PROBLEM SET 3 FOR 18.102, SPRING 2014 DUE ELECTRONICALLY BY 7AM SATURDAY 1 MARCH

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Recall that we have defined a set $E \subset \mathbb{R}$ to be 'of measure zero' if there exists an absolutely summable sequence $f_n \in \mathcal{C}_c(\mathbb{R})$ such that

$$E \subset \{\sum_n |f_n(x)| = +\infty\}. \tag{1}$$

General Hint: You might wish to recall that an open subset of $\mathbb{R}$ is the union of an at most countable collection of (uniquely defined) disjoint open intervals. Also, don't forget last week's homework.

Problem 3.1

Give a sketch of the argument (you can extract it from the notes and in any case it was given in the lecture on Thursday 20 Feb) showing that if (1) holds for an absolutely summable series of functions $f_n \subset \mathcal{L}^1(\mathbb{R})$ then E is of measure zero. By a sketch I mean a list (maybe 5-10) of successive statements that you can defend as being straightforward to check.

Problem 3.2

(1) Suppose that $O \subset \mathbb{R}$ is a *bounded* open subset, so $O \subset (-R, R)$ for some R. Show that the characteristic function of O

$$\chi_O(x) = \begin{cases} 1 & x \in O \\ 0 & x \notin O \end{cases} \tag{2}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

(2) If O is bounded and open define the length (or Lebesgue measure) of O to be $l(O) = \int \chi_O$. Show that if $U = \bigcup_j O_j$ is a (n at most) countable union of bounded open sets such that $\sum_j l(O_j) < \infty$ then $\chi_U \in \mathcal{L}^1(\mathbb{R})$; again we set $l(U) = \int \chi_U$.

(3) Conversely show that if U is open and $\chi_U \in \mathcal{L}^1(\mathbb{R})$ then $U = \sum_j O_j$ *is* the union of a countable collection of bounded open sets with $\sum_j l(O_j) < \infty$.

(4) Show that if $U \subset \mathbb{R}$ is open and there exist $g \in \mathcal{L}^1(\mathbb{R})$, non-negative, such that $g(x) \geq 1$ for all $x \in U$ then $\chi_U \in \mathcal{L}^1(\mathbb{R})$ (and so we are back in the preceding setting).

(5) Show that if $K \subset \mathbb{R}$ is compact then its characteristic function is an element of $\mathcal{L}^1(\mathbb{R})$.

Problem 3.3

Suppose $F \subset \mathbb{R}$ has the following (well-known) property:-

$$\forall\ \epsilon > 0\ \exists \text{ a countable collection of open sets } O_i \text{ s.t.}$$
$$\sum_i l(O_i) < \epsilon,\ F \subset \bigcup_i O_i. \tag{3}$$

Show that F is a set of measure zero in the sense above (the same sense as in lectures).

Problem 3.4

Suppose $f_n \in \mathcal{C}_c(\mathbb{R})$ form an absolutely summable series with respect to the L^1 norm and set

$$E = \{x \in \mathbb{R}; \sum_n |f_n(x)| = \infty\}. \tag{4}$$

(1) Show that if $a > 0$ then the set
$$\{x \in \mathbb{R}; \sum_n |f_n(x)| \le a\} \tag{5}$$
is closed.

(2) Deduce that if $\epsilon > 0$ is given then there is an open set $O_\epsilon \supset E$ with $\sum_n |f_n(x)| > 1/\epsilon$ for each $x \in O_\epsilon$.

(3) Deduce that the characteristic function of O_ϵ is in $\mathcal{L}^1(\mathbb{R})$ and that $l(O_\epsilon) \le \epsilon C$, $C = \sum_n \int |f_n(x)|$.

(4) Conclude that E satisfies the condition (3).

Problem 3.5

Show that the function with $F(0) = 0$ and

$$F(x) = \begin{cases} 0 & x > 1 \\ \exp(i/x) & 0 < |x| < 1 \\ 0 & x < -1, \end{cases}$$

is an element of $\mathcal{L}^1(\mathbb{R})$.

Problem 3.6 – Extra

(1) Recall the definition of a Riemann integrable function $g : [a, b] \longrightarrow \mathbb{R}$ – that there exist a sequence of successively finer partitions for which the upper Riemann sum approaches the lower Riemann sum. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the extension of this function to be zero outside the interval.

(2) Translate this condition into a statement about two sequences of piecewise-constant functions (with respect to the partition), u_n, l_n with $l_n(x) \le f(x) \le u_n(x)$ and conclude that $\int (u_n - l_n) \to 0$.

(3) Deduce that $f \in \mathcal{L}^1(\mathbb{R})$ and the Lebesgue integral of f on $\mathbb{R}$ is equal to the Riemann integral of g on $[a, b]$.

(4) Show that there is a Lebesgue integrable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which vanishes outside $[a, b]$ but that no function equal to it a.e. can be Riemann integrable.

Problem 3.7 – Extra

Prove that the Cantor set has measure zero.

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