A brief introduction to Imaginary Geometry

Pu Yu

May 12, 2020

1 Introduction

In studying of Riemannian geometry, a metric tensor on a two dimensional surface could be expressed as $e^{h(z)}dz$ where h is some real-valued smooth function. However, one might ask what would happen if h is now a general complex valued smooth function. Some important concept becomes invalid, like distance as we are no longer having positivity. Meanwhile some stories in real geometry are still happening here. For instance, we can define 'straight lines of angle θ' to be the solution of the ODE

$$\eta'(t) = e^{i(h(\eta(t))+\theta)}, \quad \eta(0) = z. \tag{1.1}$$

And under this definition we have many nice properties for such flow lines, including its existence and uniqueness.

Recall when we were defining the Liouville quantum gravity ([Ber16] is a good reference), in the real metric $e^{\gamma h(z)}dz$ we were trying to take h to be the Gaussian Free Field. In imaginary geometry we can actually do similar things. We can construct straight lines of angle θ and these flow lines turned out to be $\mathrm{SLE}_\kappa(\underline{\rho})$ curves nicely coupled with the Gaussian Free Field h. In this short overview we will go through the important paper [MS12] and briefly explain the main results and ideas.

The SLE/GFF couplings are originally discussed in [She10], including the coupling of Dirichlet GFF on the upper half plane $\mathbb{H}$ and forward SLE flow which gives the stationarity of AC (imaginary) surfaces, and the coupling of free boundary GFF and reverse SLE flow which implies the stationarity of LQG surfaces. In the previous case, the boundary data of the field h is given by $\mathrm{sgn}(x)$ while in latter the boundary condition is $\log|x|$, and the flow is the ordinary SLE_κ starting from the origin. Now in [MS12], the previous coupling is generalized as the boundary data of the field h is now any piecewise constant function (with finite number of changes) and force points are added with the flow becoming $\mathrm{SLE}_\kappa(\underline{\rho})$. Roughly speaking, An $\mathrm{SLE}_\kappa(\underline{\rho})$ process (where $\underline{\rho} = (\underline{\rho}^L, \underline{\rho}^R)$ with $|\underline{\rho}^L| = k$, $|\underline{\rho}^R| = l$) with force points $(\underline{x}^L, \underline{x}^R)$ is given by compact hulls K_t with Lowener map $g_t : \mathbb{H}\backslash K_t \to \mathbb{H}$ satisfying

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z \tag{1.2}$$

and centered Lowener map $f_t(z) = g_t(z) - W_t$. W_t satisfying the following SDE

$$W_t = \sqrt{\kappa}B_t + \int_0^t \sum_{q\in\{L,R\}} \sum_i \frac{\rho^{i,q}}{W_s - V_s^{i,q}} ds \tag{1.3}$$

$$V_t^{i,q} = x^{i,q} + \int_0^t \frac{2}{V_s^{i,q} - W_s} ds, \quad q \in \{L, R\}. \tag{1.4}$$

In [MS12, Section 2] the existence of $\mathrm{SLE}_\kappa(\underline{\rho})$ process is justified up till the continuation threshold (the first time t with $W_t = V_t^{i,q}$ and $\sum_{i=1}^j \rho^{i,q} \le -2$) is hit. And the $\mathrm{SLE}_\kappa(\underline{\rho})$/GFF coupling is given as follows [MS12, Theorem 1.1]:

Theorem 1.1. *Fix $\kappa > 0$, $\lambda = \frac{\pi}{\sqrt{\kappa}}$ and weights $(\underline{\rho}^L, \underline{\rho}^R)$, and let K_t be the hull at time t of the $\mathrm{SLE}_\kappa(\underline{\rho})$ process generated by (1.2)-(1.4). Also let $\mathfrak{h}_t^0$ be the harmonic function in $\mathbb{H}$ with boundary values*

$$-\lambda(1 + \sum_{i=1}^j \rho^{i,L}), \ if \ s \in [V_t^{j+1,L}, V_t^{j,L}); \quad \lambda(1 + \sum_{i=1}^j \rho^{i,R}), \ if \ s \in [V_t^{j,R}, V_t^{j+1,R}).$$

where $\rho^{0,L} = \rho^{0,R} = 0$, $x^{0,L} = 0^-$, $x^{k+1,L} = -\infty$, $x^{0,R} = 0^+$, $x^{l+1,R} = +\infty$. *We set*

$$\mathfrak{h}_t(z) = \mathfrak{h}_t^0(f_t(z)) - \chi \arg f_t'(z), \quad \chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.$$

Let $\mathcal{F}_t$ *be the filtration generated by* $(W, V^{i,q})$. *There exists a coupling* (K, h) *where* $\tilde{h}$ *is a zero boundary GFF on* $\mathbb{H}$ *and* $h = \tilde{h} + \mathfrak{h}_0$ *such that the following is true. Suppose* τ *is any* $\mathcal{F}_t$*-stopping time which almost surely occurs before the continuation threshold is reached. Then* K_τ *is a local set for* h *and the conditional law of* $h|_{H \setminus K_\tau}$ *given* $\mathcal{F}_\tau$ *is equal to the law of* $\mathfrak{h}_\tau + \tilde{h} \circ f_\tau$ *.*

With this coupling we could make sense of flow lines of the free field:

Definition 1.1. *Suppose* $\kappa \in (0, 4)$. *An* $SLE_\kappa(\underline{\rho})$ *curve (if it exists) is said to be a* ***flow line*** *(with angle 0) of the GFF* h *if* h *has boundary value specified above and the curve is coupled with* h *as in theorem 1.1. An* $SLE_\kappa(\underline{\rho})$ *curve coupled with* $h + \theta\chi$ *as above is called a flow line with angle* θ*.*

Note that for general simply connected domain D, the coordinate change formula is inherited from [She10], i.e., we take an conformal mapping $\psi : D \to \mathbb{H}$ and consider the surface $h \circ \psi - \chi \arg \psi'$. Note the branch of $\arg \psi'$ is chosen that $\arg \psi'$ is continuous within the surface.

Another concept is the counterflow line, which plays important rule in SLE duality.

Definition 1.2. *Suppose* $\kappa \in (0, 4)$ *so* $\kappa' = \frac{16}{\kappa} \in (4, \infty)$. *An* $SLE_{\kappa'}(\underline{\rho})$ *curve (if it exists) is said to be a* ***counterflow line*** *of the GFF* ***h*** *if the curve is coupled with* ***-h*** *as above in theorem 1.1.*

After making sense of 'straight lines' of the imaginary geometry, it is natural to compare them with our ordinary geometry. Indeed, in Euclidean world we have the following:

1. Given smooth function h, the solution η of ODE (1.1) starting from any z_0 is unique;
2. Moreover the solution η is continuous and extends to infinity;
3. Straight lines from the same points with different angles never cross each other;
4. Straight lines from different points with the same angles are parallel;
5. Straight lines from different points with different angles can cross each other at most once.

Surprisingly even in the world of imaginary geometry we can still establish analogs of the properties 1-5 above. These corresponds to Theorems 1.2, 1.3 and 1.5 in [MS12]:

Theorem 1.2. *Suppose that* h *is a GFF on* $\mathbb{H}$ *and that* $\eta \sim SLE_\kappa(\underline{\rho})$. *If* (η, h) *are coupled as in the statement of Theorem 1.1, then* η *is almost surely determined by* h*.*

Theorem 1.3. *Suppose that* $\kappa > 0$. *If* $\eta \sim SLE_\kappa(\underline{\rho})$ *on* $\mathbb{H}$ *from* 0 *to* ∞ *then* η *is almost surely a continuous path, up to and including the continuation threshold. On the event that the continuation threshold is not hit before* η *reaches* ∞, *we have a.s. that* $\lim_{t\to\infty} |\eta(t)| = \infty$.

Theorem 1.4. *Suppose that* h *is a GFF on* $\mathbb{H}$ *with piecewise constant boundary data. For each* $\theta \in \mathbb{R}$ *and* $x \in \partial\mathbb{H}$ *we let* η_θ^x *be the flow line of* h *starting at* x *with angle* θ. *Fix* $x_1, x_2 \in \partial\mathbb{H}$ *with* $x_1 \geq x_2$.

1. *If* $\theta_1 < \theta_2$ *then* $\eta_{\theta_1}^{x_1}$ *almost surely stays to the right of* $\eta_{\theta_2}^{x_2}$ *. If, in addition,* $\theta_2 - \theta_1 < \frac{\pi\kappa}{4-\kappa}$, *then* $\eta_{\theta_1}^{x_1}$ *and* $\eta_{\theta_2}^{x_2}$ *can bounce off of each other; otherwise the paths almost surely do not intersect (except possibly at their starting point).*
2. *If* $\theta_1 = \theta_2$, *then* $\eta_{\theta_1}^{x_1}$ *may intersect* $\eta_{\theta_2}^{x_2}$ *and, upon intersecting, the two curves merge and never separate.*
3. *Finally, if* $\theta_2 + \pi > \theta_1 > \theta_2$, *then* $\eta_{\theta_1}^{x_1}$ *may intersect* $\eta_{\theta_2}^{x_2}$ *and, upon intersecting, crosses and then never crosses back. If, in addition,* $\theta_2 - \theta_1 < \frac{\pi\kappa}{4-\kappa}$, *then* $\eta_{\theta_1}^{x_1}$ *and* $\eta_{\theta_2}^{x_2}$ *can bounce off of each other; otherwise the paths almost surely do not subsequently intersect.*

Here a difference is that flow lines could possibly merge if they start at different points with same angle, which generally do not happen for smooth h in (1.1). When the function is not smooth, say $f(x) = 2\sqrt{|x|}$, the system $x'(t) = f(x)$ can have more than 1 solutions at $x(0) = 0$. Hence the roughness of the GFF h could be an explanation of the merging phenomenon.

One other major difference is that, in $\mathbb{R}^2$, any point in the right half plane $\{\Re z > 0\}$ can be accessible by straight lines starting at 0 with angles $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. However in imaginary geometry, the points accessible by flow lines with (possibly varying in time) $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is a.s. precisely described by counterflow lines. For instance, when $\kappa \in (2, 4)$, $\kappa' \in (4, 8)$ and since SLE'_κ curves have a.s. dimension $1 + \frac{\kappa'}{8}$ [RS05, Theorem 8.1] and hence the points accessible actually have Lebesgue measure 0. This is discussed in [MS12, Theorem 1.4]:

Theorem 1.5. *Suppose that h is a GFF on $\mathbb{H}$ with piecewise constant boundary data. Let η' be the counterflow line of h starting at ∞ targeted at 0. Assume that the continuation threshold for η' is almost surely not hit. Then the range of η' is almost surely equal to the set of points accessible by SLE_κ trajectories of h starting at 0 whose angles are restricted to be in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ but may change in time. Let η_L be the flow line of h with angle $\frac{\pi}{2}$ starting at 0 and η_R the flow line of h with angle $-\frac{\pi}{2}$. It is almost surely the case that if η' is nowhere boundary filling (i.e., $\eta' \cap \mathbb{R}$ has empty interior), then η_L and η_R do not hit the continuation threshold before reaching ∞ and are the left and right boundaries of η'. A similar statement holds on the event that η' is boundary filling on one or more segments of R. In this case, η_L and η_R hit their continuation thresholds before reaching ∞, but they can be extended to describe the entire left and right boundaries of η'.*

The rest of this short overview is organized as follows. In Section 2, we review the basics for $\mathrm{SLE}_\kappa(\underline{\rho})$ and the GFF theory covered in [MS12, Section 2&3], which are foundational for the construction of flow lines. In Section 3, following the idea of [MS12, Section 4&5], we start from Dubédat's argument and establish the monotonicity result and flow/conuterflow line duality in the case the all these lines do not intersect the boundary. Finally in Section 4, we study the regularity results in [MS12, Section 6] and briefly explain how to extend to general results in Theorem 1.2-1.5 by conditioning on auxiliary flow lines.

2 $\mathrm{SLE}_\kappa(\underline{\rho})$ curves and the Gaussian Free Field

2.1 $\mathrm{SLE}_\kappa(\underline{\rho})$ and martingale characterization

As a one-parameter family of conformally invariant random curves introduced by Oded Schramm in [Sch00] SLE_κ curves are the scaling limit of many important statistical physical models. The detailed definition and as well as properties of SLE could be found in [BN16].

We start from $\mathrm{SLE}_\kappa(\rho)$ curves with only one force point at x_0 (if $\rho = 0$ this reduces to ordinary SLE_κ). Indeed, we can write out the equations (1.3) and (1.4)

$$W_t = \sqrt{\kappa} B_t + \int_0^t \frac{\rho}{W_s - V_s} ds, \quad V_t = x_0 + \int_0^t \frac{2}{V_s - W_s} ds \tag{2.1}$$

In this setting, $X_t = \frac{V_t - W_t}{\sqrt{\kappa}}$ (which corresopnds to $f_t(x_0)/\sqrt{\kappa}$) solves

$$X_t = \frac{x_0}{\sqrt{\kappa}} + \int_0^t \frac{2+\rho}{\kappa X_s} ds + B_t. \tag{2.2}$$

This is exactly a Bessel process with dimension $\delta = \frac{2(\rho+2)}{\kappa} + 1$ starting from $\frac{x_0}{\sqrt{\kappa}}$. From theory of Bessel processes (for instance in [RY99, Chapter XI]), we know (2.2) is satisfied with X_t instantaneously reflecting at 0 as long as $\delta > 1$. Moreover when $\delta \geq 2$, X_t is transient in the sense of a.s. never hitting 0. This implies when $\rho > -2$, the continuation threshold is never hit and this curve could be extended to infinite time. And if $\rho \geq \frac{\kappa}{2} - 2$, the curve should a.s. never hit the boundary (Assume $\kappa \leq 4$). In this case the law of this curve is absolute continuous w.r.t. ordinary SLE_κ curve with $\rho = 0$, so its phase is the same as described in [BN16]. If $\rho \leq -2$ then W_t and V_t a.s. collide and the curve cannot be extended after this collision.

Now for multiple force point case, roughly speaking, we could first sample $(W_t, V_t^{i,q})$ such that (1.3) is satisfied for any interval of non-collision stopping times and also the Loewner equation (1.4) is satisfied.

We also require the instantaneous reflection condition of collision of W and $V^{i,q}$. The force points $V_t^{i,q}$ is monotone in i and we relabel once they collide and merge. We fix $\tilde{\varepsilon} > 0$ and start at $S_{\tilde{\varepsilon}}$ which is the first time $|W_t - V_t^{1,q}| \geq \tilde{\varepsilon}$ and stop at $T_{\tilde{\varepsilon}}$, the minimum of a fixed time T and the first time after $S_{\tilde{\varepsilon}}$ when there exists two different force points or one force point of weight $\rho < -2$ within distance $\tilde{\varepsilon}$ of W. We set $\varepsilon \in (0, \tilde{\varepsilon})$ and $\tau_0 = S_{\tilde{\varepsilon}}$, and sample stopping times σ_k to be the first collision time after τ_{k-1} of W and some $V_t^{i,q}$, and τ_k to be the first time after σ_k such that the distance of W and the force point V^{i_k,q_k} it collided at σ_k is ε. So now for 'good intervals' $[\tau_{k-1}, \sigma_k]$ (1.3) is satisfied, i.e.

$$\sum_j (W_{T_{\tilde{\varepsilon}} \wedge \sigma_j} - W_{T_{\tilde{\varepsilon}} \wedge \tau_{j-1}}) = \sum_j \sqrt{\kappa}(B_{T_{\tilde{\varepsilon}} \wedge \sigma_j} - B_{T_{\tilde{\varepsilon}} \wedge \tau_{j-1}}) + \sum_{i,j,q} \int_{T_{\tilde{\varepsilon}} \wedge \tau_{j-1}}^{T_{\tilde{\varepsilon}} \wedge \sigma_j} \sum_i \frac{\rho^{i,q}}{W_s - V_s^{i,q}} ds. \tag{2.3}$$

Note from the instantaneous reflection condition the total length of 'bad intervals' $[\sigma_k, \tau_k]$ before $T_{\tilde{\varepsilon}}$ tends to 0 (as $\varepsilon \to 0$), therefore on the right hand side of (2.3) the quadratic variation of Brownian Motion and the change of integral $[\sigma_k, \tau_k]$ both goes to 0 (since $V^{i,q}$ satisfies Lowener equation (1.4)). Also if we set $N_{\tilde{\varepsilon}} = \min j \geq 1 : \tau_j \geq T_{\tilde{\varepsilon}}$ then the change of W on these intervals could be controlled by

$$\sum_j |W_{T_{\tilde{\varepsilon}} \wedge \tau_j} - W_{T_{\tilde{\varepsilon}} \wedge \sigma_j}| \leq N_{\tilde{\varepsilon}} \varepsilon + \sum_{i,j,q} |V^{i,q}_{T_{\tilde{\varepsilon}} \wedge \tau_j} - V^{i,q}_{T_{\tilde{\varepsilon}} \wedge \sigma_j}| \tag{2.4}$$

and the change of force points goes to 0 due to absolute continuity. Using Girsanov theorem we could now compare the process with $\text{SLE}_\kappa(\rho^{1,q})$ with one force point and it turned out $N_{\tilde{\varepsilon}}\varepsilon$ tends to 0 a.s. along some sequence $\varepsilon_k \to 0$. Therefore (1.1) is satisfied on $[S_{\tilde{\varepsilon}}, T_{\tilde{\varepsilon}}]$ and sending $\tilde{\varepsilon} \to 0$ completes the construction. More details are explained in [MS12, p.35-42].

Also similar to single force point case, as corresponding to Bessel processes with dimension at least 2, when $\sum_{i=1}^j \rho^{i,q} \geq \frac{\kappa}{2} - 2$ for any j, q , and $x^{1,L} < 0 < x^{1,R}$ then the curve η is a.s. continuous and absolutely continuous w.r.t SLE_κ curves without force points.

Another important generalization of [MS12] here is the martingale characterization of $\text{SLE}_\kappa(\underline{\rho})$ curves.

Theorem 2.1. *Suppose we are given a random continuous curve η on $\bar{\mathbb{H}}$ from 0 to ∞ whose Loewner driving function W_t is almost surely continuous. Suppose that $x^{i,q}$ and $\rho^{i,q}$ values are given and that the $V_t^{i,q}$ are defined to be the images of the $x^{i,q}$ under the corresponding Loewner evolution (1.4). Let $\mathfrak{h}_t$ be the corresponding harmonic function in the statement of Theorem 1.1. Then W_t and the $V_t^{i,q}$ can be coupled with a standard Brownian motion B_t to describe an $SLE_\kappa(\underline{\rho})$ process (up to the continuation threshold) if and only if $\mathfrak{h}_t(z)$ evolves as a continuous local martingale in t for each fixed $z \in \mathbb{H}$ until the time z is absorbed by K_t.*

Note $\mathfrak{h}_t(z)$ is $\frac{1}{\sqrt{\kappa}}$ times the imaginary part of the complex continuous local martingale $\mathfrak{h}_t^*(z)$

$$-\sum_{i=0}^{k} \rho^{i,L} \log(f_t(z) - f_t(x^{i,L})) + \sum_{i=0}^{l} \rho^{i,R}(i\pi - \log(f_t(z) - f_t(x^{i,L}))) + i\pi - 2\log f_t(z) - \frac{\pi\chi}{\lambda} \log f_t'(z).$$

Using $d \log f_t'(z) = \frac{df_t'(z)}{f_t'(z)} = -\frac{2}{f_t(z)^2} dt$ and Itô formula it is not hard to see $\mathfrak{h}_t^*(z)$ is local martingale. And if we replace f_t by forward (centered) Lowener flow and $\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}$ by $Q = \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}$ the corresponding $\mathfrak{h}_t^*(z)$ is still local martingale, and taking its real part gives couplings in LQG geometry. The proof of if part could be found in Theorem 2.4 of [MS12].

2.2 The Gaussian Free Field and its local sets

The d-dimensional Gaussian Free Field, as the $d-$dimensional time analog of the Brownian Motion, plays an important role in statistical physics and random surfaces, especially when $d = 2$ the GFF has conformal invariance. The basic definition and properties could be found in the survey paper [She07]. In particular, one significant result here is the Markov property:

Theorem 2.2. *Suppose U, D are both open sets in $\mathbb{H}$ and $U \subset D$. For the Dirichlet Gaussian Free Field h on D, we can write $h = h_U + h_{U^c}$, where h_U is a Dirichlet GFF on U and zero outside U, h_{U^c} is harmonic in U. Moreover h_U and h_{U^c} are independent.*

The key of this Markov property is we can write $H_0^1(D) = H_0^1(U) \oplus H^\perp(U)$ where $H^\perp(U)$ is the space of harmonic functions in U. From this property, we can write h_U as the projection of h to $H_0^1(U)$ and the restriction $h|_U$ is simply a 0 boundary GFF h_U plus the harmonic extension of $h|_{\partial U}$ into U.

The following proposition states that in the Markov decomposition above we could 'pretend' that h is a zero boundary valued function.

Proposition 2.3 ([MS12], Proposition 3.3)**.** *Assume that D is a non-trivial simply connected domain and let K be a deterministic closed subset of D. Let h_1 be the harmonic function on $D\backslash K$ which agrees with the projection of K onto $H^\perp(D\backslash K)$ and restricted to $D\backslash K$. Then almost surely*

$$\lim_{D\backslash K \ni z \to z_0} h_1(z) = 0, \quad \forall z_0 \in \partial D\backslash K.$$

The next proposition about absolute continuity is extremely important, especially its second part, as it allows us to compare GFFs on different domains with different boundary values. As we saw in Theorem 1.2, flow lines are almost surely determined by free fields. Hence when proving properties (say continuity) of flow lines, we could do induction on the number of force points. When we add a new force point $x^{k+1,L}$, this absolute continuity could be invoked, implying that the flow line is continuous at least before hitting $(-\infty, x^{k+1,L}]$.

Proposition 2.4 ([MS12], Proposition 3.4)**.** *Suppose that D_1, D_2 are simply connected domains with $D_1 \cap D_2 \neq \varnothing$. For $i = 1, 2$, let h_i be a zero boundary GFF on D_i and F_i harmonic on D_i. Fix a bounded simply connected open domain $U \subset D_1 \cap D_2$.*

(i) (Interior) If $dist(U, \partial D_i) > 0$ for $i = 1, 2$ then the laws of $(h_1 + F_1)|_U$ and $(h_2 + F_2)|_U$ are mutually absolutely continuous.

(ii)(Boundary) Suppose that there is a neighborhood U' of the closure $\bar{U}$ such that $D_1 \cap U' = D_2 \cap U'$, and that $F_1 - F_2$ tends to zero as one approaches points in the sets $\partial D_i \cap U'$. Then the laws of $(h_1 + F_1)|_U$ and $(h_2 + F_2)|_U$ are mutually absolutely continuous.

The proof of this proposition is again a Girsanov-type argument, i.e., if we weight the law of GFF h by $\exp(-\frac{1}{2}\|g\|_\nabla^2)\exp((h,g)_\nabla)$ then the law of h under this new measure is the same as the law of $h + g$ under original $\mathbb{P}$. Finding suitable g such $(h_1 + F_1 + g)|_U = (h_2 + F_2)|_U$ gives the proposition. Another very important property here is for flow line segment $\eta|_U$ given by η stopped upon exiting U, if we weight the law $(h|_U, \eta|_U)$ using the Radon-Nikodym derivative such that the marginal law of $h|_U$ becomes $\tilde{h}|_U$ where $\tilde{h}$ is a GFF with same boundary data on $\partial U \cap \partial D$, then the marginal law of $\eta|_U$ is absolutely continuous w.r.t. its original law under $\mathbb{P}$; moreover in this weighted measure it is still coupled with $\tilde{h}$ such that the strong Markov property is satisfied before exiting U.

Now we have established Markov property for deterministic closed sets. However in practice we may want to extend to random sets. This gives rise to the definition of local sets from [SS13]. Indeed, local sets are random variables taking values in the set Γ of closed sets in D w.r.t Borel σ-algebra of Hausdorff distance.

Definition 2.1. *Suppose (A, h) is a coupling of a GFF h on D and a random variable A taking values in Γ. Then A is said to be a **local set** of h if if there exists a law on pairs (A, h_1) where h_1 is a distribution on D with $h_1|_{D\backslash A}$ being harmonic and a sample with law (A, h) could be produced by:*

1. *Choosing the pair (A, h_1);*
2. *Sampling an (independent) instance h_2 of the zero boundary GFF on $D\backslash A$ and setting $h = h_1 + h_2$.*

The local sets defined above are random closed sets satisfying strong Markov property. Another equivalent way to define local sets is that for any deterministic open $U \subset D$, we have that given the projection of h onto $H^\perp(U)$, the event $S = \{A \cap U = \varnothing\}$ is independent of the projection of h onto $H_0^1(U)$ (i.e. h_U). Moreover if we set a random variable $\tilde{A}$ to be A on S^c and $\varnothing$ on S then $(S, \tilde{A})$ is independent of h_U given the projection of h onto $H^\perp(U)$. All other ways of definition and their equivalence are justified in [SS13, Lemma 3.9]. We may also treat h_2 as a mean zero function and define $\mathcal{C}_A = h_1$ to be the conditional mean of h on $D\backslash A$ given A.

The following lemmas on local sets will be useful when treating multiple flow lines and specifying boundary data.

Proposition 2.5 ([MS12], Proposition 3.7)**.** *Suppose h is a GFF on D, A_1, A_2 are random variables taking values in Γ, and that (A_1, h) and (A_2, h) are couplings for which A_1 and A_2 are local. Let $A =$*

$A_1 \tilde{\cup} A_2$ *denote the random variable taking values in* Γ *which is given by first sampling* h*, then sampling* A_1*,* A_2 *independently from their conditional laws given* h*, and then taking the union of* A_1 *and* A_2*. Then* A *is also a local set of* h*.*

We say a local set A is almost surely determined by h if there exists a modification of A being $\sigma(h)$ measurable. As we know in Theorem 1.2 flow lines are local sets a.s. determined by h hence unions of flow lines are local sets as they are trivially independent given h.

The next proposition is a generalization of Proposition 2.3 to local sets.

Proposition 2.6 ([MS12], Proposition 3.8)**.** *Assume that* D *is a bounded, simply connected domain. Let* A_1, A_2 *be connected local sets. Then* $\mathcal{C}_{A_1 \tilde{\cup} A_2} - \mathcal{C}_{A_2}$ *is almost surely a harmonic function in* $D \backslash A_1 \tilde{\cup} A_2$ *that tends to zero on all sequences of points in* $D \backslash A_1 \tilde{\cup} A_2$*that tend to a limit in either:*

(i) a connected component of $A_2 \backslash A_1$ *(consisting of more than a single point) or*

(ii) a connected component of $A_1 \cap A_2$ *(consisting of more than a single point) at a point that has positive distance from either* $A_2 \backslash A_1$ *or* $A_1 \backslash A_2$*.*

In the discussion of interacting flow lines, we will constantly use conformal mappings over regions zipped by flow lines, so it is important to specify the boundary value of the field h, while the proposition above greatly helps us to find it. Recall that if $\psi : D_1 \to D_2$ and h is some random surface on D_2 then the equivalent surface on D_1 is given by $h \circ \psi - \chi \arg \psi'$. Figures 1 and 2 give an example of performing this coordinate change when involving flow lines.

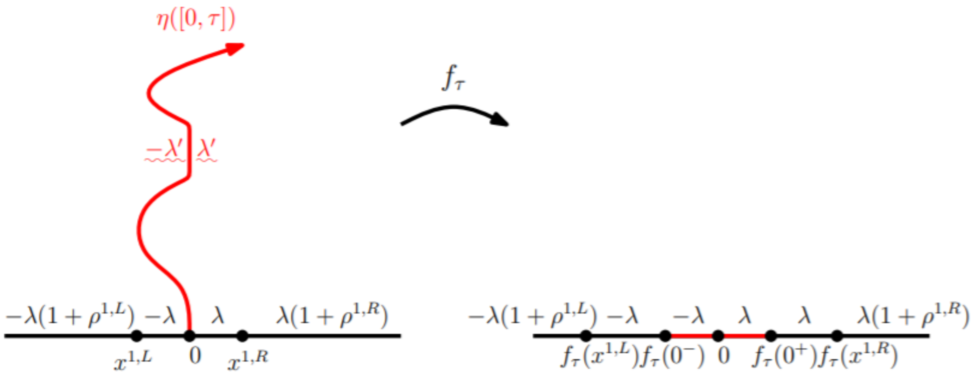

Figure 1: η is a flow line coupled with h as in Theorem 1.1 and f_τ is the centered Lowener map. Applying the coordinate change we find the boundary value of $h|_{\mathbb{H}\backslash\eta([0,\tau])}$ is $\lambda - \chi$·winding angle of η on the right side of η and $-\lambda + \chi$·winding on the left. Note $\lambda' = \frac{\pi}{\sqrt{\kappa'}} = \lambda - \frac{\pi}{2}\chi$.

The following result from the previous propositions is useful to prove independence between stories happening in different connected components of local sets.

Proposition 2.7 ([MS12], Prop 3.9)**.** *Suppose that* D *is a bounded, simply connected domain and that* A_1*,* A_2 *are connected local sets which are conditionally independent given* h*. Suppose that* C *is a* $\sigma(A_1)$*-measurable open subset of* $D \backslash A_1$ *which can be written as a union of components of* $D \backslash A_1$ *such that* $C \cap A_2 = \varnothing$ *almost surely. Then* $\mathcal{C}_{A_1 \cup A_2}|_C = \mathcal{C}_{A_1}|_C$ *almost surely. In particular,* $h|_C$ *is independent of the pair* $(h|_{D\backslash C}, A_2)$ *given* $\sigma(A_1)$*.*

The next lemma indicate that if a local set A is 'thin' then we only need to know $h|_{D\backslash A}$.

Lemma 2.8 ([MS12], Lemma 3.10)**.** *Suppose that* A *is a local set for* h *such that for every compact set* $K \subset D$ *there exists a sequence* (δ_k) *of positive numbers with* $\delta_k \to 0$ *as* $k \to \infty$ *such that we almost surely have that the number of squares with corners in* $\delta_k \mathbb{Z}^2$ *required to cover* $A \cap K$ *is* $o(\delta_k^{-2} (\log \delta_k^{-1})^{-1})$*. Then* h *is almost surely determined by the restriction* $\tilde{h}$ *of* h *to* $D \backslash A$*.*

In particular if a local set A has Hausdorff dimension strictly smaller than 2, then h is almost surely determined by the its restriction to $D \backslash A$ so we may apply this lemma to flow lines of h.

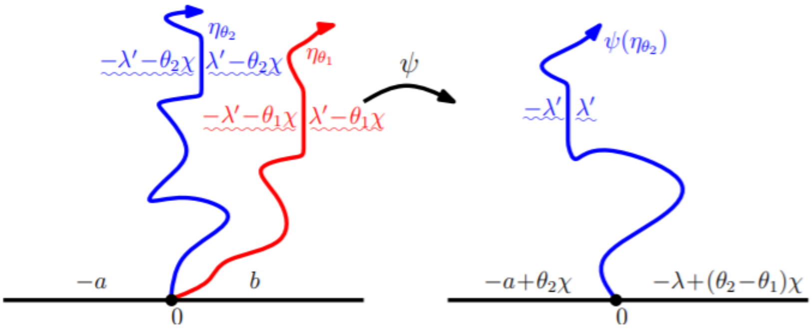

Figure 2: Consider flow lines η_{θ_1} and η_{θ_2} with $\theta_2 > \theta_1$. Assume for a while η_{θ_2} only intersects η_{θ_1} at 0 and always stays to the left of η_{θ_1}. Then the boundary data of h given both η_{θ_1} and η_{θ_2} can be computed as previous figure and using Proposition 2.6. As a result we can apply some conformal ψ taking the left component of $\mathbb{H}\backslash\eta_{\theta_1}$ to $\mathbb{H}$ and this actually gives that $\psi(\eta_{\theta_2})$ evolves as $\mathrm{SLE}_\kappa((a-\theta_2\chi)/\lambda-1;(\theta_2-\theta_1)\chi/\lambda-2)$ with force points $0^-,0^+$, i.e., this is the law of η_{θ_2} given η_{θ_1}. In the next section we will see that when a,b is large then η_{θ_2} always stays to the left of η_{θ_1}, while justification of this conformal transform even when η_{θ_1} and η_{θ_2} bounce off is given in Section 4 and we still have the same conclusion.

