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Fredholm and Trace class operators

Parial isometries

A unitary operator is an inner-product preserving bijection – possibly between two different Hilbert spaces. More generally a *partial isometry* is a bounded operator $V : H_1 \longrightarrow H_2$ which satisfies

$$\|Vu\|_{H_2} = \|u\|_{H_1} \ \forall \ u \in (Nul(V)^{\perp}. \tag{1}$$

It follows directly that the range is closed and the resulting map $V' : (\mathrm{Nul}(V)^{\perp} \longrightarrow \mathrm{Ran}(V) \subset H_2$, is a norm-preserving bijection and hence is a unitary isomorphism, since

$$\langle V'u, V'u\rangle = \langle u, u\rangle \Longrightarrow (V')^*V' = \mathrm{Id}_{\mathrm{Nul}(V)^{\perp}},\ V'(V')^* = \mathrm{Id}_{\mathrm{Ran}(V)}\,. \tag{2}$$

One of the important examples of partial isometry is the shift operator. If e_i, $i \geq 0$, is an orthonormal basis of a separable Hilbert space then there are uniquely defined bounded operators S and T determined by

$$Se_i = e_{i+1} \ \forall \ i \geq 0,\ Te_j = e_{j-1} \ \forall \ j \geq 1,\ Te_0 = 0. \tag{3}$$

Polar decomposition

If $B \in \mathcal{B}(H)$ then we can think of B as a 2×2 matrix between the two decompositions $H = (\mathrm{Nul}\, B)^{\perp} \oplus (\mathrm{Nul}\, B)$ with π_N projection off the null space and $\overline{\mathrm{Ran}(B)} \oplus \mathrm{Nul}(B^*)$ with π_R projection onto the closure of the range:

$$B = \begin{pmatrix} \pi_R B \pi_N & 0 \\ 0 & 0 \end{pmatrix}. \tag{4}$$

Then B^*B is self-adjoint and can be considered as an operator on $\mathrm{Nul}(B)^{\perp}$. Since it is non-negative, its square-root $A = (B^*B)^{\frac{1}{2}} \in \mathcal{B}(H)$ is also well-defined as an operator on H which maps $\mathrm{Nul}(B)^{\perp}$ to itself and is otherwise zero. Then we can define a linear map

$$V : \mathrm{Ran}(B) \longrightarrow \mathrm{Nul}(B)^{\perp},\ V(Bu) = Au.$$

In fact V extends by continuity to a bounded operator on the closure since

$$\|V(Bu)\|^2 = \|Au\|^2 = \langle Au, Au\rangle = \langle u, B^*Bu\rangle = \langle Bu, Bu\rangle = \|Bu\|^2.$$

So in fact $V : \overline{\mathrm{Ran}(B)} \longrightarrow \mathrm{Nul}(B)^\perp$ is an isometry. If we extend it as zero to $\mathrm{Ran}(B)^\perp$ and take $W = V^*$ we get the polar decomposition of B :

(5)

$$B = WA,\ A \text{ non-negative and } W \text{ a partial isometry with } \mathrm{Ran}(W) = \overline{\mathrm{Ran}(B)}.$$

Fredholm operators

An operator, $B \in \mathcal{B}(H)$, is said to be Fredholm if

(1) It has a finite dimensional null space $\mathrm{Nul}(B) \subset H$.
(2) It has closed range, $\overline{\mathrm{Ran}(B)} = \mathrm{Ran}(B)$
(3) It adjoint has a finite dimensional null space $\mathrm{Nul}(B^*)$.

Of course we know that $H = \overline{\mathrm{Ran}(B)} \oplus \mathrm{Nul}(B^*)$ so the second two conditions can be combined by saying that the range has a finite-dimensional complement. It is just that this is a little too easy to interpret as saying that the closure of the range has a finite dimensional complement, which is by no means enough to guarantee that the operator is Fredholm.

