PROBLEM SET 9 FOR 18.102, SPRING 2015
DUE 7AM SATURDAY 9 MAY.

RICHARD MELROSE

This of course is the last problem set for the semester; I believe it is reasonably short.

Problem 9.1

Work out the precise effect of the creation and annihilation operators on the Hermite basis – the orthonormal basis of eigenfunctions of the harmonic oscillator.

Problem 9.2

Define a subspace $H^1_{\text{iso}} \subset L^2(\mathbb{R})$ by the condition $u \in H^1_{\text{iso}}(\mathbb{R})$ if the coefficients with respect to the Hermite basis satisfy

$$\sum_{i=0}^{\infty}(2i+1)|\langle u, e_i\rangle|^2 < \infty.$$

Show that this has a norm with respect to which it is a Hilbert space and in which $\mathcal{S}(\mathbb{R})$ is dense.

Problem 9.3

Show that the creation and annilation operators extend from $\mathcal{S}(\mathbb{R})$ to define bounded operators $H^1_{\text{iso}}(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$.

Problem 9.4

Show that if $u \in H^1_{\text{iso}}(\mathbb{R})$ then $xu \in L^2(\mathbb{R})$ and $\xi\hat{u} \in L^2(\mathbb{R})$ where $\hat{u}$ is the Fourier transform of u.

Problem 9.5

Show conversely that if $u \in L^2(\mathbb{R})$, $xu \in L^2(\mathbb{R})$ and $\xi\hat{u} \in L^2(\mathbb{R})$ then $u \in H^1_{\text{iso}}(\mathbb{R})$.

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu