal{S}(\mathbb{R}^{p+n})$ (‘completed tensor product’) so the kernel is determined by the operator. The remaining issue in the proof of the kernel theorem is the continuity of the pairing of the putative kernel and the elements of the finite tensor product.

For differential operators with Schwartz coefficients the kernel is of the form

(0.11) \[ \sum_{|\alpha| \leq m} P_\alpha(x)(D^\alpha \delta)(x-y) \]

so is supported on the diagonal. Our pseudodifferential operators are really characterized by their kernels which are singular (and only in a special ‘conormal’ way) at the diagonal away from which they are given by a Schwartz function.

Exercise 1. Write out the relationship between the coefficients in the left and right forms of a differential operator

(0.12) \[ P(x,D)u(x) = \sum_{|\alpha| \leq m} P_\alpha(x)D^\alpha_x u(x) = \sum_{|\alpha| \leq m} D_x^\alpha(Q_\alpha(x)u(x)). \]

1 Harmonic Oscillator

The eigenbasis for the harmonic oscillator, which I will call the Hermite basis, gives an expansion for any element of $\phi \in \mathcal{S}(\mathbb{R}^n)$ as a series

(1.1) \[ \phi = \sum_{\gamma \in \mathbb{N}_0^n} (\phi, e_\gamma) e_\gamma \]

which is the Fourier-Bessel expansion for the orthonormal basis of eigenfunctions in $L^2(\mathbb{R}^n)$. It has the desirable property of converging in $\mathcal{S}(\mathbb{R}^n)$ precisely when $\phi \in \mathcal{S}(\mathbb{R}^n)$. 

RBM: Add sketch of alternate as suggested by Vishesh
This is used below in showing that certain spaces are ‘completed tensor products’. Consider $\mathcal{C}^\infty(\mathbb{R}^n)$, consisting of those $f \in \mathcal{C}^\infty([0,1] \times \mathbb{R}^n)$ which satisfy
\begin{equation}
\sup |y^\beta D_y^\alpha D_t^k f(t,y)| < \infty, \quad \forall \ k, \alpha, \beta.
\end{equation}
If we expand in the Hermite basis then we get
\begin{equation}
f(t,y) = \sum_\gamma f_\gamma(t)e_\gamma(y), \quad f_\gamma \in \mathcal{C}^\infty([0,1]), \quad e_\gamma \in \mathcal{S}(\mathbb{R}^n)
\end{equation}
converging in $\mathcal{C}^\infty([0,1]; \mathcal{S}(\mathbb{R}^n))$.

Such an expansion is used below in the treatment of pseudodifferential operators.

We proceed to derive the appropriate properties of the Hermite basis. You can safely skip this if you believe that for the Hermite basis, in each dimension, that (1.1) holds and if $\phi \in \mathcal{S}(\mathbb{R}^n)$ then
\begin{equation}
|\langle \phi, e_\gamma \rangle| \leq C_N (1 + |\gamma|)^{-N},
\end{equation}
where $C_N$ depends on some norm on $\phi$, $M$ depends only on $n$ and $C'_k$ depends on $n$ and $k$.

The one-dimensional harmonic oscillator can be written
\begin{equation}
H = \left( \frac{d}{dx} + x \right) \left( - \frac{d}{dx} + x \right) - 1 = -\frac{d^2}{dx^2} + x^2.
\end{equation}
So we proceed to invert $H + 1$ by inverting the two ordinary differential operators
\begin{equation}
A = \frac{d}{dx} + x, \quad C = A^* = -\frac{d}{dx} + x.
\end{equation}

Both are bounded operators
\begin{equation}
A, C : H^1_{loc}(\mathbb{R}) = H^1(\mathbb{R}) \cap (x)^{-1}L^2(\mathbb{R}) \longrightarrow L^2, \quad \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.
\end{equation}

The creation operator, $C$, is injective, since its null space on all distributions is spanned by $e^{-x^2/2}$. If $g \in \mathcal{C}^\infty(\mathbb{R})$ satisfies
\begin{equation}
\int e^{-x^2/2}g(x)dx = 0 \text{ then defining }
E_C g = -e^{x^2/2} \int_{-\infty}^{x} e^{-s^2/2}g(s)ds = e^{x^2/2} \int_{-\infty}^{x} e^{-s^2/2}g(s)ds \in \mathcal{C}^\infty(\mathbb{R})
given an operator satisfying $CE_C g = g$.
\end{equation}

Since $e^{-x^2/2}$ is monotonic decreasing near infinity it follows that if $g \in \mathcal{S}(\mathbb{R})$ satisfies the same constraint then the integrals are bounded by $C_N|x|^{-N}\exp(-x^2/2)$ near infinity. The equation then gives corresponding decay of the derivative and differentiating shows iteratively that
\begin{equation}E_C : \{ g \in \mathcal{S}(\mathbb{R}) : \int e^{-x^2/2}g(x)dx = 0 \} \longrightarrow \mathcal{S}(\mathbb{R}), \quad CE_C = 1d.
\end{equation}

If $v \in \mathcal{S}(\mathbb{R})$ then
\begin{equation}
\int |Cv(x)|^2 dx = \int_{\mathbb{R}} \left( \frac{dv}{dx}^2 + x^2|v|^2 \right) dx - \int_{\mathbb{R}} x \frac{d|v|^2}{dx} dx
\end{equation}

\begin{equation}
= \int_{\mathbb{R}} \left( \frac{dv}{dx}^2 + x^2|v|^2 + |v|^2 \right) dx.
\end{equation}
is a Hilbert norm on $H^1_{iso}(\mathbb{R})$. Thus extending by continuity gives a bounded linear operator
\begin{equation}
E_C : \{g \in L^2(\mathbb{R}) : \int e^{-x^2/2}g(x)dx = 0 \} \rightarrow H^1_{iso}(\mathbb{R})
\end{equation}
and so
\begin{equation}
C : H^1_{iso}(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \text{ is injective with range } (\exp(-x^2/2))^\perp.
\end{equation}
In particular, $C$ is Fredholm.

The dual of $H^1_{iso}(\mathbb{R})$ with respect to the $L^2$ pairing is
\begin{equation}
H^1_{iso}(\mathbb{R}) = H^{-1}(\mathbb{R}) + \langle x \rangle L^2(\mathbb{R})
\end{equation}
although this is not used below. Passing to the adjoint it follows that
\begin{equation}
A : L^2(\mathbb{R}) \rightarrow H^{-1}_{iso}(\mathbb{R}) \text{ is surjective with } \text{Nul}(A) = \text{sp}(e^{-x^2/2}).
\end{equation}
A right inverse from $C^\infty(\mathbb{R})$ to $S(\mathbb{R})$ is given by
\begin{equation}
\tilde{E}_A f(x) = u(x) = e^{-x^2/2} \int_{-\infty}^{x} e^{s^2/2} f(s)ds \in C^\infty(\mathbb{R}), \quad Au = f.
\end{equation}
In $x > a$, supp$(f) \subset [-a, a]$, $u = c e^{-x^2/2}$, $c = \int_{\mathbb{R}} e^{x^2/2} f(s) ds$, so indeed
\begin{equation}
\tilde{E}_A : C^\infty(\mathbb{R}) \rightarrow S(\mathbb{R}) \subset H^1_{iso}(\mathbb{R}).
\end{equation}

For $u \in S(\mathbb{R})$ integration by parts gives
\begin{equation}
\int_{\mathbb{R}} |Au|^2 dx = \int_{\mathbb{R}} (|du|^2 + x^2 |u|^2)dx + \int_{\mathbb{R}} x \frac{d|u|^2}{dx} dx
\end{equation}
\begin{equation}
\quad \quad \quad \quad \quad \quad \quad \quad = \int_{\mathbb{R}} (|du|^2 + 2x^2 |u|^2 - |u|^2) dx
\end{equation}
\begin{equation}
\Rightarrow \|\tilde{E}_A f\|_{H^1_{iso}}^2 \leq \|f\|_{L^2}^2 + 2 \|\tilde{E}_A f\|_{L^2}^2.
\end{equation}

Combining (1.12) and (1.14) it follows that

Lemma 1. The shifted harmonic oscillator
\begin{equation}
H + 1 = AC : H^1_{iso}(\mathbb{R}) \rightarrow H^{-1}_{iso}(\mathbb{R}) \text{ is an isomorphism}
\end{equation}
with inverse which restricts to a compact self-adjoint operator
\begin{equation}
(H + 1)^{-1} : L^2(\mathbb{R}) \rightarrow H^1_{iso}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R}).
\end{equation}

Proof. The open mapping theorem shows that $(H + 1)^{-1}$ is a bounded linear map from $H^1_{iso}(\mathbb{R})$ to $H^1_{iso}(\mathbb{R})$. That $(H + 1)^{-1}$ restricts to a compact operator on $L^2(\mathbb{R})$ follows from the compactness of the inclusion of $H^1_{iso}(\mathbb{R})$ in $L^2(\mathbb{R})$.