2.3 SLF/GFF couplings

Similar to [Ber16, Theorem 6.1], which states that under the SLE quantum zipper the random surfaces (h,η) are stationary (h is free boundary GFF coupled with reverse SLE flow η), taking the imaginary part of the continuous local martingale $\mathfrak{h}_t^*(z)$ in Theorem 2.1 could give us similar results. Applying the Itô formula it is not hard to see ($\mathfrak{h}_t = \frac{1}{\sqrt{\kappa}}\Im\mathfrak{h}_t^*$)

$$d\mathfrak{h}_t^*(z) = \frac{2\sqrt{\kappa}}{f_t(z)}dB_t, \quad d\langle\mathfrak{h}_t(z),\mathfrak{h}_t(w)\rangle = 4\Im(\frac{1}{f_t(z)})\Im(\frac{1}{f_t(w)}). \tag{2.5}$$

Note the centered Lowener map $f_t : \mathbb{H}\backslash\eta([0,t]) \to \mathbb{H}$ is conformal, and the Green function has conformal invariance property, hence the Green function on $\mathbb{H}\backslash\eta([0,t])$ is given by

$$G_t(z,w) = G_{\mathbb{H}}(f_t(z),f_t(w)) = \log|f_t(z)-f_t(w)| - \log|f_t(z)-\overline{f_t(w)}|$$

thus applying the Itô formula we see

$$dG_t(z,w) = -4\Im(\frac{1}{f_t(z)})\Im(\frac{1}{f_t(w)}) = -d\langle\mathfrak{h}_t(z),\mathfrak{h}_t(w)\rangle. \tag{2.6}$$

Now we fix $U \subset \mathbb{H}$ open and set $\tau_U = \inf\{t>0 : K_t \cap U \neq \varnothing\}$ and sample $\phi \in C_0^\infty(U)$, from (2.6) we can show that

$$d\langle(\mathfrak{h}_t,\phi),(\mathfrak{h}_t,\phi)\rangle = -dE_t(\phi), \quad t<\tau_U \tag{2.7}$$

(where $E_t(\phi) = \iint_{\mathbb{H}^2}\phi(x)\phi(y)G_t(x,y)dxdy$ is the Dirichlet energy of ϕ at time t) $\langle(\mathfrak{h}_t,\phi),(\mathfrak{h}_t,\phi)\rangle + E_t(\phi)$ is a continuous local martingale. This gives the following stationarity of imaginary surfaces:

Lemma 2.9 ([MS12], Lemma 3.11)**.** *Fix $U \subset \mathbb{H}$ open and define τ_U as above. Let τ be a $\mathcal{F}_t$-stopping time with $\mathbb{P}(\tau \le \tau_U) = 1$. Consider the random field $h_{U,\tau}$ on U generated by sampling K_τ and an independent GFF h_0 on $\mathbb{H}\backslash K_\tau$ and setting $h_{U,\tau} = (h_0+\mathfrak{h}_\tau)|_U$. Then $h_{U,\tau} \overset{d}{=} h|_U$ where $h = \tilde{h}+\mathfrak{h}_0$ and $\tilde{h}$ is a zero boundary GFF on $\mathbb{H}$.*

Indeed, this lemma can be shown by considering $(h_{U,\tau},\phi)$ and (h,ϕ) and arguing that they have the same characteristic function using (2.7). To show that in general K_τ is a local set as in Theorem 1.1, the idea is to extend this lemma to multiple open sets U_i and stopping times $\tau_i \le \tau_{U_i}$ (say countable and dense) and then use the characterization that a set A is local if any deterministic open $U \subset D$, the event $A \cap U = \varnothing$ is independent of the projection of h onto $H_0^1(U)$ given the projection of h onto $H^\perp(U)$. Full detail of this part is given in [MS12, p.77-p.82].

3 Non-boundary-intersecting flow lines and counterflow lines

In this section, we consider the properties of flow lines and counterflow lines when they do not intersect the boundary. That is, we assume that weights $\rho^{i,q}$ are large so they a.s. do not intersect $\partial\mathbb{H}$. Since we will grow flow lines and counterflow lines simultaneously sometimes it is convenient to take the strip $\mathcal{S} = \mathbb{R} \times (0, i\pi)$ with upper boundary $\partial_U \mathcal{S} = \{\Im z = pi\}$ and lower boundary $\partial_L \mathcal{S} = \mathbb{R}$ and apply a conformal map $\psi : \mathcal{S} \to \mathbb{H}$ (for example $\psi(z) = e^z - 1$.)

3.1 Critial heights for boundary intersection

We start from Dubédat's original lemma specifying parts of the boundary could $\mathrm{SLE}_\kappa(\underline{\rho})$ curves hit:

Lemma 3.1 ([Dub09], Lemma 15)**.** *Consider an $SLE_\kappa(\underline{\rho})$ with force points $0 < x_1 < \cdots < x_n < \infty$ and weights ρ_i. Set $\bar{\rho}_k = \sum_{i=1}^k \rho_i$ and $\bar{\rho}_n = \kappa - 6$. Also consider the swallowing time τ_1 when x_1 belongs to the compact hull generated by the curve $\eta[0, t]$.*

(i) Assume that for some k, $\bar{\rho}_i \geq \frac{\kappa}{2} - 2$ for $i < k$, $\bar{\rho}_i \leq \frac{\kappa}{2} - 4$ for $k \leq i < n$. Then a.s. as $t \uparrow \tau_1$, η_t accumulates at x_k and no other point in $[x_1, x_n]$;

(ii) Assume that for some k, $\bar{\rho}_i \geq \frac{\kappa}{2} - 2$ for $i < k$, $\bar{\rho}_k \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$ $\bar{\rho}_i \leq \frac{\kappa}{2} - 4$ for $k < i < n$. Then a.s. as $t \uparrow \tau_1$, η_t accumulates at a point in $[x_k, x_{k+1}]$ and no point in $[x_1, x_n] \backslash [x_k, x_{k+1}]$.

For general domains and force points we can use the coordinate change formula and apply Lemma 3.1. For instance, if we take a GFF on $\mathcal{S}$ with boundary data depicted in Figure 3 then we will find the flow line from 0 almost surely exits $\mathcal{S}$ at z_0. This result holds more generally when the boundary data of h on $\partial_U \mathcal{S}$ is piecewise constant, changes a finite number of times, and is at most $-\lambda$ to the left of z_0, at least λ to the right of z_0, and on $\partial_L \mathcal{S}$ is piecewise constant with finite change, at most $-\lambda + \pi\chi$ to the left of 0 and at least $\lambda - \pi\chi$ to the right of 0. This also holds for counterflow lines with λ, χ replaced by $\lambda', -\chi$.

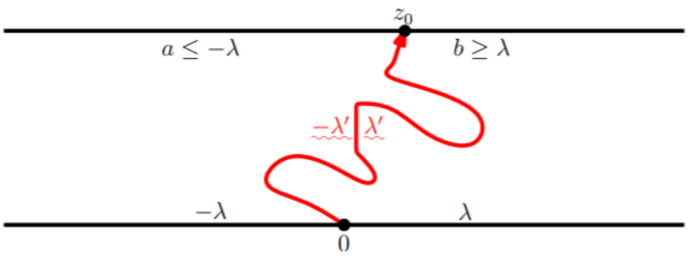

Figure 3: Assume h is a GFF on $\mathcal{S}$ with boundary data above. We take $\psi : \mathcal{S} \to \mathbb{H}$ with $\psi(z) = -e^{-z}+1$ so $\psi(-\infty) = \infty$, $\psi(+\infty) = 1$ and $\psi(\eta) \sim \mathrm{SLE}_\kappa(b/\lambda + \kappa/2 - 3; (a - b)/\lambda + \kappa/2 - 3)$ with force points at 1 and $\psi(z_0)$ is the flow line of the field on $\mathbb{H}$. Now Lemma 3.1 implies $\psi(\eta)$ will accumulate at $\psi(z_0)$.

3.2 Flow lines and counterflow lines coupling

With Dubédat's $\mathrm{SLE}_\kappa(\underline{\rho})$ first exiting criterion, we can find interesting results with Theorem 1.1. The first application is that, under non-intersecting boundary condition, flow lines of angle $\frac{\pi}{2}$ is almost surely left boundary of counterflow lines of h.

Lemma 3.2 ([MS12], Lemma 4.7)**.** *Suppose we are in the setting of Figure 4. Assume that η, η', h are coupled together so that η and η' are conditionally independent given h. Let τ be any stopping time for η. Then η' almost surely first hits $\eta([0, \tau]) \cup \partial_L \mathcal{S}$ at $\eta(\tau)$. In particular, η' contains η and hits the points of η in reverse chronological order: if $s < t$ then η' hits $\eta(t)$ before $\eta(s)$.*

To show the lemma, recall that by Theorem 1.1 η' and $\eta([0, \tau])$ are both local sets for h, and given that they are conditionally independent given h, we can apply Proposition 2.5 and 2.6 to specify the law of h given η' and $\eta([0, \tau])$ at least before η' hits $\eta([0, \tau])$ as in right panel of Figure 4. Now since η, η' are independent given h, we may take a conformal map $\mathcal{S} \backslash \eta([0, \tau]) \to \mathcal{S}$ so the law of $\psi(\eta)$ before η' hits

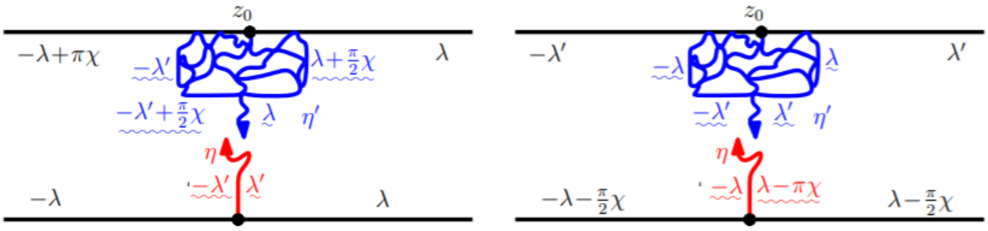

Figure 4: Assume h is a GFF on $\mathcal{S}$ with boundary data above. Consider η being the flow line of h starting from 0 (left panel) and η' being the counterflow line of $h - \frac{\pi}{2}\chi$ (right panel).

$\eta([0,\tau])$ given $\eta([0,\tau])$ is the same as η' so we only need to show $\tau = 0$ case. This follows directly by applying Dubédat's argument for counterflow lines.

With this lemma of hitting in reverse chronological order, it follows

Proposition 3.3 ([MS12], Prop 4.9)**.** *Assume we are in the setting of Lemma 3.2. Then η is almost surely equal to the left boundary of η'.*

Of course, we can extend this proposition as before to piecewise constant boundary data changing only a finite number of times, being at most $-\lambda$ to the left of z_0, at least $\lambda + \pi\chi$ to the right of z_0, and on $\partial_L\mathcal{S}$ is piecewise constant with finite change, at most $-\lambda + \pi\chi$ to the left of 0 and at least λ to the right of 0. And in this setting it is enough to show that the flow line η is a.s. determined by h:

Lemma 3.4. *Suppose X is a random variable in $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subset \mathcal{F}$ is a (complete) σ-algebra. If ψ is a measurable function on $\mathbb{R}$ such that X and $\psi(X)$ are independent given $\mathcal{G}$ then $\psi(X)$ is $\mathcal{G}$-measurable.*

Proof. Assume ψ is bounded and positive, otherwise it suffices to split $\psi = \psi^+ - \psi^-$ and consider $\psi^+ \wedge n$ with $n \to \infty$. The assumption implies $\mathbb{E}[(\psi(X) - \mathbb{E}[\psi(X)|\mathcal{G}])^2|\mathcal{G}] = 0$ and $\psi(X) = \mathbb{E}[\psi(X)|\mathcal{G}]$, a.s.. $\square$

Using arguments in type of Lemma 3.4 with X replaced by counterflow line η' and $\psi(X)$ being the left boundary of η', and $\mathcal{G}$ being $\sigma(h)$ we find

Proposition 3.5. *Suppose we are in the setting of Theorem 1.2 with weights satisfying*

$$\sum_{i=1}^{j} \rho^{i,L} \geq \frac{\kappa}{2} - 2,\ \forall 1 \leq j \leq k;\quad \sum_{i=1}^{j} \rho^{i,R} \geq 0,\ \forall 1 \leq j \leq l. \tag{3.1}$$

then η is a.s. determined by h.

Indeed, this implies that for any deterministic open set U with positive distance from 0, if we set $\tilde{\tau}_U = \inf\{t > 0 : \eta(t) \in U\}$ then $\eta([0,\tilde{\tau}_U])$ is a.s. determined by h. Moreover the properties of local set imply that given the projection of h onto $H^\perp(U)$ $\eta([0,\tilde{\tau}_U])$ is independent of h_U. Combining these two arguments we see $\eta([0,\tilde{\tau}_U])$ is a.s. determined by the projection of h onto $H^\perp(U)$.

Continuing this SLE duality argument we can actually show monotonicity of flow lines. The first lemma extends Dubédat's original argument.

Lemma 3.6 ([MS12], Lemma 5.2)**.** *Suppose that h is a GFF on $\mathcal{S}$ whose boundary data is as described in Figure 5 and let η be the flow line of h starting at 0. If $\eta|_{[0,T_0]}$ is almost surely continuous for some η-stopping time $0 < T_0 < \infty$, then $\eta|_{[0,T_0]} \cap J = \varnothing$ almost surely.*

Note that in Lemma 3.6 if we map $\mathcal{S}$ to $\mathbb{H}$, we actually find that if $\eta|_{[0,T_0]}$ is continuous then it only could possibly hit interval $(x^{j+1,L}, x^{j,L})$ with $\sum_{i=1}^{j} \rho^{i,L} \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$ or $x^{j,R}, (x^{j+1,R})$ with $\sum_{i=1}^{j} \rho^{i,R} \in (\frac{\kappa}{2} - 4, \frac{\kappa}{2} - 2)$ before T_0. This can be viewed as an extension of Lemma 3.1.

Now we know that for $\theta_1 < \theta_2$, the flow line η_{θ_1} is almost surely the left boundary of the counterflow line η'_{θ_1} of $h + (\theta_1 - \frac{\pi}{2})\chi$ so it suffices to show that η_{θ_2} almost surely stays to the left of η'_{θ_1}. The technique is similar to Lemma 3.2.

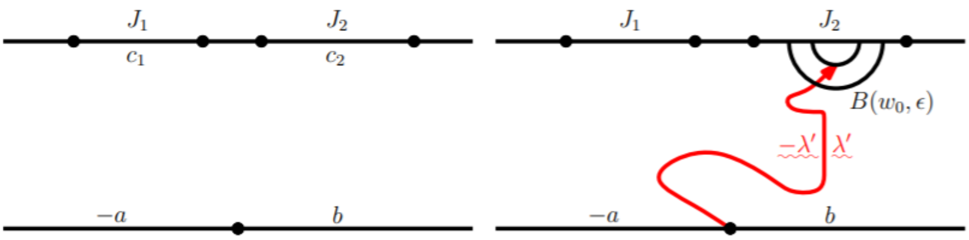

Figure 5: Assume h is a GFF on $\mathcal{S}$ and $J = \cup_k J_k$ being open subset of $\partial_U \mathcal{S}$ and on each interval J_k the boundary data is $c_k \notin (-\lambda, \lambda)$. Take $(w_0 - \varepsilon, w_0 + \varepsilon) \subset J_k$ and $0 < \tilde{\varepsilon} < \varepsilon$ along with the stopping time $\tau = \inf\{t > 0 : \eta \in B(w_0, \tilde{\varepsilon})\}$. An application of Prop 2.4 implies the law of $\eta|_{[\tau, T_0]}$ before exiting $B(w_0, \varepsilon)$ is absolutely continuous w.r.t. the flow line given boundary data c_k on $\partial_U \mathcal{S}$ and Lemma 3.1 implies that in such case $\eta|_{[\tau, T_0]}$ almost surely exits $B(w_0, \varepsilon)$ before hitting $(w_0 - \varepsilon, w_0 + \varepsilon)$. A density argument along with continuity implies a.s. η never hits J before T_0.

