Supplementary Information for

An additive algorithm for origami design

Levi H. Dudte, Gary P. T. Choi, L. Mahadevan

Corresponding Author: L. Mahadevan.
E-mail: lmahadev@g.harvard.edu

This PDF file includes:

  - Supplementary text
  - Figs. S1 to S18
  - Legends for Movies S1 to S10
  - SI References

Other supplementary materials for this manuscript include the following:

  - Movies S1 to S10
Supporting Information Text

S1. Construction

A. Completing a quad pair. Consider a potential single vertex origami of degree four, existing at the moment only as a pair of quads (i.e. two of its incident faces exist and two have yet to be designed, see Fig. S1). Let the new design angles incident to the vertex be $\theta_1$ and $\theta_2$ and the existing design angles be $\theta_3$ and $\theta_4$. Let the angle between edges at the growth boundary be $\beta \in [0, \pi]$ and the oriented angle between the $\beta$ plane and the new face containing $\theta_1$ be $\alpha \in [0, 2\pi]$. The new design angles $\theta_1$ and $\theta_2$ at the vertex must satisfy two equations:

$$\sum_{i=1}^{4} \theta_i = 2\pi, \quad \text{[S1]}$$

$$\cos \theta_2 = \cos \theta_1 \cos \beta + \sin \theta_1 \sin \beta \cos \alpha. \quad \text{[S2]}$$

Eq. (S1) guarantees the developability of the interior angles around the new interior node and Eq. (S2) expresses compatibility between $\theta_1$, $\theta_2$, $\alpha$ and $\beta$ (see Fig. S1). Eq. (S2) is the familiar law of cosines from spherical trigonometry, with $\theta_1$, $\theta_2$ and $\beta$ forming a spherical triangle with interior angle $\alpha$ opposite $\theta_2$. Notice that this system expresses the same pair of constraints as prior optimization-based origami design studies (1), namely developability in Eq. (S1) and planarity by construction in Eq. (S2).

To show existence and uniqueness of solutions to this system, we begin by observing that the existing $\theta_3$ and $\theta_4$ values at $x$ determine the value of $\theta_1 + \theta_2 = 2\pi - (\theta_3 + \theta_4) = k$ by Eq. (S1). This allows us to rewrite the system as

$$f(\theta_1) = \theta_1 + \theta_2(\theta_1) = k, \quad \text{[S3]}$$

where $\theta_2(\theta_1) = \cos^{-1}(\cos \theta_1 \cos \beta + \sin \theta_1 \sin \beta \cos \alpha)$. Now we can make several observations about this equation. For the left-hand side, note that

- $f(0) = \beta$,
- $f(\pi) = 2\pi - \beta$,
- $f(\theta_1)$ is positive monotonic on $\theta_1 \in [0, \pi]$.

Intuitively, $\theta_2(\theta_1)$ always decreases at a smaller rate than the rate of increase of $\theta_1$, except in the singular cases $\alpha = 0, \pi$ where the rates are equal and opposite and $f(\theta_1)$ is constant (see Section S1A.3 for details). These observations imply that $f(\theta_1)$ is bounded by $[\beta, 2\pi - \beta]$ and positive monotonic for $\theta_1 \in [0, \pi]$. Now for the right-hand side, observe that

- $k \in [\beta, 2\pi - \beta]$ because $\theta_3 + \theta_4 \in [\beta, 2\pi - \beta]$ (see Section S1A.2 for details). Taken together, these observations guarantee existence and uniqueness of solutions to Eq. (S3) and thereby Eq. (S1) and Eq. (S2).
This follows from the negative monotonicity and symmetry of inverse cosine about which projected angle

In general, we will forbid singular configurations throughout this work, as these growth fronts can

A.1. Genericity vs. singularity.

Now observe that \( \theta_3 + \theta_4 \) monotonically approaches \( \pi \) as \( \phi \) goes from zero to \( \pi/2 \). The sign of \( d/d\phi(\theta_3 + \theta_4) \) is determined by which projected angle \( \theta_3' \) or \( \theta_4' \) deviates most from \( \pi/2 \) (i.e. if \( |\pi/2 - \theta_3'| > |\pi/2 - \theta_4'| \) then \( \text{sgn}(d/d\phi(\theta_3 + \theta_4)) = \text{sgn}(\pi/2 - \theta_3') \)). This follows from the negative monotonicity and symmetry of inverse cosine about \( \pi/2 \). So \( \theta_3 + \theta_4 \) begins at an endpoint of the interval \( [\beta, 2\pi - \beta] \) when \( \phi = 0 \) and approaches \( \pi \) monotonically within the same interval as \( \phi \to 0 \). This parameterizes all angle pairs in space and thus \( \beta \leq \theta_3 + \theta_4 \leq 2\pi - \beta \).

Fig. S2. Genericity vs. singularity. (Left) \( \beta = 0 \) (singular) The two edge vectors at this vertex are collinear and point in the same direction, so \( \theta_2 \) increases at the same rate as \( \theta_1 \). (Middle) \( \beta = \pi/2 \) (generic for \( \alpha \neq 0, \pi \)) A qualitatively general picture of S2. \( \theta_2 \) decreases a smaller rate than that of the increase of \( \theta_1 \), so the blue curves are positive monotonic. At \( \alpha = 0, \pi \) the planes of \( \theta_1 \) and \( \theta_2 \) coincide with the plane, and rates of change in \( \theta_1 \) and \( \theta_2 \) are either equal in magnitude with opposite sign (when \( \pi \) is inside \( \beta \)) or equal (when \( \pi \) is outside \( \beta \)). (Right) \( \beta = \pi \) (singular) Again, the two vectors at the growth front are collinear, but now point in opposite directions, so the only possible value of \( \theta_1 + \theta_2 \) at this vertex is \( \pi \).

Fig. S3. Bounding the sum of two angles. Three arbitrary angles \( \theta_3, \theta_4 \) and \( \beta \) form a spherical triangle that can be divided into two spherical triangles by projecting the point between \( \theta_3 \) and \( \theta_4 \) to the \( \beta \) plane (dashed line). Denote the unsigned angle between the original point and its projection \( \phi \) and observe that \( \theta_3, \theta_4 = \pi/2 \) when \( \phi = \pi/2 \). (Left) Projection onto \( \beta \) plane of point in \( \mathbb{R}^3 \) falls inside \( \beta \): \( \theta_3' + \theta_4' = \beta \) when \( \phi = 0 \). These projections can either both be less than \( \pi/2 \) or one less and one greater. (Right) Projection onto \( \beta \) plane of point in \( \mathbb{R}^3 \) falls outside \( \beta \): \( \theta_3' + \theta_4' = 2\pi - \beta \) when \( \phi = 0 \). These projections can either both be greater than \( \pi/2 \) or one less and one greater.

A.1. Genericity vs. singularity.

In general, we will forbid singular configurations throughout this work, as these growth fronts can admit multiple solutions for a given flap angle choice. Fig. S2 illustrates the behavior of Eq. (S3) under generic (\( \beta \neq 0, \pi \)) and singular configurations (\( \beta = 0, \pi \)).

A.2. Spherical triangle inequality.

Consider a spherical triangle consisting of the angles \( \beta, \theta_3 \) and \( \theta_4 \), each of which is in \([0, \pi]\). Let the angles \( \theta_3' \) and \( \theta_4' \) be the angles of the projections of \( \theta_3 \) and \( \theta_4 \), respectively, onto the plane containing \( \beta \). Let \( \phi \) be height of the node incident to \( \theta_3 \) and \( \theta_4 \) above the \( \beta \) plane (see Fig. S3).

Now consider the sum \( \theta_3 + \theta_4 \) for all values of \( \phi \). Observe that at \( \phi = \pi \), \( \theta_3 = \theta_4 = \pi/2 \) and thus \( \theta_3 + \theta_4 = \pi \in [\beta, 2\pi - \beta] \forall \beta \in (0, \pi) \). Observe also that at \( \phi = 0 \), there are two possibilities: \( \theta_3 + \theta_4 = \beta \) (the “inside” case) and \( \theta_3 + \theta_4 = 2\pi - \beta \) (the “outside” case). By projecting \( \theta_3 \) and \( \theta_4 \) onto the \( \beta \) plane, we can construct two right spherical triangles, allowing us to write

\[
\cos \theta_3 = \cos \theta_3' \cos \phi + \sin \theta_3' \sin \phi \cos \frac{\pi}{2} = \cos \theta_3' \cos \phi, \tag{S4}
\]

\[
\cos \theta_4 = \cos \theta_4' \cos \phi + \sin \theta_4' \sin \phi \cos \frac{\pi}{2} = \cos \theta_4' \cos \phi, \tag{S5}
\]

and

\[
\theta_3 + \theta_4 = \cos^{-1}(\cos \theta_3' \cos \phi) + \cos^{-1}(\cos \theta_4' \cos \phi). \tag{S6}
\]