If we modify the generalized inverse $\tilde{E}_A$ to have range in the orthocomplement of the null space of $A$
\begin{equation}
E_A f = E_A f - \frac{1}{\pi} \langle f, \exp(-x^2/2) \exp(-x^2/2), E_A : S(\mathbb{R}) \rightarrow S(\mathbb{R})
\end{equation}
then $(H + 1)^{-1} = E_C E_A$ on $S(\mathbb{R})$. This shows that the $H + 1$ is a bijection on $S(\mathbb{R})$ and the evident symmetry of $H + 1$ on $S(\mathbb{R})$:
\begin{equation}
((H + 1)\phi, \psi) = (\phi, (H + 1)\psi) \Rightarrow (f, (H + 1)^{-1} g) = ((H + 1)^{-1} f, g), \quad f = (H + 1)\phi, \quad g = (H + 1)^{-1} \psi \in S(\mathbb{R})
\end{equation}
transfers to the symmetry of \((H + 1)^{-1}\) on the dense subspace \(S(\mathbb{R})\) from which self-adjointness follows. \(\Box\)

Thus \(L^2(\mathbb{R})\) has an orthonormal basis of eigenfunctions of \((H + 1)^{-1}\). If \(\psi \in L^2(\mathbb{R})\) and \((H + 1)^{-1} \psi = \lambda \psi\) with \(\lambda > 0\) then \(\psi \in H^1_{\text{iso}}(\mathbb{R})\) and \((H + 1) \psi = \lambda^{-1} \psi\) is an eigenfunction for \(H + 1\). The commutation relation, valid initially for the action on \(S(\mathbb{R})\)

\[ [H + 1, C] = 2C \implies [(H + 1)^{-1}, C] = -2(H + 1)^{-1}C(H + 1)^{-1} \text{ on } H^1_{\text{iso}}(\mathbb{R}) \]

using the density of \(S(\mathbb{R})\) in \(H^1_{\text{iso}}(\mathbb{R})\) which follows from the properties of \(E_C\). Thus an eigenfunction \(\psi\) satisfies

\[
(H + 1)^{-1}C \psi = C(H + 1)^{-1} \psi - 2(H + 1)^{-1}C(H + 1)^{-1} \psi \implies (H + 1)^{-1}C \psi = \frac{\lambda}{1 + 2\lambda} C \psi.
\]

So \(C \psi\) is also an eigenfunction and in particular \(C \psi \in H^1_{\text{iso}}(\mathbb{R})\). This argument can therefore be repeated and shows that \(C^k \psi \in H^1_{\text{iso}}(\mathbb{R})\) satisfies

\[
(H + 1)C^k \psi = (\frac{1}{\lambda} + 2k) C^k \psi, \ \forall \ k
\]

and these are all non-trivial eigenfunctions which are orthogonal in \(L^2(\mathbb{R})\).

The analogous commutation relation for the annihilation operator \([H + 1, A] = -2A\) shows that \((1 - 2\lambda)(H + 1)^{-1}A \psi = \lambda A \psi\), so unless \(1 - 2\lambda = 0\), \(A \psi \in H^1_{\text{iso}}(\mathbb{R})\) is again an eigenfunction. This can be iterated to show that \(A^k \psi \in H^1_{\text{iso}}(\mathbb{R})\) is an eigenfunction

\[
(H + 1)A^k \psi = (\frac{1}{\lambda} - 2k) A^k \psi,
\]

unless \(2k \lambda = 1\) in which case \(A^k \psi = 0\) since \((H + 1)^{-1}\) is injective. In fact this must occur, i.e. \(\lambda = 1/2j\) for some \(j\), since otherwise, for sufficiently large \(j\), \(A^j \psi\) is an eigenfunction of \((H + 1)^{-1}\) with a negative eigenvalue. However integration by parts shows that

\[
((H + 1)u, u) \geq 0 \ \forall \ u \in S(\mathbb{R}).
\]

The operator \(E_C E_A = (H + 1)^{-1} : S(\mathbb{R}) \rightarrow S(\mathbb{R})\) so taking \(u = E_C E_A f\) shows that \(((H + 1)^{-1} f, f) \geq 0\) for \(f \in C_c^\infty(\mathbb{R})\). It follows that \((H + 1)^{-1}\) is non-negative so such a negative eigenvalue cannot occur. So indeed the eigenvalues of \((H + 1)^{-1}\) are just \((2 + 2k)^{-1}\), of multiplicity 1 with eigenspace the multiples of the Hermite function \(\text{sp}(C^k \exp(-x^2/2)), k \in \mathbb{N}_0\).

**Proposition 1.** The eigenfunction expansion

\[
u = \sum_k (u, e_k) e_k
\]

converges in \(H^s_{\text{iso}}(\mathbb{R})\) for any \(s \in \mathbb{R}\), where as spaces

\[
H^s_{\text{iso}}(\mathbb{R}) = H^s(\mathbb{R}) \cap \langle x \rangle^{-s} L^2(\mathbb{R}), \ s \geq 0, \ H^s_{\text{iso}}(\mathbb{R}) = H^s(\mathbb{R}) + \langle x \rangle^{-s} L^2(\mathbb{R}), \ s \leq 0.
\]

Furthermore the \(L^2\) normalized Hermite functions are such that for any \(k\) continuous seminorm \(\| \cdot \|\) on \(S(\mathbb{R})\) there is a bound

\[
\|e_j\| \leq C_N (1 + j)^{N + 1} \ \forall \ j
\]
and for any $N$ there is a continuous norm $\| \cdot \|_{(N)}$ on $\mathcal{S}(\mathbb{R})$ such that
\begin{equation}
|\langle u, e_j \rangle| \leq (1 + j)^{-N} \| u \|_{(N)}, \quad \forall \phi \in \mathcal{S}(\mathbb{R});
\end{equation}
these estimates are uniform on compact subsets of $\mathcal{S}(\mathbb{R})$.

**Proof.** The rapid decay of the coefficients in (1.27) follows from the fact that $\| u \|_{(N)} = \| (H + 1)^N u \|_{L^2}$ is a continuous seminorm on $\mathcal{S}(\mathbb{R})$ so
\begin{equation}
(2j + 2)^N |\langle u, e_j \rangle| = |\langle u, (H + 1)^N e_j \rangle| = \| (H + 1)^N e_j \|_{L^2} \leq \| u \|_{(N)}
\end{equation}
is bounded and uniformly so on compact sets.

For any eigenfunction $ACe_j = (2j + 1)e_j$ so $\| Ce_j \|_{L^2} = \| A \|_{(1)} = (2j + 1)$ from which it follows that
\begin{equation}
e_{j+1} = (2j + 1)^{-\frac{1}{2}} Ce_j, \quad e_j = (2j - 1)^{-\frac{1}{2}} A e_{j+1}
\end{equation}
and
\begin{equation}
\|e_j\|_{H^1} \leq (2j + 1) \implies \sup |e_j(x)| \leq C(2j + 1)
\end{equation}
by Sobolev embedding.

The finite sums of the supremum norms on $D^k e_j$ and $x^k e_j$, $k \leq N$, give a complete set of on $\mathcal{S}(\mathbb{R})$, i.e. bound and other continuous norm. These can be expressed in terms of the creation and annihilation operators
\begin{equation}
d_k \frac{d}{dx^k} = (A - C)^k / 2^k, \quad x^k = (A + C)^k / 2^k
\end{equation}
and expanding these out and using (1.29) to evaluate the products of the $A$ and $C$ and to bound the Hermit functions shows that
\begin{equation}
\left| \frac{d^k e_j}{dx^k} \right| + \left| x^k e_j \right| \leq C(k)(2k + 2k)^k \max \sup |e_{j+l}| \leq C(k)(2j + 2k)^{k+1}.
\end{equation}
This gives the polynomial bound (1.26).

It follows by combining these two estimates that the series (1.24) is absolutely summable, and hence converges with respect to, any continuous norm on $\mathcal{S}(\mathbb{R})$. Conversely of course if the series converges in this sense then the sum is in $\mathcal{S}(\mathbb{R})$ and it is just the Fourier-Bessel series for the sum. \[\Box\]

The higher dimensional case follows by easy computation:

**Proposition 2.** The Hermite functions for the harmonic oscillator on $\mathbb{R}^n$ give a complete orthonormal basis
\begin{equation}
e_\gamma(x) = e_{\gamma_1}(x_1) \cdots e_{\gamma_n}(x_n)
\end{equation}
of $L^2(\mathbb{R}^n)$ which are polynomially bounded with respect to any continuous seminorm on $\mathcal{S}(\mathbb{R}^n)$:
\begin{equation}
\| e_\gamma \| \leq C(1 + |\gamma|)^N
\end{equation}
and for which the Fourier-Bessel coefficients are uniformly rapidly decaying
\begin{equation}
\phi_\gamma = \langle \phi, e_\gamma \rangle \implies |\phi_\gamma| \leq C_N(1 + |\gamma|)^{-N}
\end{equation}
on compact sets of $\mathcal{S}(\mathbb{R}^n)$, ensuring the convergence of (1.1) in $\mathcal{S}(\mathbb{R}^n)$. 

2 Symbols and quantization

Recall that we constructed a parameterix for an elliptic operator $P(D)$ with constant coefficients and used it to prove local elliptic regularity. Ellipticity implies that the characteristic polynomial $P(\xi)$ has real zeros (if any) only in a compact set and if we choose an appropriate $P$ that the characteristic polynomial is equal to one in a neighbourhood of the zeros then

$$a(\xi) = \frac{1 - \chi(\xi)}{P(\xi)} \in S^M(\mathbb{R}^n), \ M = -m. \tag{2.1}$$

Technically we define $a$ to be zero where the numerator vanishes.

The symbol space of order $M$ is defined by estimates on the growth of the derivatives like those of a polynomial

$$\sup_{\xi} (1 + |\xi|)^{-M+|\alpha|}|D^\alpha a(\xi)| < \infty \forall \alpha. \tag{2.2}$$

Using an iterative formula for the derivatives we showed that $a \in S^{-m}(\mathbb{R}^n)$. We showed earlier that if $a \in S^M(\mathbb{R}^n)$ then its inverse Fourier transform, $\hat{A} \in S'(\mathbb{R}^n)$, $\hat{A} = a$, satisfies

$$\text{singsupp}(A) \subset \{0\}, \ (1 - \chi(x))A(x) \in S(\mathbb{R}^n), \ A \in H^{-M-s}(\mathbb{R}^n), \ s > n/2 \tag{2.3}$$

where $\chi \in C_c^\infty(\mathbb{R}^n)$ is again equal to 1 near 0. We use this below in the stronger form (which follows from the proof) that

$$S^M(\mathbb{R}^n) \ni a \mapsto (1 - \chi)A \in S(\mathbb{R}), \ \hat{A} = a, \ F : S^M(\mathbb{R}^n) \mapsto H^{-M-s}(\mathbb{R}^n) \text{ are continuous}. \tag{2.4}$$

The distribution $\chi A$ was used as a convolution operator to deduce smoothness of solutions to $P(D)u = f$ — in particular to show that $\text{singsupp}(u) = \text{singsupp}(f)$, which is one form of elliptic regularity.

This convolution operator has kernel $\chi(x - y)A(x - y) \in S'(\mathbb{R}^{2n})$ which is only singular at the diagonal, $x = y$, and which we have arranged to be supported close to the diagonal as well. The leading estimate in (2.2) shows the boundedness of these operators on Sobolev spaces:

$$a \in S^M(\mathbb{R}^n) \implies a(D) = A* : H^t(\mathbb{R}^n) \mapsto H^{t-M}(\mathbb{R}^n), \ \forall t \in \mathbb{R}. \tag{2.5}$$

The convenient notation $a(D)$ for this convolution operator extends the standard notation for constant coefficient differential operators $P(D)$ where $P = P(\xi) \in S^k(\mathbb{R}^n)$ is a polynomial. Then, for the $a$ in (2.1), when $P$ is elliptic,

$$P(D)a(D) = a(D)P(D) = \text{Id} - e(D)$$

where $e(D)$ arises from $e \in S(\mathbb{R}^n)$ (in fact compactly supported) and so is a smoothing operator. This is what we want to extend to variable coefficient elliptic differential operators. The pseudodifferential operators used to carry this out correspond to a smooth family of such convolution operators and the main problem is just to make sense of this statement.

As noted above, we could work with the symbol spaces 'with bounds' $S^M(\mathbb{R}^n)$ and it is conventional to do so for several, good, reasons. Some of these disappear with the approach I am taking here, so instead I concentrate on the smaller space of 'classical' symbols — with which we are really more familiar. Replacing these by the full symbol spaces involves little more than change of notation. An example of a
classical symbol is precisely $a$ in (2.1). It differs from a general symbol in the sense of (2.2) by ‘having an asymptotic expansion in smooth homogeneous functions at infinity’. Rather than discuss the usual formulation of this concept let us look at $a$ in (2.1) in terms of the radial compactification of $\mathbb{R}^n$.

Recall that we discussed this compactification earlier, by identifying $\mathbb{R}^n$ with the hyperplane $\eta_{n+1} = 1$ in $\mathbb{R}^{n+1}$ and then taking the stereographic projection to the half-sphere (not the full sphere minus a point although we did that two). This means considering the bijection

$$F : \mathbb{R}^n \ni \xi \mapsto \eta = \frac{\langle \xi, 1 \rangle}{\langle \xi \rangle} \in S^n \cap \{\eta_{n+1} > 0\}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}.$$  

Since it is topologically, and differentially, a ball let me write the closure of the image as $\mathbb{B}^n$, even though it is the half-sphere $S^n \cap \{\eta_{n+1} \geq 0\}$.

**Exercise 2.** Write out an explicit ‘flattening’ of the half-sphere to a ball, making sure that you do not inadvertently introduce a square-root singularity at the boundary.

We can use $F$ to identify a function $f$ on $\mathbb{R}^n_\xi$ with the function $(F^{-1})^* f$ on the interior of $\mathbb{B}^n$. For instance $(F^{-1})^* \langle \xi \rangle^{-1} = \eta_{n+1}|_{\mathbb{B}^n}$ is smooth right up to the boundary where it vanishes simply – it is a defining function for the boundary of $\mathbb{B}^n$. The point of all this discussion is that with $a$ in (2.1),

$$(F^{-1})^* a = \eta_{n+1}^m b, \quad b \in C^\infty(\mathbb{B}^n) = C^\infty(\mathbb{R}^{n+1})|_{\mathbb{B}^n}.$$  

That is, transferred to the ball, the upper half-sphere, it becomes a smooth function which vanishes to precisely order $m$ at the boundary (in this case $m$ is an integer but we do not assume this in general).

**Definition 1.** The space of classical symbols of real order $s$ is defined to be

$$(2.7) \quad S^s_cl(\mathbb{R}^n) = F^*(\eta_{n+1}^s C^\infty(\mathbb{B}^n)) \subset S^s(\mathbb{R}^n).$$

The sign reversal in the order here is just the usual historical baggage. In particular

$$(2.8) \quad S^s_cl(\mathbb{R}^n) \subset S^{s+k}_{cl}(\mathbb{R}^n), \quad k \in \mathbb{N}_0$$

but there is no such inclusion for non-integral $k$.

Really of course (2.7) is by way of a theorem since it claims that the estimates (2.2) hold with $M = s$ for all $a \in S^s_cl(\mathbb{R}^n)$. If $a \in S^s_cl(\mathbb{R}^n)$ then $a = F^*(\eta_{n+1}^s b)$, $b \in C^\infty(\mathbb{B}^n)$, where $b$ is certainly bounded, so $|a| \leq \mathcal{C}(|\xi|^s)$ which is the top estimate in (2.2). The (infinitely many) other estimates reduce to the same bound after application of any number (repeated arbitrarily) of the vector fields $\xi_i \partial_{\xi_j}$.

**Lemma 2.** There are vector fields on $\mathbb{B}^n$ (smooth first order differential operators without constant terms) $V_{ij}$ such that $\xi_i \partial_{\xi_j}(F^* a) = F^*(V_{ij} a)$ and these vector fields are tangent to the boundary of $\mathbb{B}^n$, meaning that $V_{ij} \eta_{n+1} = \eta_{n+1} e_{ij}, \quad e_{ij} \in C^\infty(\mathbb{B}^n)$.

**Proof.** For the moment left as an exercise. □

This shows that the assertion in (2.7) does indeed hold.

One small advantage of considering classical symbols is that we are already pretty familiar with spaces of smooth functions such as $C^\infty(\mathbb{B}^n)$ although there are things to check to feel confident about it. To define $C^\infty(\mathbb{B}^n)$ there are two obvious choices, we can say $u \in C^\infty(\mathbb{B}^n)$ if $u = \tilde{u}|_{\mathbb{B}^n}$ for some $\tilde{u} \in C^\infty(\mathbb{R}^n)$ or we can just demand that $u$ is smooth in the interior of $\mathbb{B}^n$ (which remember is the closed
ball) with all derivatives uniformly bounded. By integration the second definition means all derivatives are continuous up to the boundary. If we take the second, apparently weaker definition, then the topology is given by the supremum norms on the derivatives and there is a nice theorem of Seeley that says there exists a continuous linear extension map

\[(2.9) \quad E : C^\infty(B^n) \to \{\tilde{u} \in C^\infty(\mathbb{R}^n), \ \text{supp } \tilde{u} \subset K\}, \ E u|_{|x|_B} = u\]

where we can take K to be the ball of radius 2 for instance. So these two definitions are the same. If you use Borel’s Lemma you can see that the restriction map is surjective but Seeley gives quite a bit more.

So, it is perhaps a relief to know that the space of pseudodifferential operators \(\Psi^m_{cl,S}(\mathbb{R}^n)\), as defined below, is actually identified linearly, and topologically, with the space

\[(2.10) \quad \langle \xi \rangle^m F^* C^\infty(B^n; S(\mathbb{R}^n)),\]

so ultimately with

\[(2.11) \quad C^\infty(B^n; S(\mathbb{R}^n)).\]

The idea of ‘quantization’ is to turn a function, such as one of these symbols, into an operator.

Recall that a space like \(\ref{2.11}\) is straightforward to define starting from the definition of \(C^\infty(B^n)\). We can start with the continuous maps from the ball to the Fréchet space \(S(\mathbb{R}^n)\) forming \(C(B^n; S(\mathbb{R}^n))\). Then consider once-differentiable elements, so the difference quotients defined in the interior of the ball converge (in the topology of \(S(\mathbb{R}^n)\)) pointwise and extend to define the derivatives which are required to be in \(C(B^n; S(\mathbb{R}^n))\). This definition can then be iterated to define infinite differentiability and the space \(\ref{2.11}\).

**Exercise 3.** Show that the space \(C^\infty(B^n; S(\mathbb{R}^n))\) is naturally identified with the subspace of \(C^\infty\) functions on the interior of \(\mathbb{B}^n_\alpha \times \mathbb{R}^n_\beta\) for which

\[(2.12) \quad \sup (x)^N |D^n_\alpha D^\beta_y (\eta, x)| < \infty \ \forall \ N, \ \alpha, \ \beta.\]

You might recall, or check, that \(S(\mathbb{R}^n)\) is also represented by a simple space under radial compactification. Namely \(S(\mathbb{R}^n) = F^* \hat{C}^\infty(\mathbb{R}^n)\) where \(\hat{C}^\infty(\mathbb{R}^n)\) is the space of smooth functions on \(\mathbb{R}^n\) with support in the (closed) ball \(\mathbb{B}^n\). These can also be identified with the subspace of \(C^\infty(\mathbb{B}^n)\) with all derivatives vanishing at the boundary – the existence of an extension map from this subspace is then clear! We can similarly define \(\hat{C}^\infty(\mathbb{B}^n; S(\mathbb{R}^n))\) and identify it with \(S(\mathbb{R}^{2n})\).

