

**PROBLEM SET 4 FOR 18.102, SPRING 2014  
DUE FRIDAY 14 MARCH (SO 7AM, 15 MARCH).**

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Problem 4.1

Let  $H$  be a normed space (over  $\mathbb{C}$ ) in which the norm satisfies the parallelogram law:

$$(1) \quad \|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2) \quad \forall u, v \in H.$$

Show that

$$(2) \quad (u, v) = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2)$$

is a positive-definite Hermitian form which induces the given norm.

Problem 4.2

Let  $H$  be a finite dimensional (pre)Hilbert space. So, by definition  $H$  has a basis  $\{v_i\}_{i=1}^n$ , meaning that any element of  $H$  can be written

$$(3) \quad v = \sum_i c_i v_i$$

and there is no dependence relation between the  $v_i$ 's – the presentation of  $v = 0$  in the form (3) is unique. Show that  $H$  has an orthonormal basis,  $\{e_i\}_{i=1}^n$  satisfying  $(e_i, e_j) = \delta_{ij}$  ( $= 1$  if  $i = j$  and  $0$  otherwise). Check that for the orthonormal basis the coefficients in (3) are  $c_i = (v, e_i)$  and that the map

$$(4) \quad T : H \ni v \mapsto ((v, e_1), (v, e_2), \dots, (v, e_n)) \in \mathbb{C}^n$$

is a linear isomorphism with the properties

$$(5) \quad (u, v) = \sum_i (Tu)_i \overline{(Tv)_i}, \quad \|u\|_H = \|Tu\|_{\mathbb{C}^n} \quad \forall u, v \in H.$$

Why is a finite dimensional preHilbert space a Hilbert space?

Problem 4.3

Let  $e_i$ ,  $i \in \mathbb{N}$ , be an orthonormal sequence in a separable Hilbert space  $H$ . Suppose that for each element  $u$  in a dense subset  $D \subset H$

$$(6) \quad \sum_i |(u, e_i)|^2 = \|u\|^2.$$

Conclude that  $e_i$  is an orthonormal basis, i.e. is complete.

Problem 4.4

Consider the sequence space

$$(7) \quad h^{2,1} = \left\{ c : \mathbb{N} \ni j \mapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}.$$

(1) Show that

$$(8) \quad h^{2,1} \times h^{2,1} \ni (c, d) \mapsto \langle c, d \rangle = \sum_j (1+j^2)c_j \bar{d}_j$$

is an Hermitian inner form which turns  $h^{2,1}$  into a Hilbert space.

(2) Denoting the norm on this space by  $\|\cdot\|_{2,1}$  and the norm on  $l^2$  by  $\|\cdot\|_2$ , show that

$$(9) \quad h^{2,1} \subset l^2, \quad \|c\|_2 \leq \|c\|_{2,1} \quad \forall c \in h^{2,1}.$$

#### Problem 4.5

Suppose that  $H_1$  and  $H_2$  are two different Hilbert spaces and  $A : H_1 \rightarrow H_2$  is a bounded linear operator. Show that there is a unique bounded linear operator (the adjoint)  $A^* : H_2 \rightarrow H_1$  with the property

$$(10) \quad \langle Au_1, u_2 \rangle_{H_2} = \langle u_1, A^*u_2 \rangle_{H_1} \quad \forall u_1 \in H_1, u_2 \in H_2.$$

#### Problem 4.6 – Extra

Recall (from Rudin's book for instance) that if  $F : [a, b] \rightarrow [A, B]$  is an increasing continuously differentiable map, in the strong sense that  $F'(x) > 0$ , between finite intervals then for any continuous function  $f : [A, B] \rightarrow \mathbb{C}$ , (Rudin shows it for Riemann integrable functions)

$$(11) \quad \int_A^B f(y)dy = \int_a^b f(F(x))F'(x)dx.$$

Prove the same identity for every  $f \in \mathcal{L}^1((A, B))$ , which in particular requires the right side to make sense.

#### Problem 4.7 – Extra

A subset  $E \subset \mathbb{R}$  is said to be of finite measure (resp. measurable) if the characteristic function

$$(12) \quad \chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

is in  $\mathcal{L}^1(\mathbb{R})$  (resp.  $\chi_{[-R,R]}\chi_E \in \mathcal{L}^1(\mathbb{R})$  for every  $R$ ). The measure is

$$\mu(E) = \lim_{N \rightarrow \infty} \int \chi_{[-N,N]}\chi_E$$

– so the measure of a measurable set might be infinite. Show that if  $E_i, i \in \mathbb{N}$  is a countable collection of measurable sets then  $E = \sum_i E_i$  is measurable and that

$$(13) \quad \begin{aligned} \mu(E) &\leq \sum_i \mu(E_i), \\ \mu(E) &= \sum_i \mu(E_i) \text{ if } E_i \cap E_j = \emptyset \text{ for } i \neq j. \end{aligned}$$

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