The operators of the form $\mathrm{Id} + K$, $K \in \mathcal{K}(B)$ compact are Fredholm. Indeed, the null space of $\mathrm{Id} + K$ consists of vectors $v \in H$ such that $v + Kv = 0$ which means that the unit ball in $\mathrm{Nul}(\mathrm{Id} + K)$ is mapped onto itself by K, and hence is precompact – so the null space is finite dimensional.

Since K is compact there is a finite rank operator F such that $\|F - K\| < \frac{1}{2}$. Then

$$(\mathrm{Id} + K) = \mathrm{Id} + (K - F) + F = (\mathrm{Id} + F(\mathrm{Id} + K - F)^{-1})(\mathrm{Id} + K - F). \tag{6}$$

So $\mathrm{Id} + K$ has the same range as $\mathrm{Id} + F'$ where $F' = F(\mathrm{Id} + K - F)^{-1}$ is finite rank. So the range contains the null space of F' which is a closed subspace of finite codimension, so itself is closed and of finite codimension.

These examples proved a useful restatement of the Fredholm property.

Lemma 1. *An operator $B \in \mathcal{B}(H)$ is Fredholm if and only if it has a right and a left inverse modulo compact operators, i.e. there exist L and $R \in \mathcal{B}(H)$ such that*

$$LB = \mathrm{Id} + K_L,\ BR = \mathrm{Id} + K_R,\ K_L,\ K_R \in \mathcal{K}(H). \tag{7}$$

Note that as is the case for left and right inverses in a ring, in this case if B has both a left and a right inverse modulo compacts (an ideal) then either of them is both a left and a right inverse modulo that ideal. Namely, they differ by a compact operator:

$$L = L(BR)-LK_R = (LB)R-LK_R = R+K_LR-LK_R \Longrightarrow L-R \in \mathcal{K}(H). \tag{8}$$

It follows from this criterion that the adjoint of a Fredholm operator is Fredholm and that the product of two Fredholm operators is Fredholm. It also follows directly that if $A,\ B \in \mathcal{B}(H)$ and the product $AB \in \mathcal{F}(H)$ then both A and $B \in \mathcal{F}(H)$. This is all saying that the property of being Fredholm is a sort of invertibility, namely invertibility modulo compact operators.

Proof. The existence of a right inverse modulo compacts shows that

$$\operatorname{Ran}(B) \supset \operatorname{Ran}(BR) = \operatorname{Ran}(\operatorname{Id}+K) \text{ is closed with finite-dimensional complement} \tag{9}$$

and hence so is $\operatorname{Ran}(B)$. Similarly, the existence of a left inverse modulo compact implies that

$$\operatorname{Nul}(B) \subset \operatorname{Nul}(LB) = \operatorname{Nul}(\operatorname{Id}+KL) \text{ is finite dimensional.} \tag{10}$$

Conversely, if B is Fredholm then it defines a continuous bijection between the Hilbert spaces

$$B : (\operatorname{Nul}(B))^\perp \longrightarrow \operatorname{Ran}(B) \tag{11}$$

which therefore has a continuous inverse $G : \operatorname{Ran}(B) \longrightarrow (\operatorname{Nul}(B))^\perp \subset H$. Extending G as zero to $(\operatorname{Ran}(B))^\perp$ therefore gives a bounded operator and

$$GB = \operatorname{Id} - \Pi_{\operatorname{Nul}(B)},\ \ BG = \operatorname{Id} - \Pi_{\operatorname{Nul}(B^*)}. \tag{12}$$

This is the *generalized inverse* of B and is certainly both a left and right inverse modulo compacts. It also follows that B is Fredholm, since B is its generalized inverse. □

Proposition 1. *The Fredholm operators $\mathcal{F}(H) \subset \mathcal{B}(H)$ form an open set with components labelled by the integer*

$$\operatorname{ind}(B) = \dim \operatorname{Nul}(B) - \dim \operatorname{Nul}(B^*) \tag{13}$$

and which is stable under the addition of compact operators,

$$\mathcal{F}(H) + \mathcal{K}(H) = \mathcal{F}(H). \tag{14}$$

Partial proof, not including index. The fact that the Fredholm operators form an open set follows directly from Lemma 1 above. Namel, if