It is convenient to introduce a notion of ‘support’ for our classical symbols, which distinguishes where an element of \(\ref{2.11}\) is locally equal to an element of \(\hat{C}^\infty(\mathbb{B}^n; S(\mathbb{R}^n))\). Namely we define the cone-support, as usual through its complement

\[(2.13) \quad \text{For } a \in C^\infty(\mathbb{B}^n_\alpha; S(\mathbb{R}^n)_y) \text{ conesupp}(a) = \{(\eta, x) \in \mathbb{S}^{n-1} \times \mathbb{R}^n \exists \ \tilde{a} \in \hat{C}^\infty(\mathbb{B}^n; S(\mathbb{R}^n)) \text{ s.t. } a = \tilde{a} \text{ near } (\hat{\xi}, x)\}.\]

We can think of the cone support as representing a cone in \((\mathbb{R}^n_\xi \setminus \{0\}) \times \mathbb{R}^n_\eta\), conic in the first variable, such that on the complementary open cone \(F^* a\) is rapidly
decaying. Since the ‘comparison’ functions $\hat{\mathcal{C}}^\infty(\mathbb{B}^n; S(\mathbb{R}^n))$ form an ideal, and as for the usual support of smooth functions
\begin{equation}
\text{conesupp}(ab) \subset \text{conesupp}(a) \cap \text{conesupp}(b),
\end{equation}

although this last result is not quite obvious (you can see it by checking that the absence of a point $\hat{\xi}, x$) from $\text{conesupp}(a)$ means that all derivatives of $a$ restricted to $\mathbb{S}^{n-1} \times \mathbb{R}^n$ vanish near $(\xi, x)$ so if $\text{conesupp}(a)$ is empty these boundary values are all zero, from here it is not so hard).

Exercise 4. One can refine the definition of the cone support to be ‘uniform near infinity’ in the second variable. Define the uniform cone support $\text{uconesupp} by working on the space $\mathbb{B}^n \times \mathbb{B}^n$ in which the symbols are identified as $C^\infty(\mathbb{B}^n; \hat{\mathcal{C}}^\infty(\mathbb{B}^n))$ using radial compactification in the second variable and the ‘trivial’ symbols are $\hat{\mathcal{C}}^\infty(\mathbb{R}^n; \mathcal{C}^\infty(\mathbb{B}^n))$. Then define $\text{uconesupp}(a)$ to be the closed subset of $\mathbb{S}^{n-1} \times \mathbb{B}^n$ on the complement of which a symbol is locally, on $\mathbb{B}^n \times \mathbb{B}^n$, equal to a ‘trivial’ symbol. Over the interior of the ball in the second variable, which is to say over $\mathbb{R}^n$ in that variable the definitions are the same. For classical symbols with Schwartz coefficients, which is what we are discussing here, you can check that
\begin{equation}
\text{uconesupp}(a) = \text{conesupp}(a) \text{ in } \mathbb{S}^{n-1} \times \mathbb{B}^n.
\end{equation}

Familiarity with the topology here allows us to check easily that taking the Hermite expansion in the second set of variables
\begin{equation}
C^\infty(\mathbb{B}^n; S(\mathbb{R}^n)) \ni a(\eta, y) = \sum_\gamma a_\gamma(\eta)e_\gamma(y), \quad a_\gamma = \int a(\cdot, y)e_\gamma(y)dy \in C^\infty(\mathbb{B}^n)
\end{equation}
gives a series which converges in $C^\infty(\mathbb{B}^n; S(\mathbb{R}^n))$.

Lemma 3. For any continuous norm on $C^\infty(\mathbb{B}^n)$ and any $N \in \mathbb{N}$ there is a constant $C_N$ such that the coefficients in (2.16) satisfy
\begin{equation}
\|a_\gamma\| \leq C_N(1 + |\gamma|)^{-N}.
\end{equation}

Proof. This follows directly from Proposition 2. Namely, as norms fixing the topology of $C^\infty(\mathbb{B}^n)$ we can take the supremum over $\mathbb{B}^n$ of the derivatives up to multi-order $k$. By definition, each of these derivatives defines an element of $C(\mathbb{B}^n; S(\mathbb{R}^n))$ which therefore has compact image in $S(\mathbb{R}^n)$ so the estimates (2.17) follows from (1.35). □

Each of the terms in (2.16) is already identified with an operator, since the ‘coefficients’ $a_\gamma \in S(\mathbb{R}^n)$ are multipliers on $H^s(\mathbb{R}^n)$ for any $s$.

Proposition 3. For any $a \in S^m_{\text{cl}, S}(\mathbb{R}^n; \mathbb{R}^n) = \mathcal{F}^*x_{n+1}^{-m}C^\infty(\mathbb{B}^n; S(\mathbb{R}^n))$, the series (2.16) converges in $\mathcal{B}(H^s(\mathbb{R}^n); H^{s-m}(\mathbb{R}^n))$ for each $s$ to define a bounded operator which we write as
\begin{equation}
a(D_y, y)u = \sum_\gamma a_\gamma(D)(e_\gamma(y)u(y)).
\end{equation}

Somewhat perversely the ‘coefficients’ are written on the right here. We will see below that these operators restrict to continuous linear maps $S(\mathbb{R}^n) \rightarrow S(\mathbb{R}^n)$, so the operators on the Sobolev spaces are all determined by continuous extension and it is not necessary to distinguish then by the space on which they act.
Exercise 5. Show the boundedness of the operators or order $m \leq 0$ on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ and on the Hölder spaces $C^{k,\gamma}(\mathbb{R}^n)$ for $0 < \gamma < 1$, $k \in \mathbb{N}_0$. The hard work, really the Marcinkiewicz multiplier theorem in the first case and less hard in the second (see [5 §7.9]) is to show the boundedness in this sense of the convolution operators $a_\gamma(D)$ in terms of a continuous seminorm on $S^0(\mathbb{R}^n)$. For classical symbols boundedness can be reduced to the homogenous case and the estimates in [5 §4.5] can be used instead.

Definition 2. We define the space $\Psi^m_{cl,S}(\mathbb{R}^n)$ as consisting of the operators on Sobolev spaces given by the sum (2.18) for a classical symbol $a$, so as the image of a ‘right quantization’ map

$$q_R : \mathcal{S}^m_{cl,S}(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^n)) \rightarrow \Psi^m_{cl,S}(\mathbb{R}^n).$$

The right and ‘R’ refers to fact that the coefficients are put on the right. Note that each of the summands in (2.18) has as Schwartz kernel $B_\gamma(x,y) = A_\gamma(x-y)e_\gamma(y)$. The estimates (2.17) and the continuity in (2.4) show the convergence of

$$(2.20) \quad B(x,y) = \sum_\gamma B_\gamma(x,y) = \sum_\gamma A_\gamma(x-y)e_\gamma(y) \in \mathcal{S}'(\mathbb{R}^{2n}),$$

In fact if we write $z = x - y$ and set

$$(2.21) \quad A(z,y) = B(z+y,y) = \sum_\gamma A_\gamma(z)e_\gamma(y) \text{ converges in } \mathcal{S}(\mathbb{R}^n; H^{-m-s}(\mathbb{R}^n)), \quad s > n/2,$$

and $(1 - \chi(z))A(z,y) \in \mathcal{S}(\mathbb{R}^{2n})$.

Here $\chi \in C^\infty(\mathbb{R}^n)$ is identically equal to 1 near 0.

We define a support set for these operator in terms of the symbol:-

$$(2.22) \quad \text{WF}'(q_R(a)) = \text{conesupp}(a) \subset S^{n-1} \times \mathbb{R}^n.$$  

This is also called the ‘operator wavefront set’ and has the property

$$(2.23) \quad \text{WF}'(q_R(a)) = \emptyset \implies a \in \mathcal{C}_c^\infty(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^n)) \implies B \in \mathcal{S}(\mathbb{R}^{2n})$$

where $B$ is the Schwartz kernel of $q_R(a)$ given by (2.20).

Exercise 6. If you did Exercise 4 then you can define

$$(2.24) \quad \text{WF}'_S(q_R(a)) = \text{uconesupp}(a) \subset S^{n-1} \times B^n.$$  

This definition of $\Psi^m_{cl,S}(\mathbb{R}^n)$ involves a degree of ‘handism’ – which I hope is not a real word. Namely, We can do exactly the same and define another operator, from the same symbol $a$, by ‘left quantization’

$$(2.25) \quad a(x,D)u = \sum_\gamma e_\gamma(x)(a_\gamma(D))u.$$  

In general $a(x,D) \neq a(D,y)$. As we show below, the space of operators obtained this way is indeed the same as (2.18). That is (2.25) also defines a linear bijection

$$(2.26) \quad q_L : \mathcal{S}^m_{cl,S}(\mathbb{R}^n; \mathbb{R}^n) \rightarrow \Psi^m_{cl,S}(\mathbb{R}^n)$$  

i.e. it has the same range.
There are other quantizations (especially ‘Weyl quantization’, see below). More generally we can consider

\[ (2.27) \quad S_{cl,S}^m(\mathbb{R}^n;\mathbb{R}^{2n}) = \mathcal{F}^* \left( n_{\eta+1}^{r}C^\infty(\mathbb{B}^n;\mathcal{S}(\mathbb{R}^{2n})) \right) \]

where we allow the ‘symbol’ to depend on both \(x\) and \(y\) and also \(\xi\). Then the Hermite expansion for functions on \(\mathbb{R}^{2n}\) gives

\[ (2.28) \quad S_{cl,S}^m(\mathbb{R}^n;\mathbb{R}^{2n}) \ni b(x,y,\xi) = \sum_{\gamma,\gamma'} c_{\gamma}(x)b_{\gamma,\gamma'}(\xi)e_{\gamma}(y) \]

where the convergence is strong enough that the operator

\[ (2.29) \quad b(x,\xi,\cdot) = \sum_{\gamma,\gamma'} c_{\gamma}(x)(b_{\gamma,\gamma'}(\xi))(e_{\gamma}(\cdot)) \]

converges in \(\mathcal{B}(H^s(\mathbb{R}^n); H^{s-m}(\mathbb{R}^n))\). We can write this ‘general’ quantization map as

\[ (2.30) \quad q : S_{cl,S}^m(\mathbb{R}^n;\mathbb{R}^{2n}) \rightarrow \Psi_{cl,S}^m(\mathbb{R}^n) \]

because it again has the same range.

