TEST 1 FOR 18.102, MARCH 6, 2014

Will consist (without the hints) of three of the questions below.

You may use standard results for the Riemann integral (anything from 18.100B). You may use the results we have shown up to this point and we adopt the following definition

$$\begin{aligned}(1)\quad \mathcal{L}^2(\mathbb{R}) = \{u : \mathbb{R} \longrightarrow \mathbb{C}; \exists\ F \in \mathcal{L}^1(\mathbb{R}) \text{ and } \\ u_n \in \mathcal{C}_{\mathrm{c}}(\mathbb{R}),\ u_n(x) \to u(x) \text{ a.e. and } |u_n(x)|^2 \leq F \text{ a.e.}\}.\end{aligned}$$

Question.1

Show that the function

$$u(x) = \begin{cases} 0 & x \leq 0 \\ \min(x^{-\frac{1}{3}}, x^{-3}) & x > 0 \end{cases} \tag{2}$$

is in $\mathcal{L}^1(\mathbb{R})$.

Hint: Use continuity on $[\delta, 1/\delta]$, $\delta > 0$, to deduce integrability for u multiplied by the characteristic function for this region. Estimate the integral using Riemann integrals and show that it is bounded and then the result follows by monotonicity.

Question.2

Prove from the definition above that the product of two functions in $\mathcal{L}^2(\mathbb{R})$ is in $\mathcal{L}^1(\mathbb{R})$.

Hint: Take approximating sequences U_n for f and V_n for $g \in \mathcal{L}^2(\mathbb{R})$. The product of these converges a.e. to the product and if $|U_n|^2 \leq |F|$ and $|V_n|^2 \leq |G|$ are bounds by $\mathcal{L}^1$ functions then $2|U_nV_n| \leq |F| + |G|$ gives the desired result.

Question.3

Show that the product of a bounded continuous function on $\mathbb{R}$ and an element of $\mathcal{L}^2(\mathbb{R})$ is in $\mathcal{L}^2(\mathbb{R})$.

Question.4

Give an example of a function which is in $\mathcal{L}^1(\mathbb{R})$ but not in $\mathcal{L}^2(\mathbb{R})$ and another example of a function which is in $\mathcal{L}^2(\mathbb{R})$ but not in $\mathcal{L}^1(\mathbb{R})$; justify both.

Question.5

Show that if $t \in \mathbb{R}$ and $f \in \mathcal{L}^1(\mathbb{R})$ then

$$f_t(x) = f(x - t) \tag{3}$$

is an element of $\mathcal{L}^1(\mathbb{R})$. Prove that $f \in \mathcal{L}^1(\mathbb{R})$ is *continuous-in-the-mean* in the sense that given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|t| < \delta \Longrightarrow \int |f_t - f| < \epsilon. \tag{4}$$

Hint: If U_n is an approximating sequence in $\mathcal{C}_{\mathrm{c}}$ for f the the translates, $U_n(\cdot - t)$ are an approximating sequence for f_t. Given $\epsilon > 0$ and using the convergence of the norms, we can choose n so large that $|\int(f - U_n) < \epsilon/3$. The same estimate applies to the sequence for f_t (simultaneously for all t). Having fixed n observe that the estimate with U_n in place of f follows from compactness of support and (uniform) continuity. Combining these three estimates gives what we want by the usual $\epsilon/3$ trick.

Additional hint: The approximating sequence satisfies $\int |f - U_n| \to 0$. The triangle inequality says that

$$\int |f - f(\cdot - t)| \leq \int |f - U_n| + \int |U_n - U_n(\cdot - t)| + \int |U_n(\cdot - t) - f(\cdot - t)|$$

if you get my somewhat mixed notation. Taking n large the outer two integrals can be made small independent of t (they are the same). Then for such a choice of n the middle integral can be made small by choosing $|t|$ small.

Question.6

We know that the characteristic function of a finite interval, $\chi_{[a,b)}$, $a < b$ real, is integrable. A (real-valued) step function is a finite sum of real multiples of such characteristic functions. If a function $f : [a, b] \longrightarrow \mathbb{R}$, on a finite interval, is *Riemann integrable* then it is easy to see from the definition (you do not need to check this) that given $\epsilon > 0$ there are two step functions U and L such that

$$L(x) \leq f(x) \leq U(x) \text{ on } [a, b] \text{ and } \int U - \int L < \epsilon \tag{5}$$

where we are using the Lebesgue integral. Deduce that a Riemann integrable function on an interval $[a, b]$, extended as zero outside its domain of definition, is Lebesgue integrable.

Hint: Everything here is to be extended as zero outside $[a, b]$. Choose a sequence U_n and L_n by taking $\epsilon = 1/n$. Replace U_n by the minimum of the U_k for $k \leq n$ and similarly for the L_n's. The sequence U_n is then decreasing with integrals bounded below and hence by monotonicity converges pointwise to $f_+ \in \mathcal{L}^1$ and similarly the L_n's converge up to f_- where $f_- \leq f \leq f_+$ and $\int (f_+ - f_-) = 0$. From this it follows that $f_+ = f = f_-$ a.e.

Question.7

Suppose $g \in \mathcal{L}^1(\mathbb{R})$ is non-negative and vanishes outside some bounded invterval. Show that if $t \in (0, 1)$ then

$$g_t(x) = \begin{cases} 0 & \text{if } g(x) = 0 \\ g(x)^t & \text{if } g(x) > 0 \end{cases} \tag{6}$$

is an element of $\mathcal{L}^1(\mathbb{R})$ and that $\lim_{t \downarrow 0} g_t(x) = \chi_V(x)$ is the characteristic function of some set and that the limit is in $\mathcal{L}^1(\mathbb{R})$.

Question.8

Suppose that $B : L^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a bounded linear operator. Show that if $B(\phi) = 0$ for all $\phi \in \mathcal{C}_c(\mathbb{R})$ then $B = 0$ as an operator.

Question.9

Suppose that $f_n \mathcal{L}^1(\mathbb{R})$ is absolutely summable series, $\sum_n \int |f_n| < \infty$. Show that the convergence

$$f(x) = \sum_n f_n(x) \text{ } a.e.$$

is dominated by an L^1 function.

Hint: Try $\sum_n |f_n(x)|$.

Question.10

Prove that the function

$$u(x) = \begin{cases} x^{-1} & x > 0 \\ 0 & x \le 0 \end{cases}$$

is *not* in $\mathcal{L}^1(\mathbb{R})$.