G is the generalized inverse of a Fredholm operator B and $A \in \mathcal{B}(H)$ has $\|A\| < 1/\|G\|$ then

(15)
$$G(B+A) = \operatorname{Id} + GA - \Pi_{\operatorname{Nul}(B)} \Longrightarrow ((\operatorname{Id} + GA)^{-1}G)(B+A) = \operatorname{Id} - (\operatorname{Id} + GA)^{-1}\Pi_{\operatorname{Nul}(B)},$$
$$(B+A)G = \operatorname{Id} + AG - \Pi_{\operatorname{Nul}(B^*)} \Longrightarrow (B+A)(G(\operatorname{Id} + AG)^{-1}) = \operatorname{Id} - \Pi_{\operatorname{Nul}(B^*)}(\operatorname{Id} + AG)^{-1}$$

gives left and right inverses for $B + A$ modulo compact (in fact finite rank) operators.

Similarly (14) holds since an inverse modulo compact operators for B is also an inverse modulo compact operators for $B + K$ for any $K \in \mathcal{K}(H)$. $\square$

It is certainly possible to go through the remainder of the proof, which is to show that two Fredholm operators B_0 and B_1 have the same index (13) if and only if they are in the same component of $\mathcal{F}(H)$, i.e. can be joined by a curve, a continuous map $B : [0, 1] \longrightarrow \mathcal{F}(H)$, with $B(0) = B_0$ and $B(1) = B_1$. However, this is a bit painful without using the trace functional, so I will complete the proof of Proposition 1 after discussing the trace and trace-class operators.

Hilbert-Schmidt operators

If H is a separable Hilbert space then an operator $A \in \mathcal{B}(H)$ is *Hilbert-Schmidt* if for (any) one orthonormal basis $\{e_i\}$,

$$\|A\|^2_{\text{HS}} = \sum_i \|Ae_i\|^2 < \infty. \tag{16}$$

Using Bessel's identity and the fact that the (double) series are absolutely convergent, if f_j is another orthonormal basis then

(17)
$$\sum_i \|Ae_i\|^2 < \infty = \sum_i \sum_j |\langle Ae_i, f_j\rangle|^2 = \sum_i \sum_j |\langle e_i, A^* f_j\rangle|^2 = \sum_j \|A^* f_j\|^2.$$

Applying this twice shows that the sum defining (16) is indeed independent of the orthonormal basis used and also that A^* is Hilbert-Schmidt. It follows that the Hilbert-Schmidt operators form a 2-sided $\star$-ideal $\text{HS}(H)$ since if $B \in \mathcal{B}(H)$, $\|BAe_i\| \le \|B\|\|Ae_i\|$ so

$$\|BA\|_{\text{HS}} \le \|B\|\|A\|_{\text{HS}},\ AB = (B^*A^*)^*. \tag{18}$$

In fact the Hilbert-Schmidt operators form a Hilbert space, with the inner product given in terms of any choice of orthonormal basis by

$$\langle A_1, A_2\rangle_{\text{HS}} = \sum_i \langle A_1 e_1, A_2 e_i\rangle. \tag{19}$$

The completeness follows from the fact that the finite-rank operators are dense in HS(H), since directly from (16) $A\Pi_N \to A$, in HS(H) where Π_N is projection onto the span of the first N elements of the orthonormal basis $\{e_i\}$ and that

$$\|A\| \le \|A\|_{\mathrm{HS}}.$$

Indeed, there is a sequence $u_i \in H$ with $\|u_i\| = 1$ such that $\|A\| = \lim_i \|Au_i\|$ and each u_i can be extended to an orthonormal basis, so $\|Au_i\| \le \|A\|_{\mathrm{HS}}$. Thus of A_n is Cauchy in HS(H) it is Cauchy, and hence convergent, in norm and the limit, from the boundedness of $\|A_n\|_{\mathrm{HS}}$ is in HS(H) and the sequence converges to it in HS(H). Of course it also follows from this, and the density of finite rank operators in HS(H) that HS(H) $\subset \mathcal{K}(H)$.