The Schwartz kernel of \(q(a)\) is given the series

\[ (2.31) \quad B(x,y) = \sum_{\gamma,\gamma'} c_{\gamma}(x)A_{\gamma,\gamma'}(x-y)e_{\gamma}(y) \in S'(\mathbb{R}^{2n}) \]

since this is true for the individual terms in (2.29) and the resulting series converges in \(S'(\mathbb{R}^{2n})\). If we set \(x = z + y\) it follows that

\[ (2.32) \quad A(z,y) = B(z+y,y) \in S(\mathbb{R}^n; H^{-s-m}(\mathbb{R}^n)), \quad s > n/2 \]

just as for the kernels right quantized operators.

Left and right quantization are both bijections, but from this formula it is clear that \(q\) is not.

**Exercise 7.** Find an explicit symbol in the null space of \(q\).

**Exercise 8.** If \(a \in S_{cl}^m(\mathbb{R}^n)\) and \(e \in \mathcal{S}(\mathbb{R}^n)\) show that the kernel

\[ (2.33) \quad A(x-y)e\left(\frac{x+y}{2}\right) \]

defines a bounded operator \(H^s(\mathbb{R}^n) \rightarrow H^{s-m}(\mathbb{R}^n)\) for any \(s \in \mathbb{R}\) – for instance divide \(A(z)\) into a part \(A_1\) with compact support in \(|z| < 1\) and a Schwarz part \(A_2\) and write

\[ (2.34) \quad A(x-y)e\left(\frac{x+y}{2}\right) = A_1(x-y)\phi(x-y)e\left(\frac{x+y}{1}\right) + A_2(x-y)(1-\phi(x-y))e\left(\frac{x+y}{2}\right) \]

for an appropriate cutoff \(\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)\); then apply \(q\) to the first part and observe that the second part is in \(\mathcal{S}(\mathbb{R}^{2n})\). Show that this operator is independent of the choice of cut-off and estimate the norm in terms of a seminorm on \(a(\xi)e(X) \in S_{cl,S}^m(\mathbb{R}^n;\mathbb{R}^{2n})\). Conclude that the Hermite expansion of symbols leads to a Weyl quantization map \(q_W\) with values in the bounded operators just as for \(q_R\) and \(q_L\). Use the discussion in §5 to show that this is another bijection

\[ (2.35) \quad q_W : S_{cl,S}^m(\mathbb{R}^n;\mathbb{R}^n) \rightarrow \Psi_{cl,S}^m(\mathbb{R}^n) \]

which has the useful property (among others) that \((q_W(a))^* = q_W(\bar{a})\).
3 Algebra of pseudodifferential operators

Let me collect a substantial, although not exhaustive, list of properties of the space $\Psi_{cl, S}^m(\mathbb{R}^n)$, given for definiteness sake, as above, as the image of the map $q_R$ in (2.19). Note that the subscript cl refers to the ‘classical’ symbols and the $S$ is there because the coefficients are, by assumption, in $S(\mathbb{R}^n)$. This space has certain defects, in particular $\text{Id} \notin \Psi_{cl, S}^0(\mathbb{R}^n)$ because of the assumed rapid decay of the coefficients. There are many variants of these spaces but we have to start somewhere.

**Theorem 1.** The spaces $\Psi_{cl, S}^m(\mathbb{R}^n)$ have the following properties:

(P1) (Boundedness) The elements of $\Psi_{cl, S}^m(\mathbb{R}^n)$ define bounded operators

\[ A : H^s(\mathbb{R}^n) \rightarrow H^{s-m}(\mathbb{R}^n) \quad \forall \ s \in \mathbb{R}. \]

For each $j$ and $A \in \Psi_{cl, S}^m(\mathbb{R}^n)$, $x_j A \in \Psi_{cl, S}^m(\mathbb{R}^n)$, the commutator $[x_j, A] \in \Psi_{cl, S}^{m-1}(\mathbb{R}^n)$ and in consequence

\[ A : \langle a \rangle^t H^s(\mathbb{R}^n) \rightarrow \langle x \rangle^{-t} H^{s-m}(\mathbb{R}^n) \quad \forall \ t, s \]

(P2) (Left-right reduction) The maps $q_L$ and $q_R$ are injective and $q_{R}$, $q_{L}$ and $q$ have the same range for each $m \in \mathbb{R}$. Furthermore

\[ q_{R}(a) = q_{L}(b), \quad a, b \in C(\mathbb{R}^n; S(\mathbb{R}^n)) \implies \text{conesupp}(a) = \text{conesupp}(b). \]

(P3) (Symbol map) For any $m \in \mathbb{R}$, $\Psi_{cl, S}^{m-1}(\mathbb{R}^n) \subset \Psi_{cl, S}^m(\mathbb{R}^n)$ and there is a short exact sequence

\[ \Psi_{cl, S}^{m-1}(\mathbb{R}^n) \rightarrow \Psi_{cl, S}^m(\mathbb{R}^n) \xrightarrow{\sigma} |\xi|^m C^{\infty}(\mathbb{R}^{n-1}; S(\mathbb{R}^n)) \]

where $|\xi|^m C^{\infty}(\mathbb{R}^{n-1}; S(\mathbb{R}^n))$ should be interpreted as the space of those elements of $C^{\infty}(\mathbb{R}^{n-1}; S(\mathbb{R}^n))$ which are homogeneous of degree $m$ in the first $n$ variables. So $\sigma_m$ is surjective and has null space exactly $\Psi_{cl, S}^{m-1}(\mathbb{R}^n)$.

(P4) (Multiplicativity) For any $m$ and $m'$, in terms of operator composition

\[ \Psi_{cl, S}^m(\mathbb{R}^n) \circ \Psi_{cl, S}^{m'}(\mathbb{R}^n) \subset \Psi_{cl, S}^{m+m'}(\mathbb{R}^n), \]

\[ \sigma_{m+m'}(AB) = \sigma_m(A)\sigma_{m'}(B), \quad \text{WF}'(AB) \subset \text{WF}'(A) \cap \text{WF}'(B). \]

(P5) (Adjoints) Taking adjoints with respect to Lebesgue measure,

\[ \Psi_{cl, S}^m(\mathbb{R}^n) \ni A \mapsto A^* \in \Psi_{cl, S}^m(\mathbb{R}^n) \]

is an isomorphism and $\sigma_m(A^*) = \overline{\sigma_m(A)}$, $\text{WF}'(A^*) = \text{WF}'(A)$.

(P6) (Asymptotic completeness) For any sequence $A_k \in \Psi_{cl, S}^{m-k}(\mathbb{R}^n)$, $k \in \mathbb{N}$, there exists $A \in \Psi_{cl, S}^m(\mathbb{R}^n)$ such that

\[ A - \sum_{k=0}^{N} A_k \in \Psi_{cl, S}^{m-N-1}(\mathbb{R}^n) \quad \forall \ N \in \mathbb{N}. \]

This relationship is written

\[ A \sim \sum_{k=0}^{N} A_k \implies \text{WF}'(A) \subset \bigcup_{k} \text{WF}'(A_k). \]
(P7) (Pseudo/Microlocality) For any $A \in \Psi_{cl}^*(\mathbb{R}^n)$,

\[
singsupp(Au) \subset \text{singsupp}(u), \quad \forall \ u \in \mathcal{S}'(\mathbb{R}^n),
\]

\[
WF(Au) \subset WF(u) \cap WF'(A), \quad \forall \ u \in \mathcal{S}'(\mathbb{R}^n)
\]

where the second of these implies the first.

(P8) (Residual Algebra)

\[
\bigcap_{k \in \mathbb{N}_0} \Psi_{cl,S}^{m-k}(\mathbb{R}^n) = \Psi_{S}^{-\infty}(\mathbb{R}^n) \equiv \mathcal{S}(\mathbb{R}^n) = \{ A \in \Psi_{cl,S}^{m}(\mathbb{R}^n); WF'(A) = \emptyset \}
\]

is the space of integral operators with Schwartz kernels

\[
Bu(x) = \int_{\mathbb{R}^n} B(x,y)u(y)dy, \quad B \in \mathcal{S}(\mathbb{R}^n) \text{ and } B^*: \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n).
\]

4 Elliptic regularity

Rather than proceeding directly to check all these properties, let’s see what we can achieve by using them. Suppose that

\[
P(x,D) = \sum_{|\alpha| \leq m} p_\alpha(x)D^\alpha, \quad p_\alpha \in C^\infty(\Omega),
\]

is a differential operator with smooth coefficients in some open set $\Omega \subset \mathbb{R}^n$. This is not in our algebra unless the coefficients happen to be Schwartz functions on the whole of $\mathbb{R}^n$. Still, we are interested in the regularity of solutions to

\[
P(x,D)u = f, \quad f \in C^{-\infty}(\Omega), \quad u \in C^{-\infty}(\Omega).
\]

Ellipticity for $P$ in $\Omega$ means precisely that

\[
P_m(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\xi^\alpha \neq 0 \text{ on } \Omega \times \mathbb{R}^n \setminus \{0\}.
\]

**Theorem 2.** [Elliptic Regularity] If $P$ as in (4.1) is elliptic in $\Omega$ in the sense of (4.3) then

\[
singsupp(u) = \text{singsupp}(P(x,D)u),
\]

\[
WF(u) = \text{WF}(P(x,D)u), \quad \forall \ u \in C^{-\infty}(\Omega).
\]

**Proof.** The inclusions

\[
singsupp(u) \supset \text{singsupp}(P(x,D)u),
\]

\[
WF(u) \supset \text{WF}(P(x,D)u), \quad \forall \ u \in C^{-\infty}(\Omega)
\]

are consequences of the elementary properties of singsupp and WF (even if $P(x,D)$ is not elliptic) so it is the reverse that requires proof. Moreover, these are local statements in the sense that we just need to show

\[
\bar{x} \notin \text{singsupp}(P(x,D)u) \implies \bar{x} \notin \text{singsupp}(u)
\]

and similarly for WF. So we can look near a particular point $\bar{x} \in \Omega$.

