PROBLEM SET 6 FOR 18.102
DUE FRIDAY APRIL 18, 2014.

As usual this is really due by 7AM on Saturday, April 19. There are 5 problems but you can do the extra, sixth problem, instead of one of the others (or as well of course).

Problem 6.1

Let $\mathcal{L}^2_{2\pi}(\mathbb{R}))$ denote the space of functions $u : \mathbb{R} \longrightarrow \mathbb{C}$ which are *locally* square-integrable and 2π-periodic, so $u(x + 2\pi) = u(x)$ for all $x \in \mathbb{R}$. Show that $L^2_{2\pi}(\mathbb{R}) = \mathcal{L}^2_{2\pi}(\mathbb{R})/\mathcal{N}_{2\pi}$ (where $\mathcal{N}_{2\pi}$ is the linear space of 2π-periodic null functions) is a Hilbert space with the inner product coming from

$$\langle u, v\rangle = \int \chi_{[0,2\pi]} u(x)\overline{v(x)}dx,\ u,\ v \in \mathcal{L}^2_{2\pi}(\mathbb{R}).$$

Problem 6.2

Show that the map defined by restriction

$$\mathcal{L}^2_{2\pi}(\mathbb{R}) \longrightarrow \mathcal{L}^2(0, 2\pi)$$

where $\mathcal{L}^2(0, 2\pi) \subset \mathcal{L}^2(\mathbb{R})$ is the subspace of those functions which vanish outside $(0, 2\pi)$, induces an isometric isomorphism from $L^2_{2\pi}(\mathbb{R})$ to $L^2(0, 2\pi)$.

Problem 6.3

Recall the space $h^{2,1}$ discussed in an Problem Set 4 and show, for example using the bijection

$$\mathbb{Z} \ni l \longmapsto \begin{cases} 2l & \text{if } l \geq 0 \\ -2l - 1 & \text{if } l < 0 \end{cases}$$

that this can be identified with the space of maps $d : \mathbb{Z} \longrightarrow \mathbb{C}$ satisfying

$$\sum_j (1 + |j|^2)|d_j|^2 < \infty.$$

I will denote this as $h^{2,1}$ below.

Problem 6.4

Show that the following three conditions on $u \in L^2_{2\pi}(\mathbb{R})$ are equivalent:

(1) The Fourier coefficients

$$c_j = \frac{1}{\sqrt{2\pi}} \int \chi_{[0,2\pi]} u(x)e^{-ijx}dx$$

are in $h^{2,1}$.

(2) The function u has a 'strong derivative in L^2' in the sense that

$$\lim_{t\to 0} \frac{u(x + t) - u(x)}{t}$$

exists in $L^2_{2\pi}(\mathbb{R})$.

(3) The function u has a 'weak derivative in L^2' in the sense that there exists $F \in L^2_{2\pi}(\mathbb{R})$ such that for every $g : \mathbb{R} \longrightarrow \mathbb{R}$ which is once continuously differentiable and 2π-periodic

$$\int_{[0,2\pi]} u\frac{dg}{dx} = -\int_{[0,2\pi]} Fg.$$

Hint: Try to show the first implies the second, the second implies the third and the third implies the first. The first step is probably the trickiest. So assume the first condition and guess what the Fourier series of the 'strong derivative' F should be. Let u_N be the sum of the terms with $|j| \le N$ in the Fourier series for u. Write out the difference quotient for u in terms of the difference quotient for u_N and 'error terms' and estimate the difference with F in L^2. For the second step, show that the strong derivative F is a weak derivative by replacing dg/dx by the uniform limit of its difference quotient on the left, reorganizing the integral and using the convergence to F. For the third to first just insert $g = \exp(ijx)$.

Problem 6.5

Show that the unit ball in $H^1_{2\pi}(\mathbb{R})$, the space defined by the three conditions in the preceding problem, (with respect to the inner product on $h^{2,1}$) considered as a subset of $L^2_{2\pi}(\mathbb{R})$ has compact closure (with respect to the L^2 norm).

Problem 6.6 – extra

For any positive real number s define the space $h^{2,s} \subset l^2$ as consisting of those sequences for which

$$\sum_{j\in\mathbb{Z}} (1 + |j|^2)^s |c_j|^2 < \infty.$$

Show that if $H^s_{2\pi}(\mathbb{R}) \subset L^2_{2\pi}(\mathbb{R})$ consists of the functions with Fourier coefficients in $h^{2,s}$ then provided $s > k + \frac{1}{2}$ where $k \in \mathbb{N}_0$, the elements of $H^s_{2\pi}(\mathbb{R})$ have unique representatives which are k times continuously differentiable.