SECOND TEST IN 18.102
FOR 3 APRIL, 2014

RICHARD MELROSE

PreTest2.1

Let H be a separable Hilbert space. Show that $K \subset H$ is compact if and only if it is closed, bounded and has the property that any sequence in K which is weakly convergent in H is (strongly) convergent.

PreTest2.2

Show that, in a separable Hilbert space, a weakly convergent sequence $\{v_n\}$, is (strongly) convergent if and only if

$$\|v\|_H = \lim_{n\to\infty} \|v_n\|_H \tag{1}$$

where v is weak limit.

PreTest2.3

Show that a subset of a separable Hilbert space is compact if and only if it is closed and bounded and has the property of 'finite dimensional approximation' meaning that for any $\epsilon > 0$ there exists a linear subspace $D_N \subset H$ of finite dimension such that

$$d(K, D_N) = \sup_{u\in K} \inf_{v\in D_N} \{d(u,v)\} \le \epsilon. \tag{2}$$

PreTest2.4

Strong convergence of a sequence of bounded operators $A_n \in \mathcal{B}(H)$ means that for each $u \in H$, $A_n u$ converges in H. Show that $Au = \lim_n A_n u$ is necessarily a bounded linear operator on H (called the strong limit of the sequence).

PreTest2.5

If H is a separable, infinite dimensional, Hilbert space set

$$l^2(H) = \{u : \mathbb{N} \longrightarrow H; \|u\|^2_{l^2(H)} = \sum_i \|u_i\|^2_H < \infty\}. \tag{3}$$

Show that $l^2(H)$ has a Hilbert space structure and construct an explicit isometric (norm-preserving) isomorphism (bijection) from $l^2(H)$ to H.

PreTest2.6

Let H be a separable Hilbert space and let $\mathcal{C}(\mathbb{R}; H)$ be the linear space of continuous maps from $\mathbb{R}$ to H which vanish outside some interval $[-R, R]$ depending on the function. Show that

$$\|u\|^2 = \int_{\mathbb{R}} \|u(x)\|^2_H \tag{4}$$

defines a norm which comes from a preHilbert structure on $\mathcal{C}(\mathbb{R}; H)$. Show that if u_n is a Cauchy sequence in this preHilbert space and $h \in H$ then $\langle u_n(x), h\rangle_H$ converges in $L^2(\mathbb{R})$.

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu