Reversibility of chordal $\mathrm{SLE}_\kappa(\underline{\rho})$

Pu Yu

Massachusetts Institute of Technology

Random Geometry and Statistical Physics workshop
joint work with Xin Sun
March 4, 2023

Outline

1. The chordal SLE process and its reversibility

2. Conformal welding of LQG surfaces

The SLE_κ processes

- Fix $\kappa > 0$, and let $\{B_t\}_{t\geq 0}$ be the standard Brownian motion.
- The SLE_κ curve η from 0 to ∞ on the upper half plane $\mathbb{H}$ can be characterized by

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - W_t}; \quad g_0(z) = z \tag{1}$$

  where $W_t = \sqrt{\kappa} B_t$ and g_t is the conformal map from $\mathbb{H} \backslash \eta([0,t])$ to $\mathbb{H}$ with $\lim_{|z|\to\infty} |g_t(z) - z| = 0$.
- The definition is extended to other domains via *conformal invariance.*

$\mathrm{SLE}_\kappa(\underline{\rho})$ processes

- Fix the *weights* $\rho^{0,L}, \ldots, \rho^{k,L}; \rho^{0,R}, \ldots, \rho^{\ell,R} \in \mathbb{R}$ and the *force points* $x^{k,L} < \ldots < x^{0,L} = 0^- < 0^+ = x^{0,R} < \ldots < x^{\ell,R}$.
- The $\mathrm{SLE}_\kappa(\underline{\rho})$ curve η from 0 to ∞ on the upper half plane $\mathbb{H}$ with force points $\underline{x}$ can be characterized by the Loewner equation (1) with

$$dW_t = \sum_{q \in \{L,R\}} \sum_i \frac{\rho^{i,q}}{W_t - g_t(x^{i,q})} dt + \sqrt{\kappa} dB_t \tag{2}$$

Basic properties $\mathrm{SLE}_\kappa(\underline{\rho})$ processes

- Continuation threshold: If for any $j \geq 0$ and $q \in \{L, R\}$, $\sum_{i=0}^{j} \rho^{i,q} > -2$, then the $\mathrm{SLE}_\kappa(\underline{\rho})$ process η exists all the way to the infinity.
- Boundary hitting: If for any $j \geq 0$ and $q \in \{L, R\}$, $\sum_{i=0}^{j} \rho^{i,q} \geq \frac{\kappa}{2} - 2$, then η a.s. does not hit the boundaries.
- Boundary filling: For $\kappa > 4$, if for any $j \geq 0$ and $q \in \{L, R\}$, $\sum_{i=0}^{j} \rho^{i,q} \geq \frac{\kappa}{2} - 4$, then η is non-boundary filling.
- η is a.s. simple when $\kappa \leq 4$ and space-filling when $\kappa \geq 8$.

The reversibility of $\mathrm{SLE}_\kappa(\underline{\rho})$ processes

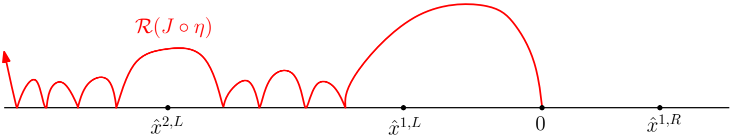
- For $\kappa \in (0,4]$, Zhan'08 proved that the time reversal of an SLE_κ process is an SLE_κ.
- For $\kappa \in (0,8]$, Miller-Sheffield'12 have proved that the time reversal of an $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ process with force points at $0^-,0^+$ (i.e., $k=\ell=0$) is an $\mathrm{SLE}_\kappa(\rho^R;\rho^L)$, provided $\rho^L,\rho^R > -2 \vee (\frac{\kappa}{2}-4)$. They also showed that when $\kappa > 8$, SLE_κ curve is not reversible.
- When the force points are all on the same side, Zhan'19 gave a description of the time reversal of $\mathrm{SLE}_\kappa(\underline{\rho})$ when (i) $\kappa \in (0,4]$ $\sum_{i=0}^j \rho^{i,q} > -2$ for any $j \ge 0$ and $q \in \{L,R\}$ (ii) $\kappa \in (4,8]$ $\sum_{i=0}^j \rho^{i,q} \ge \frac{\kappa}{2}-2$ for any $j \ge 0$ and $q \in \{L,R\}$.
- $\kappa = 4$, $\mathrm{SLE}_\kappa(\underline{\rho})$ is reversible as proved in Wang-Wu'17 via GFF level lines.

