PROBLEM SET 7 FOR 18.102, SPRING 2014
DUE FRIDAY 2 MAY (I.E. 7AM SATURDAY 3 MAY).

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Problem 7.1

Let $A \in \mathcal{B}(H)$, H a separable Hilbert space, be such that for some orthonormal basis $\{e_i\}$

$$\sum_i \|Ae_i\|_H^2 < \infty. \tag{5.1}$$

Show that the same inequality is true for any other orthonormal basis and that A^* satisfies the same inequality.

Problem 7.2

The elements of $A \in \mathcal{B}(H)$ as in Problem 7.1 are called 'Hilbert-Schmidt operators'. Show that these form a 2-sided $*$-closed ideal [more precisely it is a 2-sided ideal in $\mathcal{B}(H)$ consisting of compact operators and the adjoint of any Hilbert-Schmidt operator is also Hilbert-Schmidt] $\mathrm{HS}(H)$, inside the compact operators and that

$$\langle A, B\rangle = \sum_i \langle Ae_i, Be_i\rangle, \tag{5.2}$$

for any choice of orthonormal basis, makes this into a Hilbert space.

Problem 7.3

Suppose E is a compact, self-adjoint and injective operator on a separable infinite-dimensional Hilbert space H and that it is positive in the sense that $(Eu, u) \ge 0$ for all $u \in H$. Show that there is a decreasing sequence of positive eigenvalues given by the minimax formula:-

$$s_j(E) = \max_{F\subset H; \dim F = j} \left(\min_{u\in F; \|u\|=1} (Eu, u) \right).$$

Problem 7.4

With E as above, suppose that $D \in \mathcal{B}(H)$ is a bounded self-adjoint injective operator. Show that

$$s_j(DED) \le \|D\|^2 s_j(E) \ \forall \ j.$$

Problem 7.5

Recall what we have shown in class:- If $V \in \mathcal{C}([0, 2\pi])$ is real-valued and non-negative then $\lambda \in \mathbb{R}$ is an eigenvalue for the Dirichlet problem

$$(-\frac{d^2}{dx^2} + V(x))u(x) = \lambda u(x) \text{ on } [0, 2\pi],\ u(0) = 0 = u(2\pi)$$

with a twice-continuously differentiable eigenfunction if and only if $s = 1/\lambda$ is an eigenvalue of the operator

$$(\mathrm{Id} + AVA)^{-\frac{1}{2}} A^2 (\mathrm{Id} + AVA)^{-\frac{1}{2}}$$

where A is positive, self-adjoint and compact with eigenvalues $\frac{2}{k}$, $k \in \mathbb{N}_0$.

Show that the eigenvalues of the Dirichlet problem, with V real-valued and continuous, repeated according to multiplicity (the dimension of the eigenspace) and arranged as a non-decreasing sequence,

$$\lambda_1 \leq \lambda_2 \leq \lambda_N \to \infty$$

satisfy

$$\lambda_k \geq k^2/4 + \min_{[0,2\pi]} V.$$

Problem 7.6-extra

An operator T on a separable Hilbert space is said to be 'of trace class' (where this is just old-fashioned language) if it can be written as a finite sum

$$T = \sum_{i=1}^{N} A_i B_i$$

where all the A_i, B_i are Hilbert-Schmidt. Show that these trace class operators form a 2-sided ideal in the bounded operators, closed under passage to adjoints and that

$$\sup_{\{e_i\},\{f_i\}} \sum_i |\langle Te_i, f_i\rangle| < \infty$$

where the sup is over all pairs of orthonormal bases.

Problem 7.7-extra

Show that the trace functonal

$$\operatorname{Tr}(T) = \sum_i \langle Te_i e_i\rangle$$

is well-defined on trace class operators, independent of the orthonormal basis $\{e_i\}$ used to compute it and that if A is self-adjoint, compact and of trace class then

$$\operatorname{Tr}(A) = \sum_i s_i$$

is the (absolutely convergent) sum of the eigenvalues repeated with multiplicity.

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