QUESTIONS 18.102 FINAL, SPRING 2014

I expect to choose 6 of these questions for the final. Before the day of the exam, you can ask questions, and I may give hints.

All Hilbert spaces should be taken to be separable and non-trivial, i.e. containing an element other than 0.

Problem 1

Let $T : H_1 \longrightarrow H_2$ be a continuous linear map between two Hilbert spaces and suppose that T is both surjective and injective.

(1) Let $A_2 \in \mathcal{K}(H_2)$ be a compact linear operator on H_2, show that there is a compact linear operator $A_1 \in \mathcal{K}(H_1)$ such that
$$A_2 T = T A_1.$$

(2) If A_2 is self-adjoint (as well as being compact) and H_1 is infinite dimensional, show that A_1 has an infinite number of linearly independent eigenvectors.

Problem 2

Suppose $P \subset H$ is a closed linear subspace of a Hilbert space so each u in H has a unique decomposition

$$u = v + v',\ v \in P,\ v' \perp P \tag{1}$$

and write $\pi_P u = v$ for the orthogonal projection onto P. If H is separable and A is a compact self-adjoint operator on H, show that there is a complete orthonormal basis of H each element of which satisfies $\pi_P A \pi_P e_i = t_i e_i$ for some $t_i \in \mathbb{R}$.

Problem 3

Let A be a compact self-adjoint operator on a separable Hilbert space and suppose that for *any* orthonormal basis

$$\sum_i |(Ae_i, e_i)| < \infty. \tag{2}$$

Show that the eigenvalues of A form a sequence in l^1.

Problem 4

Let a be a continuous function on the square $[0, 2\pi]^2$. Show that $[0, 2\pi] \ni x \longmapsto a(x, \cdot) \in \mathcal{C}^0([0, 2\pi])$ is a continuous map into the continuous functions with supremum norm. Using this, or otherwise, show that the Fourier coefficients with respect to the second variable

$$c_k(x) = \frac{1}{2\pi} \int_0^{2\pi} a(x,t) e^{ikt} dt \tag{3}$$

are continuous functions on $[0, 2\pi]$.

Problem 5

Let H_i, $i = 1, 2$ be two Hilbert spaces with inner products $(\cdot,\cdot)_i$ and suppose that $I : H_1 \longrightarrow H_2$ is a continuous linear map between them. Suppose that the range of I is dense and that I is injective.

(1) Show that there is a continuous linear map $Q : H_2 \longrightarrow H_1$ such that $(u, I(f))_2 = (Qu, f)_1 \ \forall\ f \in H_1$.
(2) Show that as a map from H_1 to itself, $Q \circ I$ is bounded and self-adjoint.

Problem 6

Let $u_n : [0, 2\pi] \longrightarrow \mathbb{C}$ be a sequence of continuous differentiable functions which is uniformly bounded, with bounded derivatives i.e. $\sup_n \sup_{x\in[0,2\pi]} |u_n(x)| < \infty$ and $\sup_n \sup_{x\in[0,2\pi]} |u_n'(x)| < \infty$. Show that u_n has a subsequence which converges in $L^2([0, 2\pi])$.

Problem 7

Consider the subspace $H \subset \mathcal{C}[0, 2\pi]$ consisting of those continuous functions on $[0, 2\pi]$ which satisfy

$$u(x) = \int_0^x U(x),\ \forall\ x \in [0, 2\pi] \tag{4}$$

for some $U \in L^2(0, 2\pi)$ (depending on u of course). Show that the function U is determined by u (given that it exists) and that

$$\|u\|_H^2 = \int_{(0,2\pi)} |U|^2 \tag{5}$$

turns H into a Hilbert space.

Problem 8

Let $e_j = c_j C^j e^{-x^2/2}$, $c_j > 0$, where $C = -\frac{d}{dx} + x$ is the creation operator, be the orthonormal basis (you may assume this) of $L^2(\mathbb{R})$ consisting of the eigenfunctions of the harmonic oscillator discussed in class. Define an operator on $L^2(\mathbb{R})$ by

$$Au = \sum_{j=0}^{\infty} (2j+1)^{-\frac{1}{2}} (u, e_j)_{L^2} e_j.$$

(1) Show that A is compact as an operator on $L^2(\mathbb{R})$.
(2) Suppose that $V \in \mathcal{C}^0_\infty(\mathbb{R})$ is a bounded, real-valued, continuous function on $\mathbb{R}$. Show that $L^2(\mathbb{R})$ has an orthonormal basis consisting of eigenfunctions of $K = AVA$, where V is acting by multiplication on $L^2(\mathbb{R})$.

Problem 9

Consider the space of those complex-valued functions on $[0, 1]$ for which there is a constant C (depending on the function) such that

$$|u(x) - u(y)| \le C|x - y|^{\frac{1}{2}}\ \forall\ x, y \in [0, 1]. \tag{6}$$

Show that this is a Banach space with norm

$$\|u\|_{\frac{1}{2}} = \sup_{[0,2\pi]} |u(x)| + \inf_{(6)\text{ holds}} C. \tag{7}$$

Problem 10

Let $B : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \longrightarrow \mathbb{C}$ be a bilinear form (meaning it is linear in each factor when the other is held fixed) such that there is a constant $C > 0$ and

$$|B(u,v)| \le C\|u\|_{L^2}\|v\|_{L^2} \ \forall\ u,v \in L^2(\mathbb{R}).$$

Show that there is a bounded linear operator $T : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} T(u)(x)v(x) = B(u,v) \ \forall\ u,v \in L^2(\mathbb{R}).$$