The time reversal of $\mathrm{SLE}_\kappa(\underline{\rho})$ processes

Let $x^{k+1,L} = -\infty$, $x^{\ell+1,R} = +\infty$, $\rho^{k+1,L} = -\sum_{i=0}^{k} \rho^{i,L}$, $\rho^{\ell+1,R} = -\sum_{i=0}^{\ell} \rho^{i,R}$. Let $J(z) = -1/z$. For $0 \le i \le \ell$, let $\hat{x}^{i,L} = J(x^{\ell+1-i,R})$, $\hat{\rho}^{i,L} = -\rho^{\ell+1-i,R}$. For $0 \le i \le k$, let $\hat{x}^{i,R} = J(x^{k+1-i,L})$, $\hat{\rho}^{i,R} = -\rho^{k+1-i,R}$. For $i \ge 1$ and $q \in \{L, R\}$, let $\hat{\alpha}^{i,q} = \frac{\hat{\rho}^{i,q}(\kappa-4)}{2\kappa}$.

Theorem (Sun-Y. 22')

Up to reparametrization, the law of the time reversal of $J \circ \eta$ is the probability measure proportional to $\widetilde{\mathrm{SLE}}_\kappa(\underline{\hat{\rho}}; \underline{\hat{\alpha}})$ with force points $\underline{\hat{x}}$.



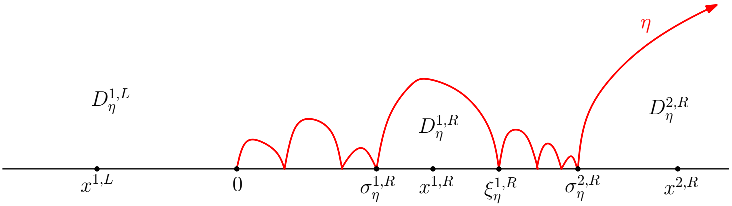
$\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ processes

- Let η be an $\mathrm{SLE}_\kappa(\underline{\rho})$ process with force points $\underline{x}$.
- For each $i \geq 1$ and $q \in \{L, R\}$, let $D_\eta^{i,q}$ be the connected component of $\mathbb{H} \backslash \eta$ containing $x^{i,q}$, and $\sigma_\eta^{i,q}, \xi_\eta^{i,q}$ be the first and the last point on $\partial D_\eta^{i,q}$ traced by η.
- Consider the conformal map $\psi_\eta^{i,q} : D_\eta^{i,q} \to \mathbb{H}$ sending $(\sigma_\eta^{i,q}, x_\eta^{i,q}, \xi_\eta^{i,q})$ to $(0, \pm 1, \infty)$ where we take the $+$ sign when $q = R$ and take the $-$ sign when $q = L$.

$\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ processes

- We associate a power parameter $\alpha^{i,q} \in \mathbb{R}$ for each $x^{i,q}$ with $\alpha^{0,L} = \alpha^{0,R} = 0$.
- Define $\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})$ by

$$\frac{d\widetilde{\mathrm{SLE}}_\kappa(\underline{\rho};\underline{\alpha})}{d\mathrm{SLE}_\kappa(\underline{\rho})}(\eta) = \prod_{q\in\{L,R\}} \prod_i \left|x^{i,q} \cdot (\psi_\eta^{i,q})'(x^{i,q})\right|^{\alpha^{i,q}}. \tag{3}$$



Liouville quantum gravity (LQG) surfaces

- Let $\gamma \in (0,2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$ and ϕ be a variant of the Gaussian free field on some domain D.
- Area measure: $\mu_\phi(d^2z) =$ "$e^{\gamma\phi(z)}d^2z$" and length measure: $\nu_\phi(dx) =$ "$e^{\frac{\gamma}{2}\phi(x)}dx$".
- Two canonical perspectives: scaling limits of random planar maps/Liouville conformal field theory.
- Liouville field on $\mathbb{H}$: sample $(h, \mathbf{c})$ from $P_{\mathbb{H}} \times [e^{-Qc}dc]$, and set $\phi(z) = h(z) - 2Q\log|z|_+ + \mathbf{c}$. Let $\mathrm{LF}_{\mathbb{H}}$ be the law of ϕ [David-Kupiainen-Rhodes-Vargas '14].

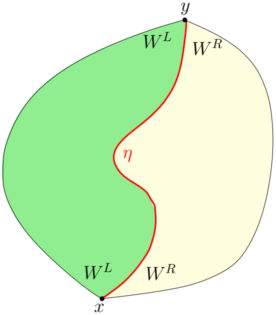
LQG surfaces via LCFT

- Let $x_j \in \mathbb{R}, \beta_j < Q$. Liouville field with boundary insertions: $\mathrm{LF}_{\mathbb{H}}^{(\beta_j, x_j)}(d\phi) = \prod_j e^{\frac{\beta_j}{2}\phi(x_j)} \mathrm{LF}_{\mathbb{H}}(d\phi)$.
- Weight W quantum disks: (Duplantier-Miller-Sheffield '14) can be viewed as *uniform embedding of* $\mathrm{LF}_{\mathbb{H}}^{(\beta,0),(\beta,\infty)}$ (Ang-Holden-Sun'21).
- Weight (W_1, W_2, W_3) quantum triangles: defined in [Ang-Sun-Y. '22] via $\mathrm{LF}_{\mathbb{H}}^{(\beta_1,0),(\beta_2,\infty),(\beta_3,1)}$.
- Relation: $\beta = \gamma + \frac{2-W}{\gamma}$.
- Thick-thin duality: when $W < \frac{\gamma^2}{2}$, a weight W (thin) quantum disk is the concatenation of weight $\gamma^2 - W$ (thick) quantum disks.