Lemma 3.7 ([MS12], Lemma 5.4)**.** *Assume the boundary data of GFF h is given as Figure 6 with flow line η_θ starting from 0 where $\theta > \frac{\pi}{2}$ and η' counterflow line from z_0. Let τ' be an η' stopping time such that η' a.s. has not hit 0. Let K'_t be the hull of $\eta'([0,t])$ and τ be any stopping time for $\mathcal{F}_t = \sigma(s \leq t : \eta(s); \eta'([0,t]))$. Then on $\{\mathrm{dist}(\eta(\tau), K'_{\tau'}) > 0\}$, $\eta|_{[\tau,\infty)}$ intersects neither the right side of $\eta'([0,\tau'])$ nor part of $\partial_U \mathcal{S}$ to the right of z_0 before hitting the left side of $\eta'([0,\tau'])$ nor part of $\partial_U \mathcal{S}$ the left of z_0.*

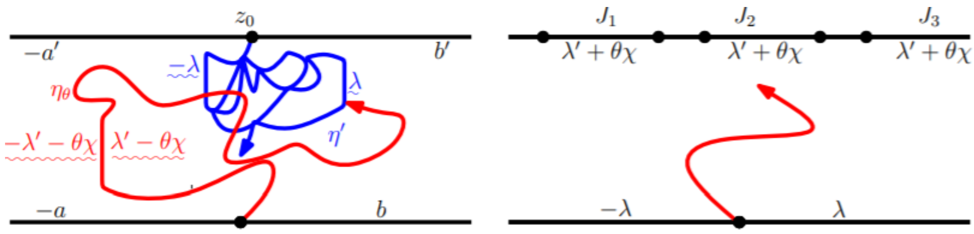

Figure 6: Assume the boundary data of GFF h is given above with $a, b \geq \lambda'$ and $a', b' \geq \lambda + \pi\chi$. We take the component D of $\mathcal{S} \backslash (\eta([0,\tau]) \cup K'_{\tau'})$ such that for small $\varepsilon > 0$ $\eta(\tau + \varepsilon)$ stay within D. We take $\psi : D \to \mathcal{S}$ sending the left and right boundaries of $\eta([0,\tau])$ to $(-\infty, 0)$ and $(0, +\infty)$ and $\psi(\eta(\tau)) = 0$. Prop 2.5 implies $\eta([0,\tau]) \cup \eta'([0,\tau'])$ is local and Prop 2.6 allows us to specify the conditional mean of h given $\eta([0,\tau]) \cup \eta'([0,\tau'])$ on parts of ∂D at least with positive distance from $\eta([0,\tau]) \cap \eta'([0,\tau'])$. The boundary value after applying ψ is depicted on right panel so the conclusion followes from Lemma 3.6.

Given Lemma 3.7, we can establish the following proposition regarding monotonicity of flow lines:

Proposition 3.8 ([MS12], Prop 5.1)**.** *Suppose h is a GFF on $\mathcal{S}$ with boundary data as Figure 6. Assume η_θ is flow line of $h + \theta\chi$ starting from 0 and η' counterflow line of h from z_0. Fix θ such that*

$$\frac{\lambda - \pi\chi - b}{\chi} \leq \theta \leq \frac{-\lambda + \pi\chi + a}{\chi} \tag{3.2}$$

Then if $\theta > \frac{\pi}{2}$ (resp. $\theta < -\frac{\pi}{2}$)then η_θ almost surely passes to the left (resp. right) of η'.

Now in the setting of Figure 6, if $\theta_1 < \theta_2$ we fit in counterflow line η'_{θ_1} of $h + (\theta_1 - \frac{\pi}{2})\chi$. If we add the assumption $\theta_1 \geq \frac{\lambda - b}{\lambda}$ then Prop 3.8 is applicable to η'_{θ_1} and η_{θ_2} given $\theta_2 \leq \frac{-\lambda + \pi\chi + a}{\chi}$ and if τ_i is the first time of η_{θ_i} accumulates in $\partial_U \mathcal{S}$ then $\eta_{\theta_2}|_{[0,\tau_2]}$ a.s. lies to the left of $\eta_{\theta_1}|_{[0,\tau_1]}$.

3.3 Light cone construction of counterflow lines

Now we have developed several properties of flow lines. Since flow lines are local sets, we can use this Markov property to construct line segments, i.e., angle-varying flow lines, giving light cone duality.

Definition 3.1. *Given angles $\theta_1, \cdots, \theta_l$, let η_{θ_1} be the flow line of h starting at 0 with angle θ_1, τ_1 be η_{θ_1} stopping time, $\eta_{\theta_1}^{\tau_1}$. For each $2 \le j \le l$ set $\eta_{\theta_1 \ldots \theta_j}^{\tau_1 \cdots \tau_{j-1}}$ be the flow line of h conditional on $\eta_{\theta_1 \ldots \theta_{j-1}}^{\tau_1 \cdots \tau_{j-1}}|_{[0,\tau_{j-1}]}$ starting at $\eta_{\theta_1 \ldots \theta_{j-1}}^{\tau_1 \cdots \tau_{j-1}}(\tau_{j-1})$ with angle θ_j, and let τ_j be a stopping time for $\eta_{\theta_1 \ldots \theta_{j-1}}^{\tau_1 \cdots \tau_{j-1}}$. We call $\eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_j}$ an **angle-varying flow line** with angles $\theta_1, \ldots, \theta_l$ w.r.t. stopping times $\tau_1 \ldots \tau_l$. The **light cone** $\boldsymbol{L}$ of h starting at 0 is defined to be the closure of set of points accessible by angle-varying flow lines with rational θ restricted in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and positive rational angle changing times. More generally, if $\eta_{\phi_1 \ldots \phi_k}^{\sigma_1 \cdots \sigma_k}$ is any angle-varying flow line with $\eta_{\phi_1 \ldots \phi_k}^{\sigma_1 \cdots \sigma_k}$-stopping time σ then $\boldsymbol{L}(\eta_{\phi_1 \ldots \phi_k}, \sigma)$ starting at $\eta_{\phi_1 \ldots \phi_k}^{\sigma_1 \cdots \sigma_k}(\sigma)$ is the closure of the set of points accessible by angle-varying flow lines starting at $\eta_{\phi_1 \ldots \phi_k}^{\sigma_1 \cdots \sigma_k}(\sigma)$ with rational angles in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and positive rational angle change time.*

Note that this definition is different from the **fan** of h starting from h, which is the points accessible by flow lines starting at 0 with fixed angles $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The main result is that the range of η' stopped when hitting the tip $\eta_{\phi_1 \ldots \phi_k}^{\sigma_1 \cdots \sigma_k}(\sigma)$ of $\eta_{\phi_1 \ldots \phi_k}^{\sigma_1 \cdots \sigma_k}|_{[0,\sigma]}$ is almost surely equal to $\mathbf{L}(\eta_{\phi_1 \ldots \phi_k}, \sigma)$. (The fan of h, however, is almost surely smaller than the light cone).

First, using an induction argument regarding Prop 2.4 and Prop 3.5 we can argue that under non-boundary intersecting condition angle-varying flow lines are continuous and a.s. determined by h:

Lemma 3.9 ([MS12], Lemma 5.6)**.** *Let $\eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_j}$ be an angle-varying flow line of h with angles $|\theta_i - \theta_j| \le \pi$. Assume $\eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_j}$ is non-boundary intersecting, $\eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_l}$ is almost surely simple and continuous. If we assume further that the boundary data for $h + \theta_1 \chi$ is at least λ on $(0, \infty)$ and at most $-\lambda + \pi\chi$ on $(-\infty, 0)$ (or at least $\lambda - \pi\chi$ on $(0, \infty)$ and at most $-\lambda$ on $(-\infty, 0)$), then $\eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_l}$ is almost surely determined by h.*

Next, arguing the same as Lemma 3.2, for $\theta_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we can perform a conform transform $\psi : \mathcal{S} \backslash \eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_l}|_{[0,\sigma]} \to \mathcal{S}$ fixing $\pm\infty$ with $\psi(\backslash \eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_l}(\sigma)) = 0$, and applying Prop 2.5 & 2.6 along with Dubédat's argument implies that the counterflow line η' a.s. exits $\mathcal{S} \backslash \eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_l}|_{[0,\sigma]}$ at $\backslash \eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_l}(\sigma)$. Similarly a density argument along with continuity implies that η' a.s. contains $\backslash \eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_l}$ and hitting the points of $\backslash \eta_{\theta_1 \ldots \theta_l}^{\tau_1 \cdots \tau_l}$ in reverse chronological order [MS12, Lemma 5.7].

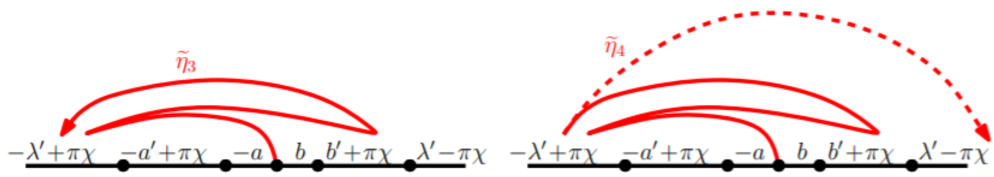

Figure 7: We grow angle-varying flow lines with $\theta_j = (-1)^{j+1}\frac{\pi}{2}$ and consider $\psi : \mathcal{S} \backslash \eta'([0, \tau']) \to \mathbb{H}$ with $\psi(\eta'(\tau')) = \infty$. The boundary data implies $\psi(\eta)$ is hitting $\mathbb{R}$ on $\psi(\eta'([0, \tau'])$ and all we need is that $\psi(\eta)$ is unbounded. This is done by stochastically bound from below each turn the amount of capacity time it takes $\psi(\eta)$ to traverse from left to right by i.i.d. nonnegative random variable with positive mean.

For the reverse inclusion, it suffices to show that for any η' stopping time τ' and $\varepsilon > 0$ there exists an angle-varying flow line with angle $\theta_j = (-1)^{j+1}\frac{\pi}{2}$ intersecting $B(\eta'(\tau'), \varepsilon)$. We take a conformal ψ mapping $\mathcal{S} \backslash \eta'([0, \tau']) \to \mathbb{H}$ fixing 0 and sending $\eta'(\tau')$ to ∞. We may use a scaling to assume the image of $\partial_U \mathcal{S}$ is contained in $\mathbb{R} \backslash (-3, 3)$. For each j we let $\tilde{\eta}_j$ be the flow line of angle $(-1)^{j+1}\frac{\pi}{2}$ starting from the tip of $\tilde{\eta}_{j-1}$ and stop once it get close to $(-1)^j[3, +\infty)$. More precisely, the curve is stopped at τ_j if its Loewner driving function $\tilde{W}_j$ enters $(-1)^j[2, +\infty)$. Then we can actually find nonnegative i.i.d. random variables Z_j with positive mean and $\tau_{2j} - \tau_{2j-1} \ge Z_j$ and hence $\tau_{2j} \to \infty$, a.s.. Then using the diameter-capacity bound $\operatorname{diam}(\tilde{\eta}([0, \tau_{2j}])) \ge \frac{1}{4}\sqrt{\operatorname{hcap}(\tilde{\eta}([0, \tau_{2j}])} = \frac{\sqrt{2}}{4}\tau_{2j}$ we find $\tilde{\eta}$ a.s. goes to ∞ which finishes the proof. The construction of Z_j is based on the following lemma:

Lemma 3.10 ([MS12], Lemma 5.10)**.** *Suppose that $(W_t, V_t^{i,q})$ is an $SLE_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ process with $W_0 = 0$ force points $(\underline{x}^L; \underline{x}^R)$. Let τ be the first time W exits $[-1, 1]$, and $C > 0$ with*

$$|\sum_{i=1}^{k} \frac{\rho^{i,L}}{W_t - V_t^{i,L}} + \sum_{i=2}^{l} \frac{\rho^{i,R}}{W_t - V_t^{i,R}}| \le C, \ \forall t \in [0, \tau]. \tag{3.3}$$

Then $\mathbb{P}(\tau \ge 1) \ge \rho_0 > 0$ with ρ_0 only depending on $C, \kappa, \rho^{1,R}$.

Indeed, we could compare the $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ process with the single point $\mathrm{SLE}_\kappa(\rho^{1,R})$ using Girsonov theorem, and condition (3.3) implies bounds on the Radon-Nykodim derivative. Thus we may reduce the lemma to the single force point case and the claim follows.

Going back to the construction of $\tilde{\eta}$, although at each turning we are add two additional force points, their weight $\rho^{i,q}$ are of same absolute value with alternating sign. Therefore we can find some universal C such that (3.3) is satisfied and Lemma 3.10 could be applied to find Z_j.

Putting all our discussion above together, we arrive at the final conclusion:

Proposition 3.11 ([MS12], Prop 5.9)**.** *Let $\eta^{\sigma_1\cdots\sigma_k}_{\phi_1\ldots\phi_k}$ be an angle-varying flow line of h with angles in $[-\frac{\pi}{2},\frac{\pi}{2}]$. Let σ be any $\eta^{\sigma_1\cdots\sigma_k}_{\phi_1\ldots\phi_k}$ stopping time. The random set $\mathbf{L}(\eta^{\sigma_1\cdots\sigma_k}_{\phi_1\ldots\phi_k},\sigma)$ is almost surely equal to the range of the counterflow line η' of h starting at z_0 stopped upon first hitting $\eta^{\sigma_1\cdots\sigma_k}_{\phi_1\ldots\phi_k}([0,\sigma])$ at $\eta^{\sigma_1\cdots\sigma_k}_{\phi_1\ldots\phi_k}(\sigma)$.*

From Prop 3.11, and in Prop 3.5 we showed that flow lines are a.s. determined by h now we have

Proposition 3.12. *Almost surely, η' is determined by h.*

Another application is that we can extend this definition of light cones to $\mathbf{L}(\underline{\theta},\bar{\theta})$ with the requirement of angles being in $[-\frac{\pi}{2},\frac{\pi}{2}]$ by in $[\underline{\theta},\bar{\theta}]$. Then for $-\frac{\pi}{2}\le\underline{\theta}<\bar{\theta}\le\frac{\pi}{2}$ $\mathbf{L}(\underline{\theta},\bar{\theta})$ is a.s. contained in η'. However when $\kappa\in(2,4)$ $\kappa'\in(4,8)$ and hence Lemma 2.8 could be applied (η' has dimension less than 2) and the law of h given η' is the same as the law of h given both η' and $h|_{\eta'}$. This in turn applies $\mathbf{L}(\underline{\theta},\bar{\theta})$ is a.s. determined by η'. A similar argument applies for the fan $\mathbf{F}$.

4 Result for general cases

So far we have established continuity, monotonicity, light cone duality of flow lines and also flow lines and counterflow lines are a.s. determined by the free field, as long as the boundary data rules out any possibility of boundary intersection. In this section we are going to extend all these results to general cases. The key here is take boundary values as before and condition on auxiliary flow lines. All properties in the previous section are inherited under conditioning and the range of flow lines satisfying the theorems is greatly extended.

4.1 Regularity of Conformal Transformation

To get the conditional law of flow lines given flow/counterflow lines, a useful tool is to consider conformal transforms ψ taking regions formed by flow/counterflow lines to $\mathbb{H}$. This is guaranteed by locality of the flow lines. After applying ψ we still need to specify the boundary data of the field and justify $\psi(\eta)$ has continuous Loewner driving functions and are corresponding flow lines for the field. This is the aim for this subsection.

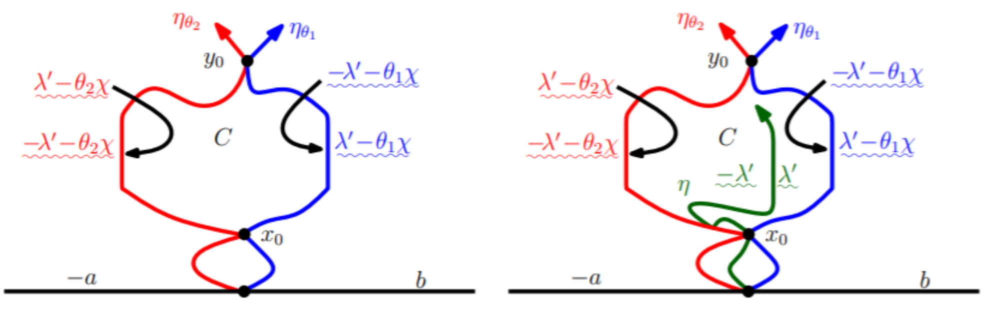

Figure 8: Suppose $\theta_1<0<\theta_2$ and η_{θ_i} are flow lines with angle θ_i. Assume a,b are chosen such that η_{θ_1} a.s. stays to right of η_{θ_2}. The conditional mean of h given all η_{θ_i} are depicted as above, at least away from intersection points. And moreover it turns out that all those intersection points are not introducing pathological behavior.

Suppose we are in the setting of Figure 8 and we want to understand the law of zero flow line η in the given pocket C formed by η_{θ_1} and η_{θ_2}. Prop 2.5 and 2.6 implies that $\eta_{\theta_1}([0,\tau_1])\cup\eta_{\theta_2}([0,\tau_2])$ is local (τ_i is some stopping time for η_{θ_i}) and the conditional mean of h on points of $\eta_{\theta_1}([0,\tau_1])\cup\eta_{\theta_2}([0,\tau_2])$

with positive distance from boundary is the same when only given $\eta_{\theta_1}([0,\tau_1])$. Therefore we need to show the intersection points x_0 and y_0 are not introducing singularities. This is done by the following proposition which allows us to perform conformal transform as before. We set $A(t)=\eta_{\theta_1}([0,t])\cup\eta_{\theta_2}$ and $\mathcal{F}_t=\sigma(\eta_{\theta_1}:s\le t;\eta_{\theta_2})$.

Proposition 4.1 ([MS12], Prop 6.1)**.** *Fix an $\mathcal{F}_T$-stopping time τ, and let $\tilde{h}$ be distributed according to the conditional law of h given $A(\tau)$ and let C be any connected component of $\mathbb{H}\backslash A(\tau)$ which is to the right of η_{θ_2}. Let $\partial C_{i,L}$(resp. $\partial C_{i,R}$) be the part of ∂C which is contained in the left (resp. right) side of η_{θ_i}. Let x_0 (resp. y_0) be the point on ∂C which is visited first (resp. last) by η_{θ_2} and let $\varphi:C\to\mathbb{H}$ be a conformal transformation which takes x_0 (resp. y_0) to 0 (resp. ∞). Let $\mathfrak{g}_C$ be the function which is harmonic in $\mathbb{H}$ with boundary values*

$$-\lambda-\theta_i\chi \text{ on } \varphi(\partial C_{i,R});\ \lambda+\theta_i\chi \text{ on } \varphi(\partial C_{i,L});\ b \text{ on } \varphi((0,\infty))$$

and set $\mathfrak{h}_C=\mathfrak{g}_C\circ\varphi-\chi\arg\varphi'$. (where the branch of $\arg\varphi'$ is chosen so that the boundary values of $\mathfrak{h}_C$ agree with those of the conditional law of h given either η_{θ_1} or η_{θ_2} on a segment of ∂C which agrees with either η_{θ_1} or η_{θ_2}). Then the law of $\tilde{h}|C$ is equal to that of the sum of a zero boundary GFF in C plus $\mathfrak{h}_C$. In particular, there is no singular contribution to $\mathfrak{h}_C$ coming from the intersection points of the paths.

The first step is justify the locality of $A(\tau)$, which is done by the following lemma via induction and characterization of local sets:

Lemma 4.2 ([MS12], Lemma 6.2)**.** *Suppose that $\eta_1,...,\eta_k$ are continuous paths such that for each $1\le i\le k$, we have that $\eta_i([0,\tau])$ is a local set for h for every η_i-stopping time τ and η_i is almost surely determined by h. Suppose that τ_1 is a stopping time for η_1 and, for each $2\le j\le k$, inductively let τ_j be a stopping time for the filtration $\mathcal{F}_t^j$ generated by $\eta_1|_{[0,\tau_1]},...,\eta_{j-1}|_{[0,\tau_{j-1}]}$ and $\eta_j(s)$ for $s\le t$. Then $\cup_{i=1}^k\eta_i([0,\tau_i])$ is a local set for h.*

We know that circle average of Gaussian Free Field at a given point evolves like a Brownian Motion when the radius is parameterized by minus log. The next lemma could be viewed as an analog for 'circle average' of local sets.

Lemma 4.3 ([MS12], Lemma 6.4)**.** *Suppose that $D\subset\mathbb{C}$ is a non-trivial, simply connected domain. Let h be a GFF on D and fix $z\in D$. Suppose that A is a local set for h such that $D\backslash A$ is simply connected and the conformal radius $C(z;D\backslash A)$ is almost surely constant and positive. Then $\mathcal{C}_A(z)$ is distributed as a Gaussian random variable with mean $C(z)$ and variance $\log C(z;D)-\log C(z;D\backslash A)$.*

The key to this lemma is to find nonrandom $\varepsilon>0$ with $B(z,2\varepsilon)\subset D\backslash A$ (by Koebe 1/4 lemma) and write $h=h_1+h_2$ as Markov decomposition for local sets. $h_\varepsilon(z)-\mathbb{E}[h_\varepsilon(z)|\sigma(h_1)]$ is equal to the average of h_2 on $\partial B(z,\varepsilon)$ hence it has distribution $N(0,-\log\varepsilon+\log C(z;D\backslash A))$. Using harmonicity $\mathcal{C}_A(z)$ is the same as $\mathbb{E}[h_\varepsilon(z)|\sigma(h_1)]$ and independent of $h_\varepsilon(z)-\mathbb{E}[h_\varepsilon(z)|\sigma(h_1)]$ and the claimed distribution follows. As a final ingredient for Prop 4.1, with this result along with the Kolmogorov extension lemma (the one we used to construct the Brownian Motion) we can modify $\mathcal{C}_{A(t)}(z)$ to be continuous in both t and z:

Proposition 4.4 ([MS12], Prop 6.5)**.** *Suppose that $D\subset\mathbb{C}$ is a non-trivial, simply connected domain. Let h be a GFF on D and suppose that $(Z(t):t\ge0)$ is an increasing family of closed sets such that $D\backslash Z(t)$ is simply connected for each $t\ge0$ and $Z(\tau)$ is local for h for every Z-stopping time τ. Suppose that $z\in D$ is such that $C(z;D\backslash Z(t))$ is almost surely continuous and strictly decreasing in t. Then $\mathcal{C}_{Z(t)}(z)-\mathcal{C}_{Z(0)}(z)$ has a modification which is a Brownian motion when parameterized by $\log C(z;D\backslash Z(0))-\log C(z;D\backslash Z(t))$ up until the first time $\tau(Z)$ that $Z(t)$ accumulates at z. Moreover, with $S=(t,z):C(z;D\backslash Z(t))>0$ we have that the map $(t,z)\to\mathcal{C}_{Z(t)}(z)$ has a modification which is almost surely continuous.*

To show Prop 4.1, using Prop 2.6 we may reduce to $\tau=\infty$ case and consider only the conditional mean near x_0 and y_0. To use the continuity given by Prop 4.4 we grow C_t to be the connected component of $\mathbb{H}\backslash A(t)$ containing C and let $\mathfrak{g}_t$ to be the function which is harmonic in $\mathbb{H}$ with boundary values given by $\lambda-\theta_2\chi$ on $\mathbb{R}^-$, $-\lambda-\theta_1\chi$ on the image of the left side of $\eta_{\theta_1}([0,t])$ under φ_t, $\lambda-\theta_1\chi$ on the image of the right side of $\eta_{\theta_1}([0,t])$ under φ_t, and b on $\varphi_t(\mathbb{R}^+)$, where φ_t is the conformal map from C_t to $\mathbb{H}$ taking x_0 to 0, y_0 to ∞ and a given point w_0 on $\partial C\cap\eta_{\theta_2}$ to -1. Then $\mathfrak{g}_t\circ\varphi_t-\chi\arg\varphi_t'$ has the same boundary behavior as $\mathcal{C}_{A(t)}$ except possibly at x_0 (again applying Prop 2.6). Now we let $t\uparrow t_0$, the time

η_{θ_1} hits η_{θ_2} at y_0, $\mathfrak{g}_t$ converges locally uniformly to the function harmonic in $\mathbb{H}$ with boundary values given by $\lambda - \theta_2\chi$ on $\mathbb{R}^-$ and $-\lambda - \theta_1\chi$ on $\mathbb{R}^+$, φ_t converges locally uniformly to the unique conformal transformation $C \to \mathbb{H}$ which takes x_0 to 0, y_0 to ∞, and w_0 to -1 and hence $\mathfrak{g}_t \circ \varphi_t - \chi \arg \varphi_t'$ converges uniformly locally to $\mathfrak{h}_C$ as $t \uparrow t_0$ and this gives regularity of $\mathfrak{h}_C$ near y_0 since $\mathcal{C}_{A(t)}$ is continuous in t and z. The behavior $\mathfrak{h}_C$ near x_0 is done by taking a counterflow line η'_{θ_1} with a.s. left boundary η_{θ_1} and using the continuity of $\mathcal{C}_{\eta'_{\theta_1} \cup \eta_{\theta_2}}$.

Besides the interaction of two flow lines, there are still many other scenarios where analogous result of Prop 4.1, including intersection between one flow line and one angle-varying flow line, flow lines contained in counterflow lines, etc. These are included in [MS12, p.117-p.122].

The next part is to argue that after conformal transform of such $\varphi : C \to \mathbb{H}$, the image $\varphi(\eta)$ still has continuous Loewner driving function. To do so we need to apply the following criterion:

Proposition 4.5 ([MS12], Prop 6.12)**.** *Suppose that $T \in (0, \infty]$. Let $\eta : [0, T) \to \mathbb{H}$ be a continuous, non-crossing curve with $\eta(0) = 0$. Assume η satisfies the following: for every $t \in (0, T)$,*

(a) $\eta((t, T))$ is contained in the closure of the unbounded connected component of $\mathbb{H} \backslash \eta((0, t))$ and

(b) $\eta^{-1}(\eta([0, t]) \cup \mathbb{R})$ has empty interior in (t, T).

For each $t > 0$, let g_t be the Loewner map taking the unbounded connected component of $\mathbb{H} \backslash \eta((0, t))$ to $\mathbb{H}$. After reparameterization, (g_t) solves the Loewner equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z$$

with continuous driving function U_t.

Now suppose we are in the setting of Figure 9 and we want to specify the conditional law of η_{θ_1} given η_{θ_2}. We take a conformal transform φ taking the right part of $\mathbb{H} \backslash \eta_{\theta_2}$ to $\mathbb{H}$. In previous part we justified such conformal transform is similar as non-boundary intersecting case and now we need to show $\varphi(\eta_{\theta_1})$ has continuous Loewner driving function. Since (at least when a, b is large) η_{θ_1} is a.s. a simple path lying to the right of η_{θ_2}, the first condition applies for $\varphi(\eta_{\theta_1})$. The second condition can be checked by arguing that if η_{θ_1} traces η_{θ_2} during time $t \in I$ then considering the conditional mean of h given $\eta_{\theta_1} \cup \eta_{\theta_2}$ will lead to a contradiction as explained in Figure 9. Hence we have

Lemma 4.6 ([MS12], Lemma 6.13)**.** *Let φ be a conformal map which takes the right connected component of $\mathbb{H} \backslash \eta_{\theta_2}$ to $\mathbb{H}$ with $\varphi(0) = 0$ and $\varphi(\infty) = \infty$. Then $\varphi(\eta_{\theta_1})$ (viewed as a path in $\bar{\mathbb{H}}$ from 0 to ∞) has a continuous Loewner driving function.*

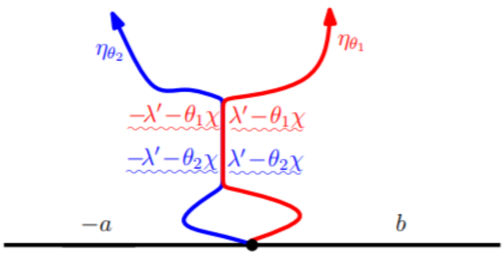

Figure 9: Suppose $\eta_{\theta_1}(t)$ is contained in η_{θ_2} for $t \in [\tau_1, \tau_2]$. Applying Prop 2.6, the conditional mean of h given $\eta_{\theta_1} \cup \eta_{\theta_2}$ on $\eta_{\theta_1}((\tau_1, \tau_2))$ is the same when only given η_{θ_1} or η_{θ_2}. This implies $\lambda - \theta_1\chi - \chi \cdot \text{winding} = \lambda - \theta_2\chi - \chi \cdot \text{winding}$, a contradiction.

Again similar to the previous part of this section, we can extend this result to a number of configurations involving angle-varying flow lines and counterflow lines. These are explained in [MS12, p.124-p.127].

So far we have justified that conformal maps ψ taking regions formed by interacting non-boundary-intersecting flow lines to $\mathbb{H}$ is well-defined and under ψ flow lines still have continuous Loewner driving function. Therefore to find the conditional law of η it remains to show that $\psi(\eta)$ is still an $\text{SLE}_\kappa(\underline{\rho})$ process and is flow line of the corresponding field. This is done by the following lemma:

Lemma 4.7 ([MS12], Lemma 7.1)**.** *Suppose we are in the setting of Figure 10. Conditional on η_{θ_1} , η_{θ_2} and η up until the first time that it hits ∂C, we have that $\eta_\psi \sim SLE_\kappa(\rho^L;\rho^R)$ where the weights ρ^R, ρ^L are given by*

$$\rho^R = -\frac{\theta_1\chi}{\lambda} - 2; \quad \rho^L = \frac{\theta_2\chi}{\lambda} - 2$$

and correspond to force points at 0^+ and 0^-, respectively. Moreover, η_ψ is almost surely continuous with $\lim_{t\to\infty}\eta_\psi(t) = \infty$ and (η_ψ, h_ψ) are coupled as in Theorem 1.1.

Also note here that the law of (η_ψ, h_ψ) only depends on θ_1, θ_2 and hence independent of $(\eta_{\theta_1}, \eta_{\theta_2})$.

To prove Lemma 4.7, note from previous discussion η_ψ is well defined as a path in $\mathbb{H}$ with continuous Loewner driving function W_t. Since $\eta, \eta_{\theta_1}, \eta_{\theta_2}$ are both a.s. determined by h, using Lemma 4.2 $\eta([0,\tau]) \cup \eta_{\theta_1} \cup \eta_{\theta_2}$ is a local set for h. Given any open $U \subset \mathbb{H}$, the events $U \subset C$ and $U \subset C\backslash\eta([0,\tau])$ is determined by the projection h_{U^c} of h onto $H^\perp(U)$ thus on $U \subset C$ the event $\eta_\psi \cap \psi(U) = \infty$ is a.s. determined by $(h_\psi)_{\psi(U)^c}$ and we can see that $\eta_\psi([0,\tau])$ is local for h_ψ for every η-stopping time τ. Furthermore in light of Prop 4.1, $\mathcal{C}^\psi_{\eta_\psi([0,t])}(z)$ is the harmonic function in $\mathbb{H}\backslash\eta_\psi([0,t])$ with boundary value given in right panel of Figure 10. Note $\mathcal{C}^\psi_{\eta_\psi([0,t])}(z)$ can be viewed as the conditional expectation of h_ψ under $\mathcal{F}^\psi_t = \{\eta_\psi([0,s]) : s \le t\}$ until z is absorbed by η_ψ and hence the martingale characterization in Theorem 2.1 is satisfied and $\eta_\psi \sim \mathrm{SLE}_\kappa(\rho^L;\rho^R)$ from 0 to ∞ (as the segment of η in C starts with x_0 and ends with y_0) with $(\rho^L;\rho^R)$ given in the lemma. The continuity of η_ψ follows from that of η.

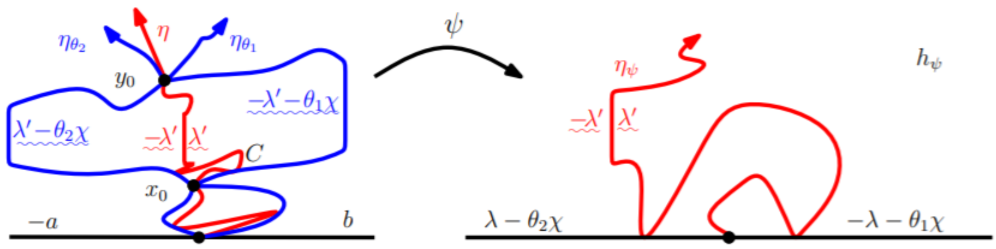

Figure 10: Suppose h is GFF on $\mathbb{H}$ with boundary data above on the left panel, $\theta_1 < 0 < \theta_2$ and η_{θ_i} is the flow line of h starting from 0 with angle θ_i, and a, b is large so they are non-boundary intersecting and η is a.s. to the left of η_{θ_1} and to the right of η_{θ_2}. Then conditionally on η_{θ_1}, η_{θ_2}, the law of η in every connected component C of $\mathbb{H}\backslash(\eta_{\theta_1} \cup \eta_{\theta_2})$ which lies between η_{θ_1} and η_{θ_2} is independently that of an $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ process with $\rho^R = -\frac{\theta_1\chi}{\lambda} - 2$; $\rho^L = \frac{\theta_2\chi}{\lambda} - 2$ and is almost surely determined by $h|_C$.

Now η_ψ is the flow line of h_ψ and it turned out that η_ψ is a.s. determined by h_ψ [MS12, Lemma 7.2]. To argue this, set h' to be the restriction of h to $\mathbb{H}\backslash\bar{C}$ and $Q = (\eta_{\theta_1}, \eta_{\theta_2}, h')$. C, ψ is determined by Q and from Lemma 2.8 (Q, h_ψ) determines the whole GFF h (and hence η_ψ). As the law of η_ψ given $\eta_{\theta_1}, \eta_{\theta_2}$ does not depend on $\eta_{\theta_1}, \eta_{\theta_2}$, it remains to show the independence of η_ψ and h'. This follows by applying Prop 2.7 for a multiple times on the components of $\mathbb{H}\backslash\bar{C}$.

