

**THE PROBLEMS FOR THE SECOND TEST FOR 18.102
WILL BE SELECTED FROM THIS LIST**

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Question.1

Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of ‘finite dimensional approximation’ meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u \in K} \inf_{v \in D_N} \{d(u, v)\} \leq \epsilon.$$

Question.2

Strong convergence of a sequence of bounded operators $A_n \in \mathcal{B}(H)$ means that for each $u \in H$, $A_n u$ converges in H . Show that $Au = \lim_n A_n u$ is necessarily a bounded linear operator on H (called the strong limit of the sequence).

Question.3

If H is a separable, infinite dimensional, Hilbert space set

$$(1) \quad l^2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|_{l^2(H)}^2 = \sum_i \|u_i\|_H^2 < \infty\}.$$

Show that $l^2(H)$ has a Hilbert space structure and construct an explicit isometric (norm-preserving) isomorphism (bijection) from $l^2(H)$ to H .

Question.4

Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if

$$(2) \quad \|v\|_H = \lim_{n \rightarrow \infty} \|v_n\|_H$$

where v is the weak limit.

Question.5

Let e_k , $k \in \mathbb{N}$, be an orthonormal basis in a separable Hilbert space, H . Show that there is a uniquely defined bounded linear operator $T : H \longrightarrow H$, satisfying

$$(3) \quad Te_j = e_{j-1} \quad \forall j \geq 2, \quad Te_1 = 0,$$

and that $T + B$ has one-dimensional null space if B is bounded and $\|B\| < 1$.

Question.6

Show that a continuous function $K : [0, 1] \rightarrow L^2(0, 2\pi)$ has the property that the Fourier series of $K(x) \in L^2(0, 2\pi)$, for $x \in [0, 1]$, converges uniformly in the sense that if $K_n(x)$ is the sum of the Fourier series over $|k| \leq n$ then $K_n : [0, 1] \rightarrow L^2(0, 2\pi)$ is also continuous and

$$(4) \quad \sup_{x \in [0, 1]} \|K(x) - K_n(x)\|_{L^2(0, 2\pi)} \rightarrow 0.$$

Question.7

Prove that for appropriate constants d_k , the functions $d_k \sin(kx/2)$, $k \in \mathbb{N}$, form an orthonormal basis for $L^2(0, 2\pi)$.

Question.8

Show that a separable Hilbert space in which every bounded operator is compact is finite dimensional.

Question.9

Show that if B is a compact operator on a separable Hilbert space H and A is an invertible operator then

$$(5) \quad \{u \in H; Bu = Au\}$$

is finite dimensional.

Question.10

Show that there is a complete orthonormal basis of $L^2([0, 2\pi])$ consisting of polynomials.

Question.11

Let H be a separable Hilbert space and let $\mathcal{C}(\mathbb{R}; H)$ be the linear space of continuous maps from $\mathbb{R}$ to H which vanish outside some interval $[-R, R]$ depending on the function. Show that

$$(6) \quad \|u\|^2 = \int_{\mathbb{R}} \|u(x)\|_H^2$$

defines a norm which comes from a preHilbert structure on $\mathcal{C}(\mathbb{R}; H)$. Show that if u_n is a Cauchy sequence in this preHilbert space and $h \in H$ then $\langle u_n(x), h \rangle_H$ converges in $L^2(\mathbb{R})$.