Here is one way to see this using the properties of the pseudodifferential algebra listed above

**Lemma 4.** Assuming that $P(x,D)$ is elliptic:

\[
\text{If } \phi \in C_c^\infty(\Omega) \text{ and } \psi \in C_c^\infty(\Omega), \quad \text{supp}(\phi) \subset \{ \psi = 1 \} \text{ then } \exists A \in \Psi_{cl,S}^{m}(\mathbb{R}^n), \quad WF'(A) \subset \mathcal{S}^{n-1} \times \text{supp}(\phi) \text{ s.t. } A \circ P(x,D)\psi = \phi \text{Id} - E, \quad E \in \Psi_{S}^{-\infty}(\mathbb{R}^n).
\]
Notice that \( P(x,D)\psi \in \Psi^m_{c_1,S}(\mathbb{R}^n) \) since now the coefficients are compactly supported in \( \Omega \), so the composition in (4.7) is defined and covered by (3.5).

So, suppose we have obtained such an \( A \) where we choose \( \phi \) to be identically 1 near \( \bar{x} \). We can choose another \( \psi' \in C^\infty(\Omega) \) (this is being a bit pedantic) so that \( \psi\psi' = \psi \) and then we see that

\[
(4.8) \quad P(x,D)\psi u = P(x,D)\psi(\psi'u) = \phi f + g, \quad f = P(x,D)u,
\]

\[
\text{supp}(g) \subset \{ \phi \neq 1 \} \implies \bar{x} \notin \text{supp}(g),
\]

and where everything is now in \( C^{-\infty}_c(\Omega) \subset S'(\mathbb{R}^n) \). So we can apply \( A \) on the left and see that

\[
(4.9) \quad AP(x,D)\psi(\psi'u) = A\phi f + Ag = \phi u - E(\psi'u) \implies
\]

\[
\phi u = A(\phi f) + Ag + E(\psi'u).
\]

The last term is in \( S(\mathbb{R}^n) \) by (P8). Since \( \bar{x} \notin \text{supp}(g) \), and in particular \( \bar{x} \notin \text{singsupp}(g) \), pseudolocality shows that \( \bar{x} \notin \text{singsupp}(Ag) \). Thus indeed we get (4.6) since \( \bar{x} \notin \text{singsupp}(P(x,D)u) \) implies \( \bar{x} \notin \text{singsupp}(\phi f) \), \( f = P(x,D)u \), so \( \bar{x} \notin \text{singsupp}(A(\phi f)) \) and then \( \bar{x} \notin \text{singsupp}\phi u \) and so finally \( \bar{x} \notin \text{singsupp} u \) since \( \phi(\bar{x}) \neq 0 \).

The proof for WF is the same, using microlocality instead of pseudolocality. \( \square \)

This shows the efficacy of the calculus of pseudodifferential operators, although you might rightly object that the proof is a little clumsy because of the need to insert all these cutoffs. This is indeed the case and we can get rid of that annoyance by working with pseudodifferential operators on the open set \( \Omega \); this just requires a bit more machinery, but really only about supports and such.

So to the

Proof of Lemma

Directly from the definitions

\[
(4.10) \quad \phi \text{Id} \in \Psi^n_{c_1,S}(\mathbb{R}^n), \quad \sigma_0(\phi \text{Id}) = \phi(\cdot),
\]

\[
P(x,D)\psi \in \Psi^n_{c_1,S}(\mathbb{R}^n), \quad \sigma_m(P(x,D)\psi) = P_m(x,\xi)\psi(x) = \sum_{|\alpha|=m} p_\alpha(x)\psi(x)\xi^\alpha.
\]

Now we are looking for \( A \in \Psi_{c_1,S}^{-m}(\mathbb{R}^n) \) and for any choice

\[
(4.11) \quad AP(x,D)\psi \in \Psi^n_{c_1,S}(\mathbb{R}^n), \quad \sigma(\psi'(AP(x,D)\psi)) = \sigma_{-m}(A)P_m(x,\xi)\psi(x).
\]

Since \( P(x,D) \) is elliptic, the function

\[
(4.12) \quad a_0 = \frac{\phi(x)}{P_m(x,\xi)\psi(x)} \in |\xi|^{-m}C^\infty(S^{n-1};S(\mathbb{R}^n))
\]

(with the usual caveat that the function is taken to be zero outside the support of the numerator and remembering that \( \text{supp}(\phi) \) is contained in the set where \( \psi = 1 \)).

So this is a legitimate choice for the symbol, being a smooth function homogeneous of degree \( -m \) as claimed.

Set

\[
A_0 = q_R(a_0(1 - \mu(\xi))
\]

where \( \mu(\xi) \in C^\infty_c(\mathbb{R}^n) \) is equal to 1 in a neighbourhood of 0 - here we are using the surjectivity of the symbol map in (3.4) in a slightly stronger sense since we also
have $\text{WF}'(A_0) \subset S^{n-1} \times K$ where $K = \text{supp}(\phi)$. With this initial choice

\begin{equation}
E_1 = A_0 P(x, D) \psi - \phi \text{Id} \in \Psi_0^\omega(S(R^n)), \sigma_0(E_1) = a_0 P_m \psi - \phi = 0 \implies E_1 \in \Psi_0^\omega(S(R^n)) \text{ and } \text{WF}'(E_1) \subset S^{n-1} \times K.
\end{equation}

Then we can continue by induction to construct a succession of operators $A_j$ so that

\begin{equation}
A_j \in \Psi_{cl,S}^{-m-j}(R^n), \text{WF}'(A_j) \subset S^{n-1} \times K \text{ s.t.}
\end{equation}

\[ \sum_{j=0}^{N} A_j P(x, D) \psi = \phi \text{Id} + E_{N+1}, \]

\[ E_{N+1} \in \Psi_{cl,S}^{-N-1}(R^n), \text{WF}'(E_{N+1}) \subset S^{n-1} \times K. \]

Then we can choose $A_{N+1} = q_R(a_{N+1})$

\begin{equation}
a_{N+1} = \frac{\sigma_{-N-1}(E_{N+1})}{P_m(x, \xi)} \in |\xi|^{-m-N-1}C^\infty(S^{n-1}; S(R^n)) \implies \text{WF}'(A_{N+1}) \subset S^{n-1} \times K.
\end{equation}

This makes sense because of the ellipticity and the last condition in (4.14) which ensures that $\text{WF}'(E_{N+1}) \subset S^{n-1} \times K$.

Now we use the asymptotic completeness result to show that we can choose one

\begin{equation}
A \sim \sum_j A_{-j}, \quad A \in \Psi_{cl,S}^{-m}(R^n).
\end{equation}

The inductive arrangement means that

\begin{equation}
AP(x, D) \psi - \phi \text{Id} \in \Psi_{cl,S}^{-N}(R^n) \forall N
\end{equation}

which implies, by the last property above, that it is a smoothing operator, with kernel in $S(R^{2n})$.

Note that we do not really need all the properties listed above to prove elliptic regularity. For instance, it is enough to stop at a sufficiently large $N$ in the construction above, depending on the regularity of the distribution $u$ involved. Still, it is neater to have one operator which works for all – just as it is convenient to dispense with the cutoffs as we can do with a bit of work.

**Exercise 9.** Carry through the microlocal version of this construction. That is, if $P_m(\tilde{x}, \tilde{\xi}) \neq 0$ for some $\tilde{x} \in \Omega$ and $\tilde{\xi} \in S^{n-1}$ show that there exists $A \in \Psi_{cl,S}^{-m}(R^n)$, $\psi, \phi \in C^\infty_c(\Omega)$ such that

\begin{equation}
AP(x, D) \psi = \phi \text{Id} - E, \quad (\tilde{x}, \tilde{\xi}) \notin \text{WF}(E), \quad \phi(\tilde{x}) \neq 0.
\end{equation}

Denoting the set of such ‘elliptic points’ of $P$ as $\text{Ell}(P) \subset S^{n-1} \times \Omega$ conclude that for any $u \in C^\infty_c(\Omega)$,

\begin{equation}
\text{WF}(u) \cap \text{Ell}(P) \subset \text{WF}(P(x, D)u) \subset \text{WF}(u).
\end{equation}
5 Proof of properties of pseudodifferential operators

We start with \([P1]\). Since \(y_j\) is a multiplier on \(S(\mathbb{R}^n)\) it follows immediately that
\[ q_R(a)y_j = q_R(ay_j) \in \Psi^m_{cl,S}(\mathbb{R}^n) \]; this applies to any such multiplier, i.e. any smooth function of slow growth so for instance
\[ a(D,y)(y)^M \in \Psi^m_{cl,S}(\mathbb{R}^n) \forall M. \]

For a convolution operator \(a(D)\) where \(a \in S^m(\mathbb{R}^n)\) we know that
\[ [x_j, A] = x_j a(D) - a(D)y_j = b(D), \quad b(\xi) = i\partial_{\xi_j} a(\xi). \]