Combining all the arguments above, letting θ_1, θ_2 range over $\theta_1 < 0 < \theta_2$, we obtain Theorems 1.2 and 1.3 for the case when all the force points are $0^-, 0^+$:

Proposition 4.8 ([MS12], Prop 7.3)**.** *Suppose that η is an $SLE_\kappa(\rho^L;\rho^R)$ process in $\mathbb{H}$ with $\rho^L > -2$, $\rho^R > -2$ with force points 0^- and 0^+. Then η is almost surely continuous and $\lim_{t\to\infty}\eta(t) = \infty$. Moreover, in the coupling of η with a GFF h as in Theorem 1.1, η is almost surely determined by h.*

So far in this subsection we have established the conditional law of flow lines given flow lines. Indeed as we argued before, Lemma 4.7 could be extended to other scenarios with angle-varying flow lines and counterflow lines and compute the corresponding conditional laws. We can also use this technique to extend the monotonicity of flow lines in non-boundary-intersecting regime to flow lines with two force points at $0^-, 0^+$, i.e., in the setting of Figure 10 if $\theta_1 < \theta_2$ satisfy

$$\theta_1 > -\frac{\lambda + b}{\chi}; \quad \theta_2 < \frac{\lambda + a}{\chi}$$

then η_{θ_1} a.s. lies to the right of η_{θ_2} and the conditional law of η_{θ_1} given η_{θ_2} is that of an $\mathrm{SLE}_\kappa((a - \theta_2\chi)/\lambda - 1; (\theta_2 - \theta_1)\chi/\lambda - 2)$ with force points $0^-, 0^+$.

We end this subsection with Theorem 1.2 for flow lines (i.e. $\kappa \in [0,4]$) using Prop 4.8.

Lemma 4.9 ([MS12], Lemma 7.5)**.** *In the setting of Theorem 1.1 for $\kappa \in [0,4]$, the $SLE_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ flow line η of h is almost surely determined by h.*

To establish this result, we may add force points at 0^+ and 0^- (with possibly zero weight) and assume the force points are $x^{k,L} < ... < x^{1,L} = 0^-$ and $0^+ = x^{1,R} < ... < x^{l,R}$. The case $k = l = 1$ is done and we want to show the $k+1, l$ case given the result for k, l. If the hull K_t of $\eta([0,t])$ never intersect $(-\infty, x^{k+1,L}]$ then we are finished since Prop 2.4 could be applied (recall the law of (h,η) after Girsanov transform is mutually absolutely continuous to law of (h,η)). Let τ be the first time K_t accumulates in $(-\infty, x^{k+1,L}]$. Then $\eta([0,\tau])$ is a.s. determined by h. Assume τ happens before the continuation threshold and the rightmost point of K_τ is contained in $[x^{j0,R}, x^{j0+1,R})$. Apply the centered Loewner map f_τ, the conditional law of $f_\tau(K_t)$ for $t \geq \tau$ given $K|_{[0,\tau]}$ is an $\mathrm{SLE}_\kappa(\bar{\rho}^L; \underline{\tilde{\rho}}^R)$ process in $\mathbb{H}$ with

$$\bar{\rho}^L = \sum_{i=1}^{k+1} \rho^{i,L}; \quad \tilde{\rho}^{1,R} = \sum_{i=1}^{j_0} \rho^{i,R}; \quad \tilde{\rho}^{2,R} = \rho^{j_0+1,R}, ..., \tilde{\rho}^{l-j_0+1,R} = \rho^{l,R}.$$

and $(f_\tau(K_t) : t \geq \tau)$ has force points $\bar{x}^L = 0^-$ and $x^{\tilde{1},R} = 0^+$ hence by induction is determined by $h \circ f_\tau - \chi \arg f'_\tau$ and the result follows.

4.2 Monotonicity, Merging and Crossing

In the previous part we have justified that conditioning on flow lines should give flow lines and proved Theorem 1.2 for flow lines. This technique could definitely extend to flow lines growing from different seeds, which is the aim of this subsection.

Now suppose $x_2 < x_1$ are on $\partial\mathbb{H}$, h is a GFF on $\mathbb{H}$ with piecewise constant boundary data which changes a finite number of times, fix angles $\theta_1 < \theta_2 + \pi$. We grow flow lines $\eta^{x_i}_{\theta_i}$ with angle θ_i from x_i. Let T_i be a stopping time for $\eta^{x_i}_{\theta_i}$, $i = 1,2$, such that $\eta^{x_i}_{\theta_i}|_{[0,T_i]}$ is almost surely continuous. Suppose $\tilde{\tau}_2 \leq T_2$ is a stopping time for $\mathcal{F}_t = \sigma(\eta^{x_2}_{\theta_2}(s) : s \leq t; \eta^{x_1}_{\theta_1}([0,T_1]))$ such that if ξ_2 is the largest time before $\tilde{\tau}_2$ such that $\eta^{x_2}_{\theta_2}(\tilde{\tau}_2)$ is contained in the unbounded connected component of $\mathbb{H}\backslash(\eta^{x_1}_{\theta_1}([0,T_1]) \cup \tilde{\tau}_2([0,\xi_2]))$. Also let ξ be first time t with $\eta^{x_1}_{\theta_1}(t) \in \tilde{\tau}_2([0,\tilde{\tau}_2])$. Then $\eta^{x_2}_{\theta_2}|_{[\tilde{\tau}_2,T_2]}$ cannot first exit $\mathbb{H}\backslash\eta^{x_1}_{\theta_1}([0,T_1])$ at $\eta^{x_1}_{\theta_1}([0,\xi_1))$[MS12, Lemma 7.6]. This argument is shown in Figure 11. Note the assumption on $\tilde{\tau}_2$ allows us to apply Prop 2.6 to specify the boundary data.

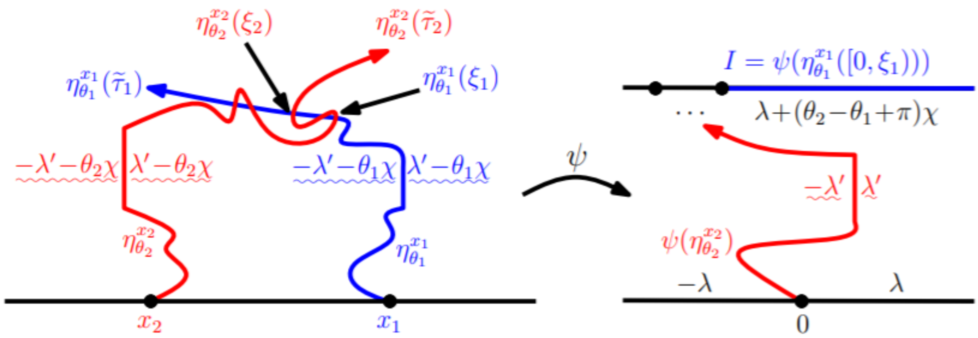

Figure 11: Suppose h is GFF on $\mathbb{H}$ with piecewise constant boundary data changing finite times. Assume $x_2 < x_1$ and $\theta_1 < \theta_2 + \pi$. We grow flow lines $\eta^{x_i}_{\theta_i}$ with angle θ_i from x_i and take stopping times T_i such that $\eta^{x_i}_{\theta_i}([0,T_i])$ is continuous. Take conformal mapping taking D, the unbounded connected component of $\mathbb{H}\backslash(\eta^{x_1}_{\theta_1}([0,T_1]) \cup \tilde{\tau}_2([0,\tilde{\tau}_2]))$ to the strip $\mathcal{S}$ with $\psi(\eta^{x_2}_{\theta_2}(\tilde{\tau}_2)) = 0$. The boundary data after taking ψ can be computed via Prop 2.4 and is depicted on the right panel and it follows from Lemma 3.6 $\psi(\eta^{x_2}_{\theta_2})$ cannot exit the strip on $\psi(\eta^{x_1}_{\theta_1}([0,\xi_1))$.

Applying this argument for a collection of countable dense stopping times, we find that if $\eta^{x_1}_{\theta_1}$ and $\eta^{x_2}_{\theta_2}$ intersect, then they must intersect in the chronological order (recall counterflow lines include (angle-varying) flow lines in reverse chronological order).

Lemma 4.10 ([MS12], Lemma 7.7). *In the setting of Figure 11, let τ_1 be the first time that $\eta_{\theta_1}^{x_1}([0,T_1])$ intersects $\eta_{\theta_2}^{x_2}([0,T_2])$ and τ_2 be the first time that $\eta_{\theta_2}^{x_2}[0,T_2]$ intersects $\eta_{\theta_1}^{x_1}[0,T_1]$. Let $K=\eta_{\theta_1}^{x_1}[0,\tau_1\wedge T_1]\cup\eta_{\theta_2}^{x_2}[0,\tau_2\wedge T_2]$. Then K is local for h, and if for $i=1,2$, $\tau_i\le T_i$, $\tau_i<\infty$ then $\eta_{\theta_1}^{x_1}(\tau_1)=\eta_{\theta_2}^{x_2}(\tau_2)$ with $\eta_{\theta_i}^{x_i}|_{[\tau_i,T_i]}$ a.s. contained in the unbounded connected component of $\mathbb{H}\backslash K$.*

Indeed, analogs of Prop 4.1 and Lemma 4.7 could be established in this setting.

Lemma 4.11 ([MS12], Lemma 7.8). *In the setting of previous lemma, let D be the unbounded connected component of $\mathbb{H}\backslash K$ and $\varphi:D\to\mathbb{H}$ be conformal transform fixing ∞ and $\mathfrak{g}$ be the harmonic function on $\mathbb{H}$ with boundary conditions given by $-\lambda-\theta_i\chi$ (resp. $\lambda-\theta_i\chi$) on the φ image of the left (resp. right) side of $\eta_i|_{[0,\tau_i\wedge T_i]}$ for $i=1,2$ and otherwise the same as $h\circ\varphi^{-1}$. Then the conditional mean $\mathcal{C}_K$ of h given $\sigma(K)$ restricted to D is $\mathfrak{g}\circ\varphi-\chi\arg\varphi'$. Moreover, $\tau_i<\infty$ for $i=1,2$ then $\eta_{\theta_i}^{x_i}|_{[\tau_i,T_i]}$ is the flow line of the conditional field h given $\sigma(K)$ restricted to D with angle θ_i starting at $w_0=\eta_{\theta_1}^{x_1}(\tau_1)=\eta_{\theta_2}^{x_2}(\tau_2)$.*

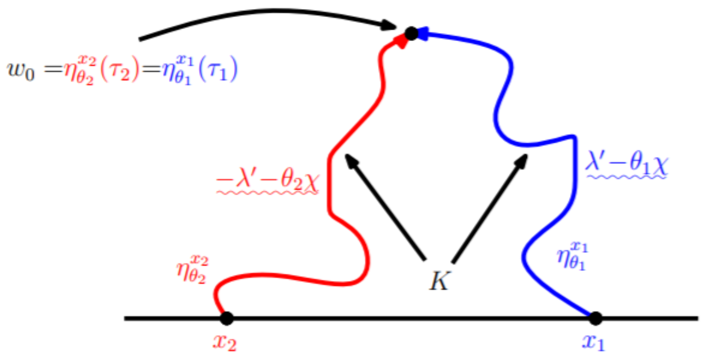

Figure 12: As a continuation of Figure 11, as in the setting of Lemma 4.10, if the flow lines $\eta_{\theta_i}^{x_i}$ hit then the first hittings at τ_i must happen at the same place $w_0=\eta_{\theta_1}^{x_1}(\tau_1)=\eta_{\theta_2}^{x_2}(\tau_2)$, and the boundary data of $h|_D$ given $\sigma(K)$ is as depicted above. Moreover, $\eta_{\theta_i}^{x_i}|_{[\tau_i,T_i]}$ is the flow line of $h|_D$ starting at w_0 with angle θ_i.

We note that in Lemmas 4.10 and 4.11, if we take a stopping time σ_2 for $\eta_{\theta_2}^{x_2}$ such that $\eta_{\theta_1}^{x_1}|_{[0,T_1]}$ lies to the right of $\eta_{\theta_2}^{x_2}|_{[0,\sigma_2]}$ and replace $\eta_{\theta_2}^{x_2}|_{[0,T_2]}$ by $\eta_{\theta_2}^{x_2}|_{[\sigma_2,T_2]}$ when defining τ_1 and τ_2, the results still hold. Hence we can extend the monotonicity for flow lines with two force points at $0^-,0^+$ to general cases:

Proposition 4.12 ([MS12], Prop 7.10). *Suppose that h is a GFF on $\mathbb{H}$ with piecewise constant boundary data which changes a finite number of times, $x_1,x_2\in\partial\mathbb{H}$, and fix angles θ_1,θ_2. For $i=1,2$, let T_i be a stopping time for $\eta_{\theta_i}^{x_i}$ such that $\eta_{\theta_i}^{x_i}([0,T_i])$ is almost surely continuous. If $\theta_1<\theta_2$ and $x_1\ge x_2$, then $\eta_{\theta_1}^{x_1}([0,T_1])$ almost surely lies to the right of $\eta_{\theta_2}^{x_2}([0,T_2])$.*

Similar results could be established for angle-varying flow lines via induction on angle-change times.

Proposition 4.13 ([MS12], Prop 7.11). *Suppose that h is a GFF on $\mathbb{H}$ with piecewise constant boundary data which changes a finite number of times. Fix angles $\theta_1,...,\theta_k$ and $\tilde\theta>\max_i\theta_i$. Assume*

$$|\theta_i-\theta_j|<\frac{2\lambda}{\chi},\quad\forall 1\le i,j\le k. \tag{4.1}$$

Let $\eta:=\eta_{\theta_1\dots\theta_k}^{\tau_1\dots\tau_k}$ be an angle varying flow line of h starting at 0 and $\tilde\eta$ be the flow line of h starting at 0 with angle $\tilde\theta$. Assume that T, $\tilde T$ are stopping times for η, $\tilde\theta$, respectively, such that $\eta|_{[0,T]}$, $\tilde\eta|_{[0,\tilde T]}$ are both almost surely continuous. Then $\eta|_{[0,T]}$ almost surely lies to the right of $\tilde\eta|_{[0,\tilde T]}$.

Note in previous section we assumed $|\theta_i-\theta_j|\le\pi$, which implies that η is a simple curve not hitting itself; (4.1) is the relaxed condition requiring η to be non-selfcrossing.

Now we are ready to complete the proof Theorem 1.4 under the assumption that flow lines are continuous.

Proposition 4.14 ([MS12], Prop 7.12). *We continue the setting of Prop 4.12 except the assumption $\theta_1<\theta_2$. If $\theta_2<\theta_1<\theta_2+\pi$, then $\eta_{\theta_1}^{x_1}([0,T_1])$ a.s. crosses $\eta_{\theta_2}^{x_2}([0,T_2])$ upon intersecting. After crossing $\eta_{\theta_1}^{x_1}([0,T_1])$ and $\eta_{\theta_2}^{x_2}([0,T_2])$ may continue to bounce off of each other, but will never cross again. If $\theta_1=\theta_2$ then $\eta_{\theta_1}^{x_1}([0,T_1])$ and $\eta_{\theta_2}^{x_2}([0,T_2])$ merges upon intersecting.*

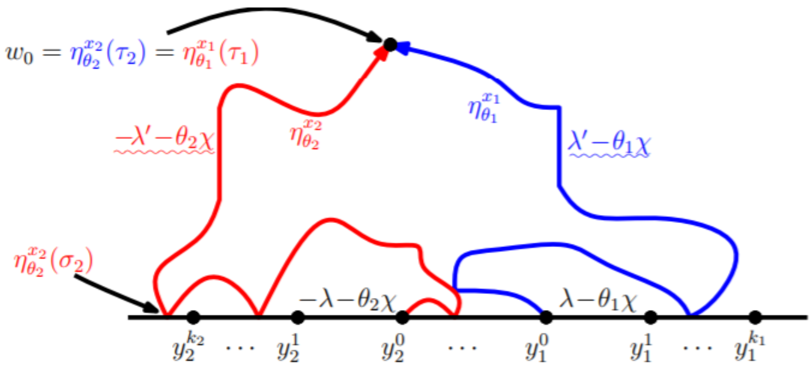

Figure 13: The proof for Prop 4.12 is again an induction argument. Suppose that h is a GFF on $\mathbb{H}$ with piecewise constant boundary data which changes a finite number of times, $x_1, x_2 \in \partial\mathbb{H}$ with $x_1 > x_2$, and fix angles $\theta_1 < \theta_2$. Let $y_2^{k_2} < .. < y_2^1 \leq y_2^0 = x_2$ and $x_1 = y_1^0 \leq y_1^1 < ... < y_1^{k_1}$ such that outside (x_2, x_1) h only changes at points listed above and the boundary data of h in $[y_2^1, x_2]$ is $-\lambda - \theta_2\chi$ and in $[x_1, y_1^1]$ is $\lambda - \theta_1\chi$. If $k_1, k_2 \leq 1$ then the result is clear from Lemmas 4.10 and 4.11 and the monotonicity result for flow lines with points at $0^-, 0^+$. Now assume the result for $k_2 - 1$ and k_1, and let σ_2 be the first time t such that $\eta_{\theta_2}^{x_2}([0, T_2])$ hits $(-\infty, y_2^{k_2}]$ then using absolute continuity result Prop 2.4 $\eta_{\theta_2}^{x_2}([0, \sigma_2 \wedge T_2])$ a.s. lies to the left of $\eta_{\theta_1}^{x_1}([0, T_1])$. Now let τ_1 be the first time $\eta_{\theta_1}^{x_1}([0, T_1])$ hits $\eta_{\theta_2}^{x_2}([\sigma_2, T_2])$ and τ_2 be the first time (after σ_2) $\eta_{\theta_2}^{x_2}([\sigma_2, T_2])$ hits $\eta_{\theta_1}^{x_1}([0, T_1])$. The $\tau_1 = \tau_2 = \infty$ case is clear and when $\tau_1, \tau_2 < \infty$ applying Lemmas 4.10 and 4.11 $\eta_{\theta_1}^{x_1}([\tau_1, T_1])$ almost surely lies to the right of $\eta_{\theta_2}^{x_2}([\tau_2, T_2])$ by induction hypothesis.