Exercise 1. Show that for the case of $H = L^2(\mathbb{R}^n)$ that HS(H) may be identified with $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ with the function on the product giving the operator as an integral operatorL

$$Au(x) = \int_{\mathbb{R}^n} A(x,y)u(y)dy. \tag{20}$$

Trace class operators

Now the 'trace class' operators are by definition those compact operators which can be written in the form of a finite sum of products of Hilbert-Schmidt operators

$$T = \sum_{i=1}^{N} A_i B_i,\ A_i,\ B_i \in \mathrm{HS}(H). \tag{21}$$

This is a more complicated definition since the presentation (21) is clearly not unique. On the other hand it follows that TC(H) is a 2-sided $\star$-ideal contained in the compact operators.

Now, notice that using a presentation (21) it follows that for *any* two orthonormal bases e_j and f_j,

$$\sum_j |\langle Te_j, f_j\rangle| \le \sum_j \sum_i |\langle B_i e_j, A_i^* f_j\rangle| \le \sum_i \|B_i\|_{\mathrm{HS}} \|A_i\|_{\mathrm{HS}}. \tag{22}$$

Thus for any $T \in \mathrm{TC}(H)$,

$$\|T\|_{\mathrm{TC}} = \sup \sum_j |\langle Te_j, f_j\rangle| < \infty. \tag{23}$$

Here the supremum is over all pairs of orthonormal bases. That this is indeed a norm follows directly – with the triangle inequality following from the fact that for any pair of orthonormal bases

$$\sum_j |\langle (T_1+T_2)e_j, f_j\rangle| \le \|T_1\|_{\mathrm{TC}} + \|T_2\|_{\mathrm{TC}}. \tag{24}$$

Note that as for the Hilbert-Schmidt norm

$$\|T\| = \sup_{\|u\|=\|v\|=1} |\langle Tu, v\rangle| \le \|T\|_{\mathrm{TC}}. \tag{25}$$

Again by looking at TP_N it follows that $\mathrm{TC}(H) \subset \mathcal{K}(H)$.

Now, in fact $\mathrm{TC}(H)$ is a Banach space with respect to the trace norm (23). To see this, apply the polar decomposition to some T for which the norm (23) is finite:

$$T = VA,\ A = (T^*T)^{\frac{1}{2}}. \tag{26}$$

Then let f_j be an orthonormal basis containing the eigenvectors for A (which is compact) and e_j one that includes the Ve_j. It follows that if $\lambda_j \ge 0$ are the eigenvalues of A repeated with multiplicity then

$$\sum_j \lambda_j \le \|T\|_{\mathrm{TC}} < \infty. \tag{27}$$

Then if we define $E = A^{\frac{1}{2}}$, with eigenvalues $\lambda_j^{\frac{1}{2}}$ and use an orthonormal basis including an eigenbasis, $E \in \mathrm{HS}(H)$ since (27) is just the Hilbert-Schmidt norm of E. Thus in fact

$$T = (VE)E \in \mathrm{TC}(H). \tag{28}$$

Summing all this up we have shown most of:

Proposition 2. *The operators for which the Trace Class norm* (25) *is finite form the Banach space of Trace class operators, which can be written in the form* (21) *or* (28) *and satisfy* $\|BT\|_{\mathrm{TC}} \le \|B\|\|T\|_{\mathrm{TC}}$; *the trace functional*

$$\mathrm{Tr}(T) = \sum_i \langle Te_i, e_i \Longrightarrow \tag{29}$$

is continuous on $\mathrm{TC}(H)$, *independent of the orthonormal basis used to define it and satisfies*

$$\mathrm{Tr}([T,B]) = 0 \text{ if } T \in \mathrm{TC}(H),\ B \in \mathcal{B}(H). \tag{30}$$