Since \(\partial_{\xi_j} : S^m_{cl,S}(\mathbb{R}^n) \rightarrow S^{m-1}_{cl,S}(\mathbb{R}^n)\), the convergence of the operator series (2.18) shows that
\[ x_j a(D, y) = b(D,y), \quad b(\xi, y) = i\partial_{\xi_j} a(\xi, y). \]

Thus \(x_j a(D, y) \in \Psi^m_{cl,S}(\mathbb{R}^n)\) as well and, iterating, it follows that \(x^\alpha a(D, y) \in \Psi^m_{cl,S}(\mathbb{R}^n)\). Thus
\[ x^\alpha a(D, y)(y)^M : H^s(\mathbb{R}^n) \rightarrow H^{s-m}(\mathbb{R}^n), \quad \forall \alpha, M, s. \]

This shows the first part of (3.2) and the second part follows from the fact that
\[ S(\mathbb{R}^n) = \bigcap_m \langle x \rangle^{-m} H^m(\mathbb{R}^n), \quad S(\mathbb{R}^n) = \bigcup_m \langle x \rangle^{-m} H^m(\mathbb{R}^n). \]

The injectivity of \(q_R\) in \([P2]\) is a weak form of the Schwartz kernel theorem, here just amounting to the fact that \(a\) can be recovered from the kernel of the operator \(q_R(a)\). The kernel of \(q_R(a)\) is given by \(B\) in (2.20). Using (2.21) we can recover the expansion of the symbol by taking the Hermite expansion of \(A(z, y)\) in \(y\) and then the Fourier transform of the resulting distribution.

From this the exactness of the symbol sequence in (3.4) follows with \(\sigma_m(q_R(a)) = |\xi|^m a_m, a_m = x^m a|_{S^{-m-1}S^{m-1}}\) being the restriction of \(a \in x^{-m}C^\infty(\mathbb{R}^n; S(\mathbb{R}^n))\) to the boundary. That is to say, after removal of a factor of \(\eta_{n+1}\) just arises from the exact ‘restriction sequence’
\[ \eta_{n+1}C^\infty(\mathbb{R}^n; S(\mathbb{R}^n)) \rightarrow C^\infty(\mathbb{R}^n; S(\mathbb{R}^n)) \rightarrow C^\infty(S^{n-1}; S(\mathbb{R}^n)). \]

Exercise 10. (Maybe on a second reading ...) Show that for the more general operators
\[ \Psi^m_S(\mathbb{R}^n) = q_R(S^m_S(\mathbb{R}^n; \mathbb{R}^n)) \]
defined by ‘non-classical symbols’ the discussion above carries over except that the principal symbol sequence has to be interpreted in terms of a quotient
\[ \Psi^{m-1}_S(\mathbb{R}^n) \rightarrow \Psi^m_S(\mathbb{R}^n) \xrightarrow{\sigma_m} (S^m_S / S^{m-1}_S)(\mathbb{R}^n; \mathbb{R}^n) \]

Go through the rest of the proofs checking what happens in the non-classical case.

Next consider the ‘residual algebra’ defined to be
\[ \Psi^{\infty}_S(\mathbb{R}^n) = \bigcap_k \Psi^{m-k}_{cl,S}(\mathbb{R}^n), \quad m \in \mathbb{R} \implies \Psi^{-\infty}_S(\mathbb{R}^n) = \bigcap_M \Psi^M_{cl,S}(\mathbb{R}^n) \]
\[ B : \Psi^{-\infty}_S(\mathbb{R}^n) \leftarrow S(\mathbb{R}^{2n}) \]
where \(B\) is the map to the Schwartz kernel. Here the first part is the definition and the claim is that the result is independent of \(m\) hence the second part holds.

So, fix \(m\) and consider \(q_R(a) \in \Psi^{\infty}_S(\mathbb{R}^n)\). The symbol map, applied iteratively,
shows that all terms in the Taylor series of $a$ at the boundary of $\mathbb{B}^n$ vanish. Thus $a \in \mathcal{C}^\infty(\mathbb{B}^n; \mathbb{R}^n)$. This in turn means that $F^*a \in \mathcal{S}(\mathbb{R}^{2n})$ and so the kernel of $q_R(a)$ is given by (2.20) where the series now converges in $\mathcal{S}(\mathbb{R}^{2n})$. Conversely, taking a kernel $B \in \mathcal{S}(\mathbb{R}^{2n})$ and writing it as $B(x, y) = A(x - y, y)$ with $A \in \mathcal{S}(\mathbb{R}^{2n})$ the inverse Fourier transform in $z$ of $A(z, y)$ is a ‘trivial’ symbol $F^*a$ with $a \in \mathcal{C}^\infty(\mathbb{B}^n; \mathcal{S}(\mathbb{R}^n))$ such that $q_R(a)$ has kernel $B$. This proves (5.7) and hence (P8).

Asymptotic completeness as in (P6) is then a form of Borel’s Lemma. Namely, if $A_k = q_R(a_k) \in \Psi_{cl,S}^{n-f}(\mathbb{R}^n)$ then $x_n^m a_k \in x_n^k C^\infty(\mathbb{B}^n; \mathcal{S}(\mathbb{R}^n))$. An appropriate form of Borel’s Lemma shows that there exists $b \in \mathcal{C}^\infty(\mathbb{B}^n; \mathcal{S}(\mathbb{R}^n))$ such that $b - \sum_{k=0}^N x_n^m a_k \in x_n^N C^\infty(\mathbb{B}^n; \mathcal{S}(\mathbb{R}^n))$. Then $A = q_R(x_n^m b)$ satisfies (5.7). From the discussion of the residual algebra above, the difference of two such ‘asymptotic sums’ is in $\Psi_{cl,S}^{n-f}(\mathbb{R}^n)$ for all $N$ and hence in $\Psi_{S}^{\infty}(\mathbb{R}^n)$.

Next consider the quantization map $q$ with the objective being to show that it has range contained in that of $q_R$; this is really the crucial result here. The map $q$ turns a symbol $F^*a$, $a \in x_n^m C^{1}I(\mathbb{B}^n; \mathcal{S}(\mathbb{R}^{2n}))$ into an operator $q(a)$ by expanding it in Hermite series on $\mathbb{R}^{2n}$

$$a(x, y, \xi) = \sum_{\gamma, \gamma'} e_\gamma(x) a_{\gamma, \gamma'}(\hat{\xi}) e_{\gamma'}(y).$$

(5.8)

Then each term becomes the operator

$$u \mapsto q(e_\gamma a_{\gamma, \gamma'} e_{\gamma'}) u = e_\gamma(x)(A_{\gamma, \gamma'}) e_{\gamma'} u.$$  

(5.9)

Then we use the convergence of (5.8) to sum (5.9) over $\gamma, \gamma'$ in $\mathcal{B}(H^s, H^{s-m})$ for any $s$. The continuity of this operator can be expressed by

$$\|q(a)\|_{\mathcal{B}(H^s, H^{s-m})} \leq \|a\|_{s,m}$$

(5.10)

for some continuous seminorm $\|\cdot\|_{s,m}$ on $x_n^m C^\infty(\mathbb{B}^n; \mathcal{S}(\mathbb{R}^{2n}))$. The Schwartz kernel is given by the corresponding series (2.31).

Given $a \in S_{cl,S}^{n}(\mathbb{R}^n; \mathbb{R}^{2n})$ we ‘prepare’ it by writing

$$a = a_1 + a_2, \quad F^*a_1(x, \cdot, y) = \hat{A}(x, \cdot, y), \quad F^*a_2(x, \cdot, y) = \hat{A}(x, \cdot, y),$$

$$A_1(x, z, y) = \phi(z) A(x, z, y), \quad \phi \in C^{\infty}(\mathbb{R}^n), \quad \phi(z) = 1 \text{ in } |z| < 1.$$  

(5.11)

That is, treating $z$ and $y$ as parameters we divide $a$ into a part for which $F^*a_1$ has Fourier transform compactly supported near 0 in the dual variable $z$. Now, the properties of symbols imply that $A_2 \in \mathcal{S}(\mathcal{S}^3)$ is Schwartz in all variables – being a Schwartz function of $z$ with values in the Schwartz functions in the other variables.

From

The Hermite expansion respects this division so

$$q(a) = q(a_1) + q(a_2), \quad q(a_2) \in \Psi_{S}^{\infty}(\mathbb{R}^n)$$

where the last statement follows from the formula for the Schwartz kernel and (5.7).

The continuity of (5.10) and the fact that it is explicitly defined on products $e(x) a(\hat{\xi}) f(y)$, $e, f \in \mathcal{S}(\mathbb{R}^n), a \in C^\infty(\mathbb{B}^n)$ means that we can use any expansion as in (5.8).

$$a(x, \hat{\xi}, y) = \sum_{\gamma, \gamma'} L_\gamma(x) a_{\gamma, \gamma'}(\hat{\xi}) e_{\gamma'}(y)$$

(5.13)
provided it converges in $x_n^{-m}C^\infty(B^n; \mathcal{S}(\mathbb{R}^{2n}))$ and has the correct limit. In particular this shows that

(5.14) \[ [q_i, q(a)] = q((-i\partial_{\xi_i})a) = q(-i\partial_{\xi_i}a), \quad \partial_{\xi_i}a \in x_n^{-m+1}C^\infty(B^n; \mathcal{S}(\mathbb{R}^{2n})) \]

comes from a symbol one order lower. Iterating this statement we see that

(5.15) \[ q((x_i - y_i)^{\alpha}a) = q((-i\partial_{\xi_i})^{\alpha}a), \quad \partial_{\xi_i}^n a \in x_n^{-m+|\alpha|}C^\infty(B^n; \mathcal{S}(\mathbb{R}^{2n})) \]

We showed that $q(a_2) \in \Psi^{-m}_S(\mathbb{R}^n)$ so are reduced to examining $q(a_1)$ for the first part $a_1$ in (5.11); let me again denote this $a$ to simplify the notation but now we know that it is ‘prepared’ in the sense that

(5.16) \[ q(a(\cdot, x, y)) = q(a(\cdot, x, y)\phi(x - y)), \quad \phi \in C^\infty_c(\mathbb{R}^n) \]

since this formula holds for the Schwartz kernels.