Again let τ_1 and τ_2 defined as in Lemma 4.10, and applying Lemma 4.10 and Lemma 4.11, if $\tau_1, \tau_2 < \infty$ then $\eta_{\theta_i}^{x_i}|_{[\tau_i, T_i]}$ is the flow line of $h|_D$ given $\sigma(K)$ starting at $w_0 = \eta_{\theta_1}^{x_1}(\tau_1) = \eta_{\theta_2}^{x_2}(\tau_2)$ with angle θ_i. If $\theta_2 < \theta_1 < \theta_2 + \pi$ then the result follows from Prop 4.12 (flow lines from w_0), and if $\theta_1 = \theta_2$ then the merging is from Theorem 1.2 as flow lines are a.s. determined by the free field.

With Prop 4.14 we are able to compute the conditional law of $\eta_{\theta_1}^{x_1}$ given $\eta_{\theta_2}^{x_2}$ before $\eta_{\theta_1}^{x_1}$ crosses $\eta_{\theta_2}^{x_2}$ as we did in Lemma 4.7. For instance, assume for simplicity the boundary value of h is simply a fixed constant c, then $\eta_{\theta_i}^{x_i}$ is continuous and that conditional law is nothing but an $\mathrm{SLE}_\kappa(\rho^{1,L}, \rho^{2,L}; \rho^{1,R})$ process with

$$\rho^{1,L} = -\frac{\theta_1\chi + c}{\lambda} - 1; \quad \rho^{1,L} + \rho^{2,L} = \frac{(\theta_2 - \theta_1)\chi}{\lambda} - 2; \quad \rho^{1,R} = \frac{\theta_1\chi + c}{\lambda} - 1.$$

In particular, assume $\theta_1\chi + c \in (-\lambda, \lambda)$. If $\theta_1 \geq \theta_2$ then $\rho^{1,L} + \rho^{2,L} \leq -2$ and $\eta_{\theta_1}^{x_1}$ almost surely intersects and crosses $\eta_{\theta_2}^{x_2}$.

4.3 Continuity for flow lines with many force points

In this part we establish the continuity of flow lines for general cases. Again by using Prop 2.4, we start from the two-force-point case and use induction to extend to general settings. To complete the induction step, we need to show that η is continuous when interacting with a force point or hitting the boundary at the continuation threshold. For the two force point case, we work on the $\mathrm{SLE}_\kappa(\rho^{1,R}, \rho^{2,R})$ case.

The first result is that for any $\mathrm{SLE}_\kappa(\rho^L; \rho^R)$ curve η with $\rho^L > -2$ and $\rho^R \in (-2, \frac{\kappa}{2} - 2)$ and force point at $0^+, 0^-$, $\eta \cap \partial\mathbb{H}$ a.s. has zero Lebesgue measure [MS12, Lemma 7.16]. The proof is again based on conditioning, i.e., we construct auxiliary flow lines $\mathrm{SLE}_\kappa(\rho_n^L; \rho_n^R)$ such that the conditional law of η_{θ_k} given $\eta_{\theta_{k-1}}$ is $\mathrm{SLE}_\kappa(\rho_n^L; \rho^R)$ and n is large with $\rho_n^R \geq \frac{\kappa}{2} - 2$. In this case we apply Lemma 3.6 trace back so that the probability of η to hit a given $x \in \mathbb{R}$ is 0.

The next result given in [MS12, Lemma 7.17] is an extension of continuity result in Prop 4.8 to angle-varying flow lines with force points at 0^+ and 0^-, i.e., if h is a GFF on $\mathbb{H}$ with boundary data $-a$ on $(-\infty, 0)$ and b on $(0, \infty)$, $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$ is an angle-varying flow line with $|\theta_1 - \theta_2| < \frac{2\lambda}{\chi}$ (as in Prop 4.13) and $b + \theta_i\chi > -\lambda$, $-a + \theta_i\chi < \lambda$ (as in Prop 4.8 this corresponds to $\rho > -2$) then $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$ is a.s. continuous. The proof is once again take auxiliary flow lines η_θ and $\eta_{\tilde\theta}$ such that $-\lambda - \theta_1\chi = \lambda - \theta\chi$ and $-\lambda - \tilde\theta\chi = \lambda - \theta_1\chi$ (this implies that when we perform a conformal transform at τ_1 no new force point other than 0^- and 0^+

is created) and conditioning. An analog of the proof for Lemma 4.7 could be applied and the continuity follows as the conditional law of $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$ after τ_1 given $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}([0,\tau_1])$, η_θ and $\eta_{\tilde\theta}$ is specified by $\mathrm{SLE}_\kappa(\tilde\rho^L;\tilde\rho^R)$ with $\tilde\rho^L = \frac{\theta_1-\theta_2}{\lambda}$, $\tilde\rho^R = -\frac{\theta_1-\theta_2}{\lambda}$ and force points 0^-, 0^+. (Thus Prop 4.8 is applicable.)

With the two results above we can establish continuity for the first case of two-force-point flow lines.

Lemma 4.15 ([MS12], Lemma 7.18)**.** *Suppose η is an $SLE_\kappa(\rho^{1,R},\rho^{2,R})$ process in $\mathbb{H}$ with $\rho^{1,R} > -2$, $\rho^{1,R}+\rho^{2,R} > -2$ and force points at $0^+, 1$. Assume $|\rho^{2,R}| < 2$ or $\rho^{1,R} < \frac{\kappa}{2}-2$. Then η almost surely does not hit 1 and is generated by a continuous curve.*

When $|\rho^{2,R}| < 2$ then we grow an angle-varying flow line $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$ with $\tau_1 = 1$,

$$\theta_1 = -\frac{\lambda}{\chi}(2+\rho^{1,R});\quad \theta_2 = -\frac{\lambda}{\chi}(2+\rho^{1,R}+\rho^{2,R})$$

with respect to GFF h on $\mathbb{H}$ given boundary data $-a=-\lambda$ on $(-\infty,0)$ and $b>0$ on $(0,\infty)$ sufficiently large. Also let η be the zero angle flow line of h starting from 0. Then $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$ and η are both continuous and η a.s. lies to the left of $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$ by previous argument and $\theta_1,\theta_2 < 0$. Now take the left component of $\mathbb{H}\backslash\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$ and $\psi: C\to\mathbb{H}$ conformal fixing 0 and ∞. Then from Section 4.1 $\psi(\eta)$ is an $\mathrm{SLE}_\kappa(\rho^{1,R},\rho^{2,R})$ process and continuity follows. The second part follows by taking $0>\theta>\theta_1,\theta_2$ and conditioning on θ_θ and $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$. In this case the conditional law of η is reduced to single force point case and [MS12, Lemma 7.16] is applicable.

When it comes to $\rho^{1,R} < \frac{\kappa}{2}-2$ case, we may assume $\rho^{1,R}+\rho^{2,R} \geq \frac{\kappa}{2}-2$ (otherwise back to the previous case). Using absolute continuity from Prop 2.4 we can compare with $\mathrm{SLE}_\kappa(\rho^{1,R},\frac{\kappa}{2}-2-\rho^{1,R})$, the first case we argued before as long as η a.s. does not hit $[1,\infty)$. To show this we take the flow line η_1 of the same h as η but starting from 1. If η hits $[1,\infty)$ then it has to merge with η_1. However after merging η evolves as $\mathrm{SLE}_{\rho^{1,R}+\rho^{2,R}}$ (in the corresponding unbounded component) and a.s. does not hit the boundary, which leads to a contradiction.

Note that for $\rho^{1,L} > -2$, if in the $|\rho^{2,R}| < 2$ case we condition furthermore on both $\eta_{\theta_1\theta_2}^{\tau_1\tau_2}$ and $\eta_{\tilde\theta}$ where $\eta_{\tilde\theta}$ is the flow line of h with angle $(2+\rho^{1,L})\frac{\lambda}{\chi} > 0$ then we obtain the continuity for $\mathrm{SLE}_\kappa(\rho^{1,L};\rho^{1,R},\rho^{2,R})$ (with left force point $x^L = 0^-$).

The following lemma completes the continuity of $\mathrm{SLE}_\kappa(\rho^{1,R},\rho^{2,R})$ process when $\rho^{1,R} > -2$ and $\rho^{1,R}+\rho^{2,R} > -2$.

Lemma 4.16 ([MS12], Lemma 7.20)**.** *Assume we are in the setting of Lemma 4.15 with $\rho^{1,R} \geq \frac{\kappa}{2}-2$ and $\rho^{1,R}+\rho^{2,R} > -2$. Then η almost surely does not hit 1 and is generated by a continuous curve.*

Indeed we may assume $\rho^{1,R}+\rho^{2,R} \in (-2,\frac{\kappa}{2})$ and using Prop 2.4 again η is a.s. continuous at least before hitting $[1,\infty)$ at time τ. From Dubédat's argument η a.s. accumulate at $(1,\infty)$, and after the hitting time τ η evolves as $\mathrm{SLE}_(\rho^{1,R}+\rho^{2,R})$ process in the unbounded connected component C of $\mathbb{H}\backslash\eta([0,\tau])$ and is continuous.

The the following lemma deals with the case when the continuity threshold is possibly hit.

Lemma 4.17 ([MS12], Lemma 7.21)**.** *Suppose η is an $SLE_\kappa(\rho^{1,R},\rho^{2,R})$ process in $\mathbb{H}$ with force points $0<x^{1,R}<x^{2,R}<\infty$. If $\rho^{1,R}\leq -2$ or $\rho^{1,R}+\rho^{2,R}\leq -2$, then η is a.s. a continuous curve.*

The proof is again conditioning on auxiliary flow lines and is done in [MS12, p.149-p.151]. Note that if $\rho^{1,R}+\rho^{2,R}\leq\frac{\kappa}{2}-4$ then we may take conformal map ψ with $\psi(x^{2,R})=\infty$ (say $\psi(z)=\frac{1}{x^{2,R}-z}$) then the boundary data on $(-\infty,0)$ becomes lesser or equal to $-(\frac{\kappa}{2}-2)\lambda$ which corresponds to the well-behaved case.

Now to prove Theorem 1.3 for $\kappa\in(0,4)$, assume the continuation threshold is not hit at the starting point 0 and we may run η for a small amount of time (such that force point at 0^- will not bother; This is allowed by Prop 2.4 and the two force point case) and then apply a conformal mapping to make sure that all the force points are located to the right of 0. Denote the location of force points by $\underline{x}^R$ with $|\underline{x}^R| = n$. $n=2$ case is done and suppose that we want to prove the $n+1$ case from the result for $|\underline{x}^R|=n$ case.

Suppose that there exists $j_0\geq 2$ with $\sum_{i=1}^{j_0-1}\rho^{i,R} > \frac{\kappa}{2}-4$ and $\sum_{i=1}^{j_0}\rho^{i,R} > \frac{\kappa}{2}-2$ as the flow line coupled with the GFF as in Figure 14. We apply a conformal transform ψ with $\tilde\eta=\psi(\eta)$, force points $\underline{\tilde{x}}$ and Loewner driving function $\tilde W_t$. Let $\tilde g_t$ be Lowener map with $\tilde V_t^{i,q} = \tilde g_t(\tilde x^{i,q})$. We sample the stopping times $\tilde\xi_j$ and $\tilde\zeta_j$ as follows: let $\tilde\xi_1$ be the first time $\tilde W_t = 0$ and $\tilde\zeta_1$ be the first time t after $\tilde\xi_1$ that $\tilde\eta$

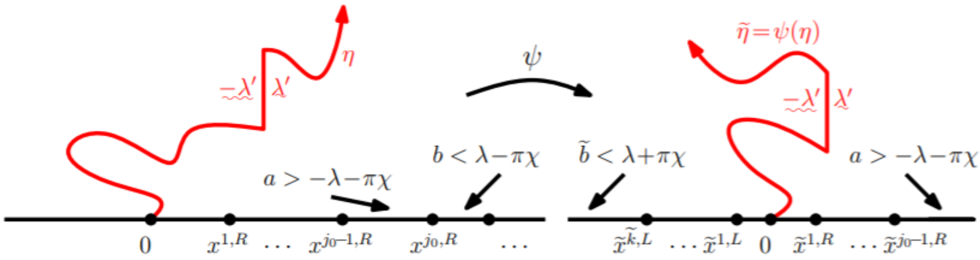

Figure 14: Let h be GFF on $\mathbb{H}$ with boundary data as the left side and η be the corresponding flow line from 0. Let $\psi : \mathbb{H} \to \mathbb{H}$ the conformal map taking ∞ to -1, $x^{1,R}$ to 1 and $x^{j_0,R}$ to ∞ while $\underline{x}^R$ is taken to $\underline{\tilde{x}}$. Then it is possible for $\tilde{\eta} = \psi(\eta)$ to hit $(-\infty, \tilde{x}^{\tilde{k},L}) \cup (\tilde{x}^{j_0-1,R}, +\infty)$ or reach ∞ before hitting other points of $\partial\mathbb{H}$.

comes within distance $\frac{1}{2}$ of either $[\tilde{V}_0^{\tilde{k},L}, \tilde{V}_0^{1,L}]$ or $[\tilde{V}_0^{1,R}, \tilde{V}_0^{j_0-1,R}]$. For $k \geq 2$ let $\tilde{\xi}_k$ be the first time after $\tilde{\eta}_{k-1}$ such that $\tilde{W}_t = 0$, and $\tilde{\eta}_k$ be the first time after $\tilde{\xi}_k$ such that $g_{\tilde{\xi}_k}(\tilde{\eta}(t))$ comes within distance $\frac{1}{2}$ of either $[\tilde{V}_{\tilde{\xi}_k}^{\tilde{k},L}, \tilde{V}_{\tilde{\xi}_k}^{1,L}]$ or $[\tilde{V}_{\tilde{\xi}_k}^{1,R}, \tilde{V}_{\tilde{\xi}_k}^{j_0-1,R}]$. Let $\tilde{\tau}$ be the first time $\tilde{\eta}$ either hits $(-\infty, \tilde{V}_0^{2,L}]$, $[\tilde{V}_0^{2,R}, \infty)$, or the continuation threshold, or escapes to ∞. We apply the following uniform estimate:

Lemma 4.18 ([MS12], Lemma 7.22)**.** *Suppose that η is an $SLE_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ process in $\mathbb{H}$ starting from 0, with $k = |\underline{\rho}^L|$, $l = |\underline{\rho}^R|$, $\sum_{i=1}^k \rho^{i,L}, \sum_{i=1}^l \rho^{i,R} > \frac{\kappa}{2} - 4$ and $|x^{1,L}|, |x^{1,R}| > 1$. Fix $M > 0$ such that the force points satisfy $x^{i,R}/x^{1,R} \leq M$ and $x^{i,L}/x^{1,L} \leq M$ for all i and the weights satisfy $|\rho^{i,q}| \leq M$. Now let E_1 be the event that either $\lim_{t\to\infty} \eta(t) = \infty$ or η disconnects $x^{k,L}$ or $x^{l,R}$, and let E_2 be the event that $\mathrm{dist}(\eta([0,\infty)), [x^{1,R}, x^{l,R}] \cup [x^{k,L}, x^{1,L}]) \geq \frac{1}{2}$. Then there exists a $\rho_0 > 0$ depending only on M, $\sum_{i=1}^k \rho^{i,L}$ and $\sum_{i=1}^l \rho^{i,R}$ such that $\mathbb{P}(E_1 \cap E_2) \geq \rho_0 > 0$.*

Now let $E = \cup_k \{\tilde{\tau} \leq \tilde{\eta}_k\}$ and we can show that $\tilde{\eta}|_{[0,\tilde{\tau}]}$ is continuous. Indeed $\tilde{\eta}|_{[0,\tilde{\tau}\wedge\tilde{\xi}_k]}$ is continuous as a consequence of induction hypothesis and absolute continuity from Girsanov Theorem when $k = 1$. For $t \in (\tilde{\xi}_k, \tilde{\zeta}_k]$ we may use Prop 2.4 to compare with the two-force point case, while for $t \in (\tilde{\zeta}_k, \tilde{\xi}_{k+1}]$ again we can apply the Girsanov Theorem along with the induction hypothesis. Using monotonicity of force points in $\mathrm{SLE}_\kappa(\underline{\rho}^L; \underline{\rho}^R)$ process the assumption of Lemma 4.18 is satisfied and thus $\mathbb{P}(\tilde{\tau} < \tilde{\eta}_{k+1} | \tilde{\tau} > \tilde{\eta}_k) \geq \rho_0$ and $\mathbb{R}(E^c) = 0$ as desired.

