PROBLEM SET 5 FOR 18.102, SPRING 2014 DUE FRIDAY MARCH 28 (SO 7AM SATURDAY MARCH 29).

RICHARD MELROSE

Since it will not be returned until after Spring Break you can have until March 29 to complete this problem set. You are of course welcome to send it in early.

Problem 5.1

Show that for any $f \in L^2(\mathbb{R})$ the zero extension outside an interval, $\chi_{[-R,R]}f \in L^2(\mathbb{R})$ and that

$$\lim_{R\to\infty} \|f - \chi_{[-R,R]}f\|_{L^2} = 0. \tag{5.1}$$

Problem 5.2

Show that any element of $L^2(\mathbb{R})$ is continuous-in-the-L^2-mean in the sense that

$$\lim_{t\to 0} \int |f(\cdot) - f(\cdot - t)|^2 = 0 \tag{5.2}$$

[including that the norm is well-defined].

Problem 5.3

Show that $S \subset L^2(\mathbb{R})$ is compact if and only if it is closed, bounded and both 'uniformly small at infinity' and 'uniformly continuous in the mean' [where you need to properly formulate these conditions].

Hints: Use the fact that $L^2(\mathbb{R})$ is separable to conclude it has an orthonormal basis. Now, use the characterization of compact sets in terms of such a basis and the fact that the two conditions above hold for each element of the basis. Using linearity it follows that they hold for any bounded set in a finite dimensional subspace. Now, check that you can make these hold for the small tail.

Problem 5.4

Work out the Fourier coefficients $c_k(t) = \int_{(0,2\pi)} f_t e^{-ikx}$ of the step function

$$f_t(x) = \begin{cases} 1 & 0 \le x < t \\ 0 & t \le x \le 2\pi \end{cases} \tag{5.3}$$

for each fixed $t \in (0, 2\pi)$.

Problem 5.5

Give an example of a closed subset of a Hilbert space which is not weakly closed – which contains a weakly convergent subsequence which has weak limit not in the set.

Problem 5.6 – Extra

Now, suppose that you know that the Fourier basis $e^{ikx}/\sqrt{2\pi}$ is complete in $L^2(0,2\pi)$. Use this to prove that the functions $d_k\sin(kx/2)$ form an orthonormal basis for $L^2(0,2\pi)$ for choices of the constants d_k, $k\in\mathbb{N}$. (Hint think of extending functions to $(-2\pi,2\pi)$ to be odd, use the corresponding Fourier basis and see what this means).

Problem 5.7 – Extra

At this stage we have NOT proved that the Fourier functions $e^{ikx}/\sqrt{2\pi}$ form an orthonormal basis – we have not shown they are complete. So, without assuming this explain why the Fourier series in Problem 5.4 converges to f_t in $L^2(0,2\pi)$ if and only if

$$2\sum_{k>0}|c_k(t)|^2=2\pi t-t^2,\ t\in(0,2\pi). \tag{5.4}$$

Write the condition (5.4) out as a Fourier series and apply the argument again to show that the completeness of the Fourier basis implies identities for the sum of k^{-2} and k^{-4}.

Can you explain how reversing the argument, that knowledge of the sums of these two series might imply the completeness of the Fourier basis?

Department of Mathematics, Massachusetts Institute of Technology
E-mail address: rbm@math.mit.edu