Consider Taylor’s formula with remainder for $g(\cdot, x, y) = g^{m+1}(\cdot, x, y)$ (just removing the leading factor to make everything smooth) around $x = y$:

(5.17) \[ g(\cdot, x, y) = \sum_{|\alpha| \leq N} (x - y)^{\alpha}g_\alpha(\cdot, y) + \sum_{|\alpha| = N+1} (x - y)^{\alpha}h_\alpha(\cdot, x, y). \]

This identity only holds in $|x - y| < 2$. If we insert the cutoff we see that

(5.18) \[ a(\cdot, x, y)\phi(x - y) = \sum_{|\alpha| \leq N} (x - y)^{\alpha}a_\alpha(\cdot, y)\phi(x - y) + \sum_{|\alpha| = N+1} (x - y)^{\alpha}b_\alpha(\cdot, x, y)\phi(x - y) \]

where $a_\alpha = \eta^{m}_{n+1}g_\alpha, b_\gamma = \eta^{m}_{n+1}b_\gamma$ have the leading powers restored. Now all terms are in the symbol space $S^m_S(B^n; \mathbb{R}^{2n})$ and the identity (5.18) holds globally.

The identities (5.16) and (5.15) show that

(5.19) \[ q(a) = q(\phi(x - y)a) = \sum_{|\alpha| \leq N} q((-i\partial_{\xi})^{\alpha}a_\alpha(\cdot, y)\phi(x - y)) \]

+ \[ \sum_{|\alpha| = N+1} q((-i\partial_{\xi})^{\alpha}b_\alpha(\cdot, x, y)\phi(x - y)) \]

for any $N$. As noted above, the Schwartz kernel an operator such as

\[ q((-i\partial_{\xi})^{\alpha}a_\alpha(\cdot, y)\phi(x - y) \]

can be computed from any convergent expansion (5.13) for its symbol. Set

(5.20) \[ f_\alpha(\cdot, y) = (-i\partial_{\xi})^{\alpha}a_\alpha(\cdot, y) \in S^m_{S_S}(B^n; \mathbb{R}^n). \]

Then taking the expansion of $f_\alpha(\cdot, y)$ as usual in a ‘single’ Hermite expansion and multiplying by $\phi(x - y)$

(5.21) \[ f_\alpha(\cdot, y)\phi(x - y) = \sum_{\gamma} \gamma e_{\gamma}(y)\phi(x - y) \]

converges in $S^m_{S_S}(B^n; \mathbb{R}^{2n})$. We can further expand each $e_{\gamma}(y)\phi(x - y) = \in \mathcal{S}(\mathbb{R}^{2n})$ in a double Hermite series and get a rapidly converging expansion as in (5.13) with more indices. Summing the series for the Schwartz kernel over this series first we find that the kernel of $f_\alpha(\cdot, y)\phi(x - y)$ is

(5.22) \[ B_\alpha(x, y) = \sum_{\gamma} E_{\alpha, \gamma}(x - y)e_{\gamma}(y)\phi(x - y), \quad E_{\alpha, \gamma} = f_{\alpha, \gamma}. \]
However, the preparation of the symbol above means that all the $E_{\alpha,\gamma}(z)$ are supported in $|z| < 1$ so in fact
\begin{equation}
q(f_\alpha(\cdot, y)\phi(x - y)) = q_R(f_\alpha(\cdot, y))
\end{equation}
by equality of the corresponding Schwartz kernels.

Thus, for a prepared symbol the sum on the right in (5.19) is of the form
\begin{equation}
\sum_{|\alpha| \leq N} A_\alpha, \quad A_\alpha = q_R(a_\alpha) \in \Psi_{cl, S}^{m-|\alpha|}(\mathbb{R}^n).
\end{equation}
So now we may invoke the asymptotic summability result shown above to find one $A \in \Psi_{cl, S}^{m}(\mathbb{R}^n)$ such that
\begin{equation}
A \sim \sum_{|\alpha|} A_\alpha.
\end{equation}
It follows now from (5.19) again that for all $N$
\begin{equation}
E = q(a) - A = q(a\phi(x - y)) - A = A(N) + \sum_{|\alpha| = N+1} q((-i\partial)^\alpha b_\alpha(\cdot, x, y)\phi(x - y)), \quad A(N) \in \Psi_{cl, S}^{m-N-1}(\mathbb{R}^n).
\end{equation}
Here there is a single operator, so with a fixed Schwartz kernel, expressed in terms of symbols all of order $m - N - 1$. The regularity of the kernels in (2.32), together with the fact that they are all supported in $|x - y| < 1$, shows that $E$ has kernel in $\mathcal{S}(\mathbb{R}^{2n})$ finally we have shown that
\begin{equation}
q(a) = q_R(\tilde{a})
\end{equation}
and the range of $q$ is contained in the range of $q_R$.

The converse, that the range of $q_R$ is contained in that of $q$, is also readily checked by a similar argument. Namely 'preparing' the symbol $a \in S_{cl, S}^m(\mathbb{R}^n; \mathbb{R}^n)$ as in (5.11) allows us to insert a cutoff $\phi(x - y)$ to see that $q_R(a) = q(\phi(x - y)a) + e$ where $e$ is residual. Thus it is enough to show that any residual operator is in the range of $q$ which (is not very important and) follows from the fact that any $B \in \mathcal{S}(\mathbb{R}^n)$ (its kernel) is the restriction to $z = x - y$ of $\tilde{B}(x, z, y) \in \mathcal{S}(\mathbb{R}^{3n})$ and so is the kernel of $q(b)$ where $b \in \mathcal{C}^\infty(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^{2n}))$ is obtained from $\tilde{B}$ by inverse Fourier transform in $z$.

Directly from the definitions taking adjoints, with respect to Lebesgue measure as usual, interchanges $q_R$ and $q_L$:
\begin{equation}
(q_R(a))^* = q_L(\tilde{a}) \implies \sigma_m(q_R(a))^* = \sigma_m(q_L(\tilde{a}))
\end{equation}
which is [P5]. Using this we conclude that the range of $q_R$ is contained in that of $q_L$, so these are equal.

The fundamental property is multiplicativity but this now follows easily. Namely if $A \in \Psi_{cl, S}^m(\mathbb{R}^n)$, $B \in \Psi_{cl, S}^{m'}(\mathbb{R}^n)$ the composite $A \circ B$ is certainly well-defined as an operator on $\mathcal{S}(\mathbb{R}^n)$. Moreover we now know that we can write $A = q_L(a), \quad B = q_R(b)$ and then from the definitions (using the series expansions and the composition of convolution operators in the middle)
\begin{equation}
A \circ B = q(a(\cdot, x)b(\cdot, y)) \in \Psi_{cl, S}^{m+m'}(\mathbb{R}^n) \implies \sigma_m+m'(A \circ B) = \sigma_m(A)\sigma_m'(B).
\end{equation}
This is [P4] with the formula for the principal symbol following from

This, I think, only leaves [P7] which is a good exercise!
6 Diffeomorphism-invariance

For any open set $\Omega \subset \mathbb{R}^n$ we can consider the

\begin{equation}
\Psi^m_{cl,c}(\Omega) \subset \Psi^m_{cl,S}(\mathbb{R}^n)
\end{equation}

with the defining support property

\begin{equation}
A \in \Psi^m_{cl,S}(\mathbb{R}^n) \text{ s.t. } A^* : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{C}_c^\infty(\Omega) \subset \mathcal{S}(\mathbb{R}^n).
\end{equation}

Lemma 5. The defining condition (6.2) that $A \in \Psi^m_{cl,c}(\Omega)$ is equivalent to the condition that the Schwartz kernel $B \in \mathcal{S}'(\mathbb{R}^{2n})$ of $A$ have compact support within the product

\begin{equation}
\text{supp}(B) \subseteq \Omega \times \Omega.
\end{equation}

Proof. \hfill \Box

Proposition 4. If $F : \Omega \rightarrow \Omega'$ is a diffeomorphism between open subsets of $\mathbb{R}^n$ with inverse $G : \Omega' \rightarrow \Omega$ then

\begin{equation}
A_F u = F^*(AG^* u), \quad u \in \mathcal{C}_c^\infty(\Omega) \text{ defines a bijection}
\end{equation}

\begin{equation}
\Psi^m_{cl,c}(\Omega') \ni A \mapsto A_F \in \Psi^m_{cl,c}(\Omega).
\end{equation}

7 Localization

We define the large space $\Psi^m_{cl}(\Omega)$ as consisting of continuous linear operators

\begin{equation}
A : \mathcal{C}_c^\infty(\Omega) \rightarrow \mathcal{C}_c^\infty(\Omega) \text{ s.t. } A\phi = \sum_{i,j} \chi_i A_{ij} \chi_j, \quad A_{ij} \in \Psi^m_{cl,S}(\mathbb{R}^n)
\end{equation}

for one (and hence any) partition of unity $\chi_i \in \mathcal{C}_c^\infty(\Omega)$ for $\Omega$ with locally finite supports (meaning that for any $K \subseteq \Omega$ there are only finitely many $j$ such that $\text{supp}(\chi_j) \cap K \neq \emptyset$).

\begin{equation}
\Psi^m_{cl,c}(\Omega) \subset \Psi^m_{cl,P}(\Omega) \subset \Psi^m_{cl}(\Omega)
\end{equation}

References