There are two remaining cases. One easy case is, $\sum_{i=1}^j \rho^{i,R} \leq \frac{\kappa}{2} - 4$ for all $j = 1, ..., n+1$. Applying a conformal transform sending $x^{1,R}$ to ∞ and other $x^{i,R}$ to the left side of 0, we get the desired continuity since now $\sum_{i=1}^j \tilde{\rho}^{i,L} \geq \frac{\kappa}{2} - 2$ for all j. The other case is that there exists a J with $\sum_{i=1}^j \rho^{i,R} \leq \frac{\kappa}{2} - 4$ for all $j = 1, ..., J-1$,$\sum_{i=1}^j \rho^{i,R} \geq \frac{\kappa}{2} - 2$ for all $j = J+1, ..., n$ and $\sum_{i=1}^J \rho^{i,R} \geq \frac{\kappa}{2} - 4$. In this case we grow an auxiliary flow line at $x^{j+1,R}$ with angle π and the argument is the same as two force point case.

Now we have established the continuity of flow lines and indeed an induction argument can let us extend to angle-varying flow lines (with angles satisfying non-selfcrossing condition (4.1)).

4.4 The counterflow lines

We have proved Theorem 1.2 and Theorem 1.3 for flow lines. We can modify these proofs and use the duality argument developed in Section 3.2 and 3.3 (plus conditioning argument) to extend to the counterflow line case. We only sketch some examples with two force points and the detailed statements could be found in [MS12, p.154-p.162]. Again as we are using duality, sometimes it is more convenient to consider the strip $\mathcal{S}$ other than $\mathbb{H}$.

The first lemma is the analog of Prop 4.8 for counterflow lines.

Lemma 4.19 ([MS12], Lemma 7.24)**.** *Suppose that $\rho^L > -2$ and $\rho^R > \frac{\kappa'}{2} - 4$. In the coupling of an $SLE_{\kappa'}(\rho^L; \rho^R)$ process η_0' with a GFF h_0 as in Theorem 1.1, η_0' is a.s. determined by h_0. Moreover, η_0' is a.s. a continuous path.*

Suppose we are in the setting of Figure 15. Then η' a.s. lies between η_{θ_1} and η_{θ_2} (Assume a, b, a', b' are large and this follows from monotonicity in Section 3.2). The first thing to observe is that for

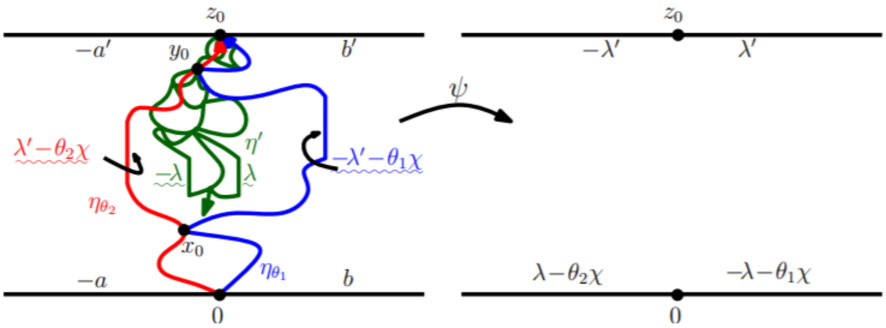

Figure 15: Let h be GFF on the strip $\mathcal{S}$ with boundary data as the left side and η_{θ_i} be the corresponding flow line from 0 with angle θ_i. Assume $\theta_1 < -\frac{\pi}{2}$ and $\theta_2 > \frac{\pi}{2}$, which implies η' a.s. intersects C and conditioning makes sense. Take C to be any connected component of $\mathcal{S}\backslash(\eta_{\theta_1} \cup \eta_{\theta_2})$. Fix a stopping time τ' for $\mathcal{F}_t : \sigma(\eta'(s) : s \leq t, \eta_{\theta_1}, \eta_{\theta_2})$ such that $\eta'(\tau') \in C$ almost surely. Let ψ be the conformal map from the connected component of $C\backslash\eta'([0,\tau'])$ containing x_0 to $\mathcal{S}$ taking $\eta'(\tau)$ to z_0 and x_0 to 0. Then the boundary data of $h \circ \psi^{-1} - \chi \arg(\psi^{-1})'$ is depicted on the right side and the conditional law of η' viewed as a path in C is an $\mathrm{SLE}'_\kappa((1/2+\theta_2/\pi)(\kappa'/2-2)-2; (1/2-\theta_1/\pi)(\kappa'/2-2)-2)$ process.

$A(t) = \eta_{\theta_1} \cup \eta'([0,t]) \cup \eta_{\theta_2}$, $A(\tau)$ is a local set for h. The arguments in Section 4.1 is applicable, i.e., the conformal transform ψ taking the connected component of $C\backslash\eta'([0,\tau'])$ containing x_0 to $\mathcal{S}$ is well defined with boundary data specified on the right panel. Using the martingale characterization Theorem 2.1 (the corresponding $\mathfrak{h}_t$ is the conditional expectation of h given $\mathcal{F}_t$) and the continuity from 4.4 gives the conditional law of η' given η_{θ_1} and η_{θ_2} in each bounded component of $\mathcal{S}\backslash$ is an $\mathrm{SLE}'_\kappa(\rho^L;\rho^R)$ process with

$$\rho^L = (\frac{1}{2}+\frac{\theta_2}{\pi})(\frac{\kappa'}{2}-2)-2;\ \rho^R = (\frac{1}{2}-\frac{\theta_1}{\pi})(\frac{\kappa'}{2}-2)-2.$$

The continuity follows since we may take a, b large where absolute continuity implies η' is a.s. continuous, and we let $\theta_1 < -\frac{\pi}{2}$, $\theta_2 > \frac{\pi}{2}$ vary. This law is independent of η_{θ_1}, η_{θ_2} and we may furthermore argue as in Prop 4.8 to find η'_ψ is a.s. determined by h_ψ.

If we replace η_{θ_1} and η_{θ_2} by suitable angle-varying flow lines then we may extend Lemma 4.19 to $\rho^L, \rho^R \in (-2, \frac{\kappa'}{2}-4)$. This is done by [MS12, Lemma 7.25]. Theorem 1.2 for counterflow lines now follows with essentially the same proof for flow lines in Lemma 4.9.

The next result is an analog of Lemma 4.15. Here for counterflow lines we are rotated 180 degrees with 'L', 'R' swapped.

Lemma 4.20 ([MS12], Lemma 7.26)**.** *Suppose η'_0 is an $SLE_{\kappa'}(\rho^{1,R},\rho^{2,R})$ process in $\mathbb{H}$ with $\rho^{1,R} > -2$, $\rho^{1,R}+\rho^{2,R} > -2$ and force points at $-1,-2$. Assume $|\rho^{2,R}| < \frac{\kappa'}{2}$. Then η'_0 is almost surely a continuous curve.*

To prove this lemma, we consider a GFF h on the strip $\mathcal{S}$ with boundary data as in Figure 15 and grow an angle-varying flow line $\eta^{\tau_1\tau_2}_{\theta_1\theta_2}$ with $\tau_1 = 1$ and $|\theta_1-\theta_2| < \frac{2\lambda}{\chi}$ (non-selfcrossing). When $\theta_1, \theta_2 < \frac{\pi}{2}$ $\eta^{\tau_1\tau_2}_{\theta_1\theta_2}$ a.s. stays to the right of the left boundary of η' and the conditional law of η' given $\eta^{\tau_1\tau_2}_{\theta_1\theta_2}$ viewed as a path in the left connected component of $\mathcal{S}\backslash\eta^{\tau_1\tau_2}_{\theta_1\theta_2}$ is that of an $\mathrm{SLE}_{\kappa'}(\rho^{1,R},\rho^{2,R})$ process with

$$\rho^{1,R} = (\frac{1}{2}-\frac{\theta_2}{\pi})(\frac{\kappa'}{2}-2)-2;\quad \rho^{1,R}+\rho^{2,R} = (\frac{1}{2}-\frac{\theta_1}{\pi})(\frac{\kappa'}{2}-2)-2.$$

Therefore the result follows from the continuity of η' and varying θ_1, θ_2.

Indeed the continuity for general two-force-point counterflow lines could be established in a similar manner and we may as well deduce Theorem 1.3 for $\kappa' > 4$ from two-force-point case.

Now if we go back to the light cone construction in Prop 3.11, all the essential inputs are:

1. The angle-varying flow line $\eta^{\sigma_1\ldots\sigma_k}_{\phi_1\ldots\phi_k}$ and the counterflow line η' are a.s. continuous. This implies that we could sample a countably dense subset of stopping times and invoke continuity such that we need not worry uncountablility issues;

2. η' a.s. exits $\mathcal{S}$ at 0 and hence by applying a conformal map the angle-varying flow line given $\phi_i \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is included in η' in reverse chronological order;

3. $\eta^{\sigma_1\dots\sigma_k}_{\phi_1\dots\phi_k}$ almost surely hits the left (resp. right) side of $\eta'([0,\tau'])$ or the side of $\partial_U\mathcal{S}$ to the left (resp. right) of z_0 when $\theta_l = \frac{\pi}{2}$ (resp. $\theta_l = -\frac{\pi}{2}$). From this if we take a conformal map taking $\mathcal{S}\backslash\eta'([0,\tau']) \to \mathbb{H}$ with $\eta'(\tau)$ taken to ∞ then we are able to grow flow lines with angles alternating in $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ and a.s. unbounded, indicating that we can construct $\eta^{\sigma_1\dots\sigma_k}_{\phi_1\dots\phi_k}$ with $\phi_i \in \{-\frac{\pi}{2}, \frac{\pi}{2}\}$ including $\eta'(\tau')$.

For the general setting all the ingredients 1-3 are able to fill in and thus Theorem 1.5 follows.

If η' hits some boundary point z almost surely (for instance in Dubédat's lemma) we may as well condition on the segment of η' before hitting z and apply a conformal transform to find the conditional law of η' after hitting z. This is done in [MS12, Prop 7.31] using similar techniques as in Lemma 4.19.

As a final epilogue, recall that in contrast to angle-varying flow lines, the fan $\mathbf{F}$ is the points accessible by flow lines starting from 0 with some fixed angle in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. For some given point in the counterflow line $\eta'(\tau')$, $\eta'(\tau')$ a.s. belongs to the light cone and a.s. does not belong to the fan.

Proposition 4.21 ([MS12], Prop 7.33)**.** *Suppose that we have a GFF h on the strip $\mathcal{S}$ with boundary value as in the left panel of Figure 15 with $a, b \geq \lambda'$ and $a', b' \geq \lambda' + \pi\chi$. Let τ' be any η' stopping time such that $\eta'(\tau') \neq 0$ almost surely. Then we have that $\mathbb{P}(\eta' \in \boldsymbol{F}) = 0$. In particular (as the fan is contained in η') $\boldsymbol{F}$ a.s. has Lebesgue measure 0.*

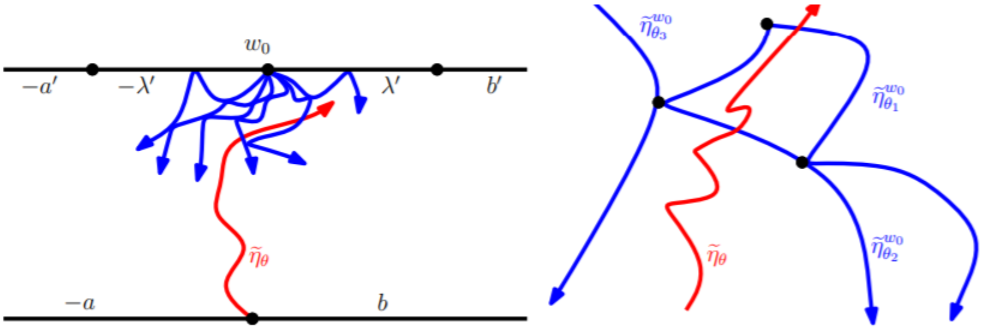

Figure 16: Take a conformal map from $\mathcal{S}\backslash\eta'([0,\tau'])$ back to $\mathcal{S}$ fixing 0 and taking $\eta'(\tau')$ to $w_0 \in \partial_U\mathcal{S}$. The boundary condition after taking ψ is specified on the left panel. We grow flow lines $\tilde{\eta}^{w_0}_{\theta_i}$ with angle θ_i for $i = 1, ..., n$. We are able to choose θ_i such that the flow lines $\tilde{\eta}^{w_0}_{\theta_i}$ almost surely hit its neighbors (when $i = 1, n$ it hits $\partial_U\mathcal{S}$) for infinite number of times in some neighborhood $B(w_0, r)$. In this setting, for a flow line $\tilde{\eta}_\theta$ from 0 with angle θ to hit w_0, it must cross an infinite number of pockets formed by $\tilde{\eta}^{w_0}_{\theta_i}$. However from Theorem 1.4 each flow line $\tilde{\eta}^{w_0}_{\theta_i}$ could only be crossed at most once by $\tilde{\eta}_\theta$, indicates $\tilde{\eta}_\theta$ hits at most $n+1$ pockets and hence cannot hit w_0.

As explained in Figure 16, we take a conformal map from $\mathcal{S}\backslash\eta'([0,\tau'])$ back to $\mathcal{S}$ fixing 0 and taking $\eta'(\tau')$ to $w_0 \in \partial_U\mathcal{S}$. This brings us to a similar setting in [MS12, Lemma 7.16] as it remains to show that for any flow line $\tilde{\eta}_\theta$ from 0 with angle θ, it will almost surely not hit w_0. The main idea of the proof is explained as in Figure 16. The infinite times hitting comes from infinite hitting of 0 in Bessel processes with dimension in $(1,2)$. We take $r > 0$ such that $\partial B(w_0, r)$ is contained in $\psi(\eta'([0,\tau']))$ then the law of $h_\psi|_{\partial B(w_0,r)}$ is absolutely continuous w.r.t. that of the field with boundary data $-\lambda'$ on $\partial_U\mathcal{S}$ to the left of w_0 and λ on $\partial_U\mathcal{S}$ to the right of w_0, and the infinite number of hitting follows from the result that for any $\text{SLE}_\kappa(\rho^L, \rho^R)$ process η with $\rho^L, \rho^R \in (-2, \frac{\kappa}{2} - 2)$ and force points at 0^+ and 0^-, $\eta([0,t])$ is hitting both $(-\infty, 0)$ and $(0, \infty)$ for a infinite number of times.

Acknowledgement

I would like to thank my advisor Prof Sheffield for offering this wonderful 18.677 course and providing me support in this difficult situation. And also every figure I used is from [MS12]. I would also like to thank Morris Ang for providing me many useful suggestions regarding the problem sets and many other general topics.

References

[Ber16] N. Berestycki. Introduction to the Gaussian Free Field and Liouville Quantum Gravity. *Lecture notes*, 2016.

[BN16] N. Berestycki and James Norris. Lectures on Schramm Lowener Evolution. *Lecture notes*, 2016.

[Dub09] J. Dubédat. Duality of Schramm-Loewner Evolutions. *Ann. Sci. Éc. Norm. Supér*, 42(5), 2009.

[MS12] J. Miller and S. Sheffield. Imaginary Geometry I: Interacting SLEs. *arXiv preprint arXiv:1201.1496*, 2012.

[RS05] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. of Math.*, 161(2), 2005.

[RY99] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293. Springer-Verlag, Berlin, third edition, 1999.

[Sch00] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math*, 118:221288, 2000.

[She07] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139, 2007.

[She10] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *arXiv preprint arXiv:1012.4797*, 2010.

[SS13] O. Schrmm and S. Sheffield. A contour line of the continuum gaussian free field. *Probab. Theory Related Fields*, 157, 2013.