Conformal welding of quantum disks

Theorem (Ang-Holden-Sun '20)

$$\mathcal{M}_2^{\mathrm{disk}}(W^L + W^R) \otimes \mathrm{SLE}_\kappa(W^L - 2, W^R - 2) = \int_0^\infty \mathrm{Weld}(\mathcal{M}_2^{\mathrm{disk}}(W^L;\ell), \mathcal{M}_2^{\mathrm{disk}}(W^R;\ell) d\ell. \quad (4)$$

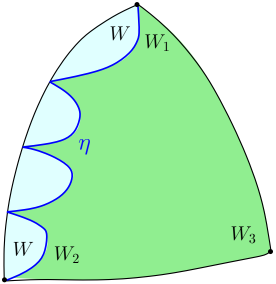


Conformal welding of quantum triangles

Theorem (Ang-Sun-Y.' 22)

$$\mathrm{QT}(W+W_1, W+W_2, W_3) \otimes \widetilde{\mathrm{SLE}}_\kappa(W-2; W_2-2, W_1-W_2; \alpha) = c\int_0^\infty \mathrm{Weld}(\mathcal{M}_2^{\mathrm{disk}}(W;\ell), \mathrm{QT}(W_1, W_2, W_3;\ell)). \quad (5)$$

where $\alpha = \frac{W_3+W_2-W_1-2}{4\kappa}(W_3+W_1+2-W_2-\kappa)$.



$\mathrm{SLE}_\kappa(\underline{\rho})$ reversibility via conformal welding

- In the above two welding pictures, one can immediately read off the law of the time reversal of $\mathrm{SLE}_\kappa(\rho^-;\rho^+)$ and $\mathrm{SLE}_\kappa(\rho^-;\rho^+,\rho_1)$ processes by looking at the welding picture in the reverse direction.
- In [Sun-Y. '22], we extend this welding result to the welding of multiple quantum disks and quantum triangles. The output surface is an LCFT type surface, with interface being $\mathrm{SLE}_\kappa(\underline{\rho})$ conditioned on hitting the boundaries.
- This justifies the reversibility of boundary-hitting $\mathrm{SLE}_\kappa(\underline{\rho})$.

An example

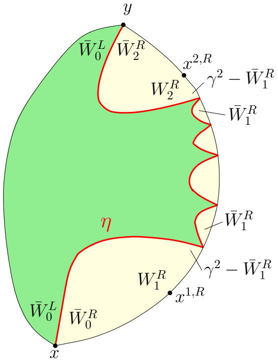


Figure: The welding of two quantum triangles with two quantum disks. The interface η is an $\widetilde{\mathrm{SLE}}_\kappa(\bar{W}_0^L - 2; \bar{W}_0^R - 2, \bar{W}_1^R - \bar{W}_0^R, \bar{W}_2^R - \bar{W}_1^R; \alpha^{1,R}, \alpha^{2,R})$ process from x to y conditioned hitting the arc between $x^{1,R}$ and $x^{2,R}$, and its time reversal can be figured out by looking from the other direction.

Non-boundary-hitting case

Let η be an $\mathrm{SLE}_\kappa(\underline{\rho})$ process, and $\tilde{\rho}^{i,q} \in \mathbb{R}$. Let $f_t(z) = g_t(z) - W_t$ be the centered Loewner flow.

Theorem (Schramm-Wilson '05)

If one weight the law of η by

$$M_t := \prod_{i,q} |f_t'(x^{i,q})|^{\frac{(\tilde{\rho}^{i,q}-\rho^{i,q})(\tilde{\rho}^{i,q}+\rho^{i,q}+4-\kappa)}{4\kappa}} |f_t(x^{i,q})|^{\frac{\tilde{\rho}^{i,q}-\rho^{i,q}}{\kappa}} \cdot \prod |f_t(x^{i,q}) - f_t(x^{i',q'})|^{\frac{\tilde{\rho}^{i,q}\tilde{\rho}^{i',q'}-\rho^{i,q}\rho^{i',q'}}{2\kappa}} \tag{6}$$

(with an appropriate stopping time), the law of η is an $\mathrm{SLE}_\kappa(\underline{\tilde{\rho}})$ process under the reweighted measure.

When η is non-boundary hitting, M_t tends to product of powers of conformal derivatives and Poisson kernels, allowing us to obtain the reversal of $\mathrm{SLE}_\kappa(\underline{\tilde{\rho}})$ from that of $\mathrm{SLE}_\kappa(\underline{\rho})$.

Outlook

- Solve the problem by theory of Imaginary Geometry and traditional methods;
- Extension to radial/annulus SLE.

Thanks for listening!