

18.102 QUESTIONS FOR FINAL EXAM, SPRING 2015

PROBLEM 1

Let $T : H_1 \longrightarrow H_2$ be a continuous linear map between two Hilbert spaces and suppose that T is both surjective and injective.

- (1) Let $A_2 \in \mathcal{K}(H_2)$ be a compact linear operator on H_2 , show that there is a compact linear operator $A_1 \in \mathcal{K}(H_1)$ such that

$$A_2 T = T A_1.$$

- (2) If A_2 is self-adjoint (as well as being compact) and H_1 is infinite dimensional, show that A_1 has an infinite number of linearly independent eigenvectors.

PROBLEM 2

Let a be a continuous function on the square $[0, 2\pi]^2$. Show that $[0, 2\pi] \ni x \longmapsto a(x, \cdot) \in C^0([0, 2\pi])$ is a continuous map into the continuous functions with supremum norm. Using this, or otherwise, show that the Fourier coefficients with respect to the second variable

$$(1) \quad c_k(x) = \frac{1}{2\pi} \int_0^{2\pi} a(x, t) e^{ikt} dt$$

are continuous functions on $[0, 2\pi]$.

PROBLEM 3

Let H_i , $i = 1, 2$ be two Hilbert spaces with inner products $(\cdot, \cdot)_i$ and suppose that $I : H_1 \longrightarrow H_2$ is a continuous linear map between them. Suppose that the range of I is dense and that I is injective.

- (1) Show that there is a continuous linear map $Q : H_2 \longrightarrow H_1$ such that $(u, I(f))_2 = (Qu, f)_1 \ \forall f \in H_1$.
 (2) Show that as a map from H_1 to itself, $Q \circ I$ is bounded, self-adjoint and injective.

PROBLEM 4

Let $u_n : [0, 2\pi] \longrightarrow \mathbb{C}$ be a sequence of continuously differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_n \sup_{x \in [0, 2\pi]} |u_n(x)| < \infty$ and $\sup_n \sup_{x \in [0, 2\pi]} |u'_n(x)| < \infty$. Show that u_n has a subsequence which converges in $L^2([0, 2\pi])$.

PROBLEM 5

Consider the subspace $H \subset \mathcal{C}[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$(2) \quad u(x) = \int_0^x U(x), \quad \forall x \in [0, 2\pi]$$

for some $U \in L^2(0, 2\pi)$ (depending on u of course). Show that the function U is determined by u (given that it exists) and that

$$(3) \quad \|u\|_H^2 = \int_{(0, 2\pi)} |U|^2$$

turns H into a Hilbert space.

PROBLEM 6

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant C (depending on the function) such that

$$(4) \quad |u(x) - u(y)| \leq C|x - y|^{\frac{1}{2}} \quad \forall x, y \in [0, 1].$$

Show that this is a Banach space with norm

$$(5) \quad \|u\|_{\frac{1}{2}} = \sup_{[0, 1]} |u(x)| + \inf_{(4) \text{ holds}} C.$$

PROBLEM 7

Let $A_j \subset \mathbb{R}$ be a sequence of subsets with the property that the characteristic function, χ_j of A_j , is integrable for each j . Show that the characteristic function of $\mathbb{R} \setminus A$, where $A = \bigcup_j A_j$ is locally integrable.

PROBLEM 8

Let A be a Hilbert-Schmidt operator on a separable Hilbert space H , which means that for some orthonormal basis $\{e_i\}$

$$(6) \quad \sum_i \|Ae_i\|^2 < \infty.$$

Using Bessel's identity to expand $\|Ae_i\|^2$ with respect to another orthonormal basis $\{f_j\}$ show that $\sum_j \|A^*f_j\|^2 = \sum_i \|Ae_i\|^2$. Conclude that the sum in (6) is independent of the orthonormal basis used to define it and that the Hilbert-Schmidt operators form a Hilbert space.

PROBLEM 9

Let B_n be a sequence of bounded linear operators on a Hilbert space H such that for each u and $v \in H$ the sequence $(B_n u, v)$ converges in $\mathbb{C}$. Show that there is a uniquely defined bounded operator B on H such that

$$(Bu, v) = \lim_{n \rightarrow \infty} (B_n u, v) \quad \forall u, v \in H.$$

PROBLEM 10

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space, with $\pi_P : H \rightarrow P$ the orthogonal projection onto P . If H is separable and A is a compact self-adjoint operator on H , show that there is a complete orthonormal basis of H each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$.

PROBLEM 11

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $j = 1, 2, \dots$, and $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. You may assume completeness in $L^2(\mathbb{R})$ and use the facts established in class that $-\frac{d^2 e_j}{dx^2} + x^2 e_j = (2j+1)e_j$, that $c_j = 2^{-j/2} (j!)^{-1/2} \pi^{-1/4}$ and that $e_j = p_j(x) e_0$ for a polynomial of degree j . Compute $C e_j$ and $A e_j$ in terms of the basis and hence arrive at a formula for de_j/dx . Use this to show that the sequence $j^{-1/2} \frac{de_j}{dx}$ is bounded in $L^2(\mathbb{R})$. Conclude that if

$$(7) \quad H_{\text{iso}}^1 = \{u \in L^2(\mathbb{R}); \sum_{j \geq 1} j |(u, e_j)|^2 < \infty\}$$

then there is a uniquely defined operator $D : H_{\text{iso}}^1 \rightarrow L^2(\mathbb{R})$ such that $D e_j = \frac{de_j}{dx}$ for each j .

PROBLEM 12

Suppose that $f \in \mathcal{L}^1(0, 2\pi)$ is such that the constants $c_k = \int_{(0, 2\pi)} f(x) e^{-ikx}$, $k \in \mathbb{Z}$, satisfy $\sum_{k \in \mathbb{Z}} |c_k|^2 < \infty$. Show that there is an element $g \in L^2([0, 2\pi])$ such that $\int (f - g) e^{ikx} dx = 0$ for all $k \in \mathbb{Z}$.

PROBLEM 13

Let A be a compact self-adjoint operator on a separable Hilbert space and suppose that for *any* orthonormal basis

$$(8) \quad \sum_i |(A e_i, e_i)| < \infty.$$

Show that the eigenvalues of A , if infinite in number, form a sequence in l^1 .

PROBLEM 14

For $u \in L^2(0, 1)$ show that

$$Iu(x) = \int_0^x u(t) dt, \quad x \in (0, 1)$$

is a bounded linear operator on $L^2(0, 1)$. If $V \in \mathcal{C}([0, 1])$, is real-valued and $V \geq 0$, show that there is a bounded linear operator B on $L^2(0, 1)$ such that

$$(9) \quad B^2 u = u + I^* M_V I u \quad \forall u \in L^2(0, 1)$$

where M_V denotes multiplication by V .