Now observe that \( \theta_3 + \theta_4 \) monotonically approaches \( \pi \) as \( \phi \) goes from zero to \( \pi/2 \). The sign of \( d/d\phi(\theta_3 + \theta_4) \) is determined by which projected angle \( \theta_3' \) or \( \theta_4' \) deviates most from \( \pi/2 \) (i.e. if \( |\pi/2 - \theta_3'| > |\pi/2 - \theta_4'| \) then \( \text{sgn}(d/d\phi(\theta_3 + \theta_4)) = \text{sgn}(\pi/2 - \theta_3') \)). This follows from the negative monotonicity and symmetry of inverse cosine about \( \pi/2 \). So \( \theta_3 + \theta_4 \) begins at an endpoint of the interval \( [\beta, 2\pi - \beta] \) when \( \phi = 0 \) and approaches \( \pi \) monotonically within the same interval as \( \phi \to 0 \). This parameterizes all angle pairs in space and thus \( \beta \leq \theta_3 + \theta_4 \leq 2\pi - \beta \).
A.3. Angle sum solutions are unique. For uniqueness, we need to show that \( f \) is positive monotonic on the interval \( \theta_1 \in [\beta, 2\pi - \beta] \). This amounts to showing that \(|d\theta_2(\theta_1)/d\theta_1| \leq 1\) or, equivalently, \((d\theta_2(\theta_1)/d\theta_1)^2 \leq 1\). Recalling that \( \theta_2(\theta_1) = \cos^{-1}(\cos \theta_1 \cos \beta + \sin \theta_1 \sin \beta \cos \alpha) \), we have
\[
\frac{d\theta_2}{d\theta_1} = \frac{\sin \theta_1 \cos \beta - \cos \theta_1 \sin \beta \cos \alpha}{\sqrt{1 - (\cos \theta_1 \cos \beta + \sin \theta_1 \sin \beta \cos \alpha)^2}} \tag{S7}
\]
and hence
\[
\left( \frac{d\theta_2}{d\theta_1} \right)^2 = \frac{(\sin \theta_1 \cos \beta - \cos \theta_1 \sin \beta \cos \alpha)^2}{1 - (\cos \theta_1 \cos \beta + \sin \theta_1 \sin \beta \cos \alpha)^2} = \frac{\cos^2 \beta + \sin^2 \beta \cos^2 \alpha - K^2}{1 - K^2} \leq 1, \tag{S8}
\]
where \( K = \cos \theta_1 \cos \beta + \sin \theta_1 \sin \beta \cos \alpha \) and equality occurs only for \( \alpha = 0, \pi \). Thus \( f(\theta_1) \) is positive monotonic on \( \theta_1 \in [\beta, 2\pi - \beta] \) and solutions \( f(\theta_1) = k \) are unique for all values \( \beta \neq 0, \pi \).

A.4. Any flap angle location explores the full design space. One detail in this construction remains, which is the question of whether choosing different locations for the flap angle will yield different spaces of possible new designs. If this were true, it would complicate the search at each additive design step, requiring the designer / algorithm to explore design spaces yielded by varying the flap angle at each new quad and to collate the results before choosing a new design. Fortunately, this is not the case: all choices of quad locations for the flap angle are equivalent, so exploring \( \alpha \) values at any location along the growth front explores the full space of possible designs at that front. Let \( \alpha_i \) be the flap angle at the chosen location along strip (dihedral angle between \( \theta_{i,1} \) face and \( \beta_i \) face, left-hand oriented about the growth front edge \( e_i \), \( \alpha_i' \) be the angle from the \( \beta_i \) plane and the \( \theta_{i,2} \) plane, left-hand oriented about the growth front edge \( e_{i+1} \) (see Fig. S4). Solutions of the angle sum constraint \( \theta_{i,1} + \theta_{i,2} = \theta_i \) form an ellipse of spherical arcs \( \gamma_i \). The foci of this ellipse are given by the edges that define the \( \beta_i \) face, i.e. the boundary edges \( e_i \) and \( e_{i+1} \) of the growth front and the axes of \( \alpha_i \) and \( \alpha_i' \), respectively. Clearly, any choice of \( \alpha_i \in [0, 2\pi] \) yields a unique \( \alpha_i' \in [0, 2\pi] \), which is related to the next flap angle \( \alpha_{i+1} \) by a constant phase shift given by the dihedral angle between the \( \beta_i \) and \( \beta_{i+1} \) planes. Thus sweeping \( \alpha_i \) through the interval \([0, 2\pi]\) also sweeps \( \alpha_{i+1} \) uniquely through the same interval. This establishes a bijective mapping between two consecutive flap angles, which in turn determine new interior angles along the entire growth front. So the choice of which quad along the strip to pick as the flap angle location is arbitrary: any sweeping the flap angle at any quad location along the strip will explore the full space of possible new interior strip geometries uniquely.

B. Adaptations of strip construction. While it is trivial to growth a simple patch of quad origami by adding strips in any order to its four boundaries, the strip construction can be adapted, in most cases by sacrificing some degrees of freedom, to alternative surface geometries and topologies. We illustrate two cases here, non-convex patches and closed loops, and leave further adaptations for future work. Notably, the degree of freedom associated with flap angle choice becomes determined in both of these examples.

B.1. Concave corner. For a surface patch with a concave corner formed by a row of boundary points \( x_i, i \in \{0, n\} \) and a column of boundary points \( y_j, j \in \{0, m\} \) such that \( x_0 = y_0 \) is the corner, we adapt the strip algorithm to add a pair of strips originating at this corner by determining growth directions \( r_i \) and \( t_j \) for the \( x \) and \( y \) boundaries, respectively (see Fig. S5).

It is clear that the plane containing \( x_0, x_1, y_0 \) and \( y_1 \) determines the flap angle needed to calculate both \( r_1 \) and \( t_1 \). So all interior growth directions \( r_i, i \in \{i, \ldots, n-1\} \) and \( t_j, j \in \{j, \ldots, m-1\} \) are determined by the existing surface geometry. The boundary growth directions \( r_n \) and \( t_m \) can be chosen freely in the plane of \( x_{n-1}, x_n \) and \( r_{n-1} \) and the plane of \( y_{m-1}, y_m \) and \( t_{m-1} \), respectively. Finally, the edge lengths at \( x_i, i \in \{2, \ldots, n\} \) and \( y_j, j \in \{2, \ldots, m\} \) may be chosen freely within the usual
We now have a relationship between one of the new design angles and the flap angle, so special vertex solutions given only by boundary growth directions will be simple rather a corner.

For a closed loop consisting of $m - 1$ quads, we treat the growth front as consisting of $m + 1$ vertices $x_i, i \in \{0, \ldots, m\}$ with two overlapping pairs $(x_0, x_{m-1})$ and $(x_1, x_m)$, such that the boundary growth directions $r_0$ and $r_m$ overlap with the interior growth directions $r_{m-1}$ and $r_1$ respectively. We then search for flap angles which give $n_1 = n_m$, where $n_i$ is the unit normal of the $i^{th}$ new quad.

local intersection bounds. The lengths of new edges at $x_1$ and $y_1$ are bounded additionally by the possible intersection of $r_1$ and $t_1$. If edge lengths at these locations are not chosen to match their intersection lengths, then a new edge will be added to the growth front at the corner, the corner face will be five-coordinated rather than four (a defect) and the next growth front will be simple rather a corner.

B.2. Closed loop. For a surface patch having a growth front formed by a single closed loop of quads, we adapt the strip algorithm to attach a new closed loop of quads at this boundary by searching for closure conditions around the loop. Let the growth front consist of $m - 1$ quads, then we can treat the growth front as consisting of $m + 1$ vertices $x_i, i \in \{0, \ldots, m\}$ where the pairs $(x_0, x_{m-1})$ and $(x_1, x_m)$ each describe the same vertex, i.e. these labels overlap (see Fig. S5). This gives a simple growth front with boundary growth directions $r_0$ and $r_m$ overlapping with interior growth directions $r_{m-1}$ and $r_1$, respectively. This setup allows us to search for flap angles which give $n_1 = n_m$, where $n_i$ is the unit normal of the $i^{th}$ new quad. The continuation solution is guaranteed to exist, and other solutions to this closure condition give non-trivial folds at the growth front.

S2. Special vertices

A. Angle conditions. We derive the flap angle values that produce flat-foldable, equal new design angles, continuation/reflection and locked single vertices. First, we derive the closed-form single vertex solution from the main text using the law of cosines for the spherical triangle formed by $\theta_{i,1}$, $\theta_{i,2}$ and $\beta_i$ and a cosine identity.

\[
\cos \theta_{i,2} = \cos (k_i - \theta_{i,1}) \quad \text{[S9]}
\]
\[
\cos \theta_{i,1} \cos \beta_i + \sin \theta_{i,1} \sin \beta_i \cos \alpha_i = \cos k_i \cos \theta_{i,1} + \sin k_i \sin \theta_{i,1} \quad \text{[S10]}
\]
\[
\cos \beta_i + \tan \theta_{i,1} \sin \beta_i \cos \alpha_i = \cos k_i + \sin k_i \tan \theta_{i,1} \quad \text{[S11]}
\]
\[
\tan \theta_{i,1} (\sin \beta_i \cos \alpha_i - \sin k_i) = \cos k_i - \cos \beta_i \quad \text{[S12]}
\]
\[
\tan \theta_{i,1} = \frac{\cos k_i - \cos \beta_i}{\sin \beta_i \cos \alpha_i - \sin k_i} \quad \text{[S13]}
\]

We now have a relationship between one of the new design angles and the flap angle, so special vertex solutions given only by design angles conditions follow, so long as $(k_i - \beta_i)/2 \leq \theta_{i,1} \leq (k_i + \beta_i)/2$. 

Levi H. Dudte, Gary P. T. Choi, L. Mahadevan
We also observe that a single vertex origami is self-intersecting when $\alpha_{i,3}$ rotates $\mathbf{n}_{i,\beta}$ to $\mathbf{n}_{i,1} = -\mathbf{n}_{i,4}$ about $\mathbf{e}_i$, shown oriented into the page. (Right) A locked right vertex has coplanar faces containing $\theta_{i,2}$ and $\theta_{i,3}$. The angle $\alpha'_{i,lr}$ rotates $\mathbf{n}_{i,\beta}$ to $\mathbf{n}_{i,2} = -\mathbf{n}_{i,3}$ about $\mathbf{e}_{i+1}$, shown oriented out of the page.

**B. Locked configurations.** A single vertex origami is trivially locked when $\alpha_i \in (0, \pi)$, i.e. the $\beta_i$, $\theta_{i,1}$ and $\theta_{i,2}$ faces are coplanar. Non-trivial locked configurations occur when one of the new faces in the vertex is coplanar with an existing face. These special vertices depend on the orientation of the existing faces in three-dimensional space, so their flap angles cannot be derived from the above design/flap angle relation. The left locked configuration occurs when face normals $\mathbf{n}_{i,1}$ and $\mathbf{n}_{i,4}$ belonging to $\theta_{i,1}$ and $\theta_{i,4}$ faces, respectively, satisfy $\mathbf{n}_{i,1} = -\mathbf{n}_{i,4}$ (see Fig. S6). We write down the flap angle $\alpha_{i,ll}$ that gives a such a vertex by inspecting the arrangement of $\mathbf{n}_{i,1}$, $\mathbf{n}_{i,4}$ and $\mathbf{n}_{i,\beta}$ about their common axis, the growth front edge $\mathbf{e}_i$ between the faces containing $\theta_{i,1}$ and $\theta_{i,4}$:

$$\alpha_{i,ll} = \text{mod}\left(\text{atan2}\left(\mathbf{n}_{i,\beta} \times \mathbf{n}_{i,4}, -\mathbf{n}_{i,\beta} \cdot \mathbf{n}_{i,4}\right), 2\pi\right).$$

[S14]

The right locked configuration occurs when face normals $\mathbf{n}_{i,2}$ and $\mathbf{n}_{i,3}$ belonging to $\theta_{i,2}$ and $\theta_{i,3}$ faces, respectively, satisfy $\mathbf{n}_{i,2} = -\mathbf{n}_{i,3}$. We write down the angle $\alpha'_{i,lr}$ by inspecting the arrangement of $\mathbf{n}_{i,2}$, $\mathbf{n}_{i,3}$ and $\mathbf{n}_{i,\beta}$ about their common axis, the growth front edge $\mathbf{e}_{i+1}$ between the faces containing $\theta_{i,2}$ and $\theta_{i,3}$:

$$\alpha'_{i,lr} = \text{mod}\left(\text{atan2}\left(\mathbf{n}_{i,\beta} \times \mathbf{n}_{i,3}, -\mathbf{n}_{i,\beta} \cdot \mathbf{n}_{i,3}\right), 2\pi\right).$$

[S15]

The flap angle $\alpha_{i,lr}$ that gives a locked right vertex can then be calculated from $\alpha'_{i,lr}$. Calculating $\theta_{i,2}$ from $\tan \theta_{i,2} = (\cos \beta_i - \cos k_i)/\sin k_i - \sin \beta_i \cos \alpha_i$ and using the spherical laws of cosines and sines for the spherical triangle formed by $\theta_{i,1}$, $\theta_{i,2}$ and $\beta_i$, we have

$$\alpha_{i,lr} = \text{mod}\left(\text{atan2}\left(\sin \theta_{i,2} / \sin \theta_{i,1}, \cos \theta_{i,2} - \cos \theta_{i,1} \cos \beta_i \sin \alpha_{i,lr}, \sin \theta_{i,1} \sin \beta_i\right), 2\pi\right).$$

[S16]

We also observe that a single vertex origami is self-intersecting when $(\mathbf{r} \cdot \hat{n}_3)(\mathbf{r} \cdot \hat{n}_4) < 0$ and $\mathbf{r} \cdot \hat{r} < 0$, which occurs when $\alpha \in (\pi, \min(\alpha_{ii}, \alpha_{il})) \cup (\max(\alpha_{il}, \alpha_{lr}), 2\pi)$ if $\hat{n}_\beta \cdot \hat{r} > 0$ or $\alpha \in (0, \min(\alpha_{ii}, \alpha_{il})) \cup (\max(\alpha_{il}, \alpha_{lr}), \pi)$ if $\hat{n}_\beta \cdot \hat{r} < 0$.

**S3. Edge length bounds**

For each new quad along the growth front incident to new design angles $\theta_{i,2}$ and $\theta_{i,i+1}$ and new edge directions $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$, if

$$\theta_{i,2} + \theta_{i,i+1} < \pi,$$

[S17]

the lines $\mathbf{p}_i(t_i) = \mathbf{x}_{i} + t_i \mathbf{r}_i$ and $\mathbf{p}_{i+1}(s_i) = \mathbf{x}_{i+1} + s_i \mathbf{r}_{i+1}$ will intersect for some positive $t_i$ and $s_i$ (see Fig. S7). Their intersection gives a bound on the edge length choices $l_i$ and $l_{i+1}$, which we can locate by minimizing the distance $D$ between $\mathbf{p}_i$ and $\mathbf{p}_{i+1}$ with respect to $t$ and $s$:

$$D(t_i, s_i) = ||\mathbf{p}_i(t_i) - \mathbf{p}_{i+1}(s_i)||_2.$$

[S18]
We then add strips to either side of the seed additively such that all nodes on the upper side of the origami surface fall exactly on the upper corrugating surface and all nodes on the lower side of the origami patch fall exactly on the lower corrugating surface. This novel approach is perfectly tailored to design origami sandwich structures that reside in the interstice of the two smooth surfaces (see Fig. S10). Consider a target surface \( X(u,v) \) with surface normal vector \( n(u,v) \). A normal surface \( X_0(u,v) \) is given by displacing the target surface in the direction of its normal field by a constant thickness \( h \): \( X(u,v) + h n(u,v) \). We construct upper \( X_{+h/2} \) and lower \( X_{-h/2} \) normal surfaces each displaced by a distance of \( \epsilon/2 \) from the target surface to give a
volume with constant thickness $\epsilon$. Note that $\epsilon$ is a tunable parameter that controls how close the origami structure is to the target surface. The surface example in Fig. 4A in the main text is given by

$$X(u, v) = <u, v, H \sin(Au) \cos(Av)>, \quad [S22]$$

with shape parameters $H = 0.2$ and $A = 1.05\pi$, thickness $\epsilon = 2/m$ and seed resolution $m = 40$.

We initialize two rows of coordinates $<u_{i,0}, v_{i,0}>$ and $<u_{i,1}, v_{i,1}>$ according to

$$<u_{i,j}, v_{i,j}> = <(-1)^i \frac{c}{2} + (-1)^j \frac{H}{m}, -1 + \frac{i}{m}>, \quad [S23]$$

with $i \in \{0, \ldots, m\}$ and $j \in \{0, 1\}$. Evaluating $X_{+\epsilon/2}(u_{i,0}, v_{i,0})$ and $X_{-\epsilon/2}(u_{i,1}, v_{i,1})$ gives two rows of points in space, one on each of the upper and lower surfaces, which define a strip of quads whose faces are nearly, but not exactly, planar. To polish the geometry of the strip before it becomes a growth seed, we collect the set of rays $r_i$ and lengths $l_i$:

$$a_i = X_{+\epsilon/2}(u_{i,1}, v_{i,1}) - X_{-\epsilon/2}(u_{i,0}, v_{i,0}), \quad [S24]$$

$$l_i = \|a_i\|_2, \quad [S25]$$

and use Matlab’s fmincon to minimize the objective function

$$C(b) = \sum_{i=0}^{m} 1 - \frac{a_i \cdot b_i}{l_i}, \quad [S26]$$
subject to the constraints

\[ \| \mathbf{b}_i \|_2 - 1 = 0, \ i \in \{0, \ldots, m\}, \quad \text{[S27]} \]
\[ (\mathbf{b}_i \times \mathbf{b}_{i+1}) \cdot \mathbf{e}_i = 0, \ i \in \{0, \ldots, m - 1\}, \quad \text{[S28]} \]

where

\[ \mathbf{e}_i = \frac{X_{-\epsilon/2}(u_{i+1,0}, v_{i+1,0}) - X_{-\epsilon/2}(u_{i,0}, v_{i,0})}{\| (X_{-\epsilon/2}(u_{i+1,0}, v_{i+1,0}) - X_{-\epsilon/2}(u_{i,0}, v_{i,0})) \|_2}. \quad \text{[S29]} \]

This allows us to construct our final Miura-type seed from the modified strip of quads with exactly planar faces whose two boundary rows are the original \( X_{-\epsilon/2}(u_{i,0}, v_{i,0}) \) and the polished \( X_{-\epsilon/2}(u_{i,0}, v_{i,0}) + l_i \mathbf{b}_i \). In practice, this results in the polished row of points being offset slightly from the bounding surface, so we adjust \( l_i \) according to the line/surface intersection routine described in Section S4A.2 below such that all points in the seed fall on the upper or lower surfaces.
We note that in the formulation above, the target surface is parameterized using two parameters. A gallery of origami surface fitting results created by our additive approach are presented in Fig. S11. It can be observed that write down bounds on the flap angle that constrain the growth of the surface to this interstice, and optimize the resultant bounds to Matlab’s `fmincon`. Boundaries design angles are chosen such that the boundary growth directions $r_0$ and $r_m$ are parallel to their adjacent interior rays $r_1$ and $r_{m-1}$, respectively.

### A.1. Choosing the angles

Now we proceed to grow the singly-corrugated seed by using the strip construction to determine compatible strips at its two boundaries, one of which lies on each of the lower and upper surfaces. In general, we seek a flap angle that gives growth directions toward the lower surface for the upper growth front and toward the upper surface for the lower growth front. That is, we seek to induce a second, transverse corrugation in the origami surface by reflecting, at least in a qualitative sense, its growth back and forth in the interstice of the upper and lower target surfaces. To accomplish this, we write down bounds on the flap angle that constrain the growth of the surface to this interstice, and optimize the resultant design angles within these bounds and with respect to the flap angle. A cost function that avoids extreme design angles is given by

$$C(\alpha_1) = \sum_{i=1}^{m-1} (\theta_{i,1}(\alpha_1) - \pi/2)^2 + (\theta_{i,2}(\alpha_1) - \pi/2)^2,$$  

[S30]

and $\alpha_i$ is bounded to the interval $(0, \pi)$ or $(\pi, 2\pi)$ depending on the orientation of the flap angle location relative to the upper or lower surface. In practice, we perform a linear grid search of $C(\alpha_i)$ over the valid interval to visualize the behavior of the cost function, then polish the optimal evaluated flap angle by passing it as an initial value along with the above cost function and bounds to Matlab’s `fmincon`.

### A.2. Choosing the edge lengths

Edge lengths are chosen such that the next row of points lies exactly on the other corrugating surface, either lower or upper, via a numerical projection routine that computes intersections between rays and a surface. Consider a surface $X(u,v)$ with unit normal $n(u,v)$ and a line $L(t) = p + tv$. For a given $t$, we can calculate the distance from a point on the line to the surface by finding $\hat{u}(t), \hat{v}(t)$ that minimizes $d(u,v) = \|X(u,v) - L(t)\|_2$. We find the intersection of the line and the surface, if it exists and is well-defined, by minimizing

$$d(t) = \|X(\hat{u}(t), \hat{v}(t)) - L(t)\|_2$$  

[S31]

using Matlab’s constrained optimization routine `fmincon` in order to bound $t \geq 0$ during minimization.

We note that in the formulation above, the target surface is parameterized using two parameters $u, v$. We remark that any other representation of the surface (in the form of an explicit function, an implicit function, a triangular mesh, a quad mesh etc.) can also be used as long as the surface normal (for constructing the upper and lower surfaces) and the point-to-surface distance (for optimizing the edge lengths) can be computed from it.

A gallery of origami surface fitting results created by our additive approach are presented in Fig. S11. It can be observed that our additive approach is capable of approximating surfaces with different curvature properties. Moreover, our approach allows for the creation of very high resolution models, which are nearly impossible to discover using a global optimization scheme due to the computational complexity. We note that increasing the resolution of the approximating origami surface naturally decreases the amount of integrated Gauss curvature each cell in the origami surface needs to accommodate. This suggests that as resolution increases and the characteristic length of the approximating Miura-ori cell becomes vanishingly small compared to the radii of principle curvatures of the target surface, the surface starts to look like its tangent plane and the surface fitting problem becomes trivial, locally.

One may ask whether every target surface admits a unique origami approximant under our framework. As shown in Fig. S11, one can change the resolution of the seed strip to achieve different fold patterns for the same target surface. Fig. S12 shows two origami structures with the same resolution (same number of vertices and same number of quads) approximating the

---

**Fig. S10. Surface fitting construction. (Left) The seed strip (outlined in black) and the target surface. (Middle) Flap angles optimization. (Right) The mountain-valley folding pattern.**

**Fig. S11. Surface fitting construction. (Left) The seed strip (outlined in black) and the target surface. (Middle) Flap angles optimization. (Right) The mountain-valley folding pattern.**

**Fig. S12. Surface fitting construction. (Left) The seed strip (outlined in black) and the target surface. (Middle) Flap angles optimization. (Right) The mountain-valley folding pattern.**
Fig. S11. Additional surface fitting results. We deploy our additive approach to approximate a large variety of surfaces, including a helicoid at two different resolutions (top left), cylinders with different Gaussian curvatures (zero/negative/positive as shown in top right), a landscape shape with mixed curvature at four different resolutions (the middle row), a paraboloid at five different resolutions (bottom left), and a hypar at five different resolutions (bottom right).

The same target surface obtained by our additive framework. The seed strips of the two structures are set to be with different corrugation widths, and it can be observed that both the 3D folded structures and the 2D crease patterns are significantly different. This shows that our additive framework is capable of producing multiple surface-fitting origami structures even with the same resolution.

B. Twisted plane models (main text Figs. 4B,C). We describe here details from the nested cone example presented in Fig. 4C in the main text and display several other examples as well. Consider an upside-down cone of height \( h \) with origin \( \mathbf{p} = < 0, 0, -h/2 > \) and base described by the circle \( \mathbf{f}(t) = < A \cos t, A \sin t, h/2 > \) where \( A = h \tan(\theta/2) \). Following the closed loop growth pattern described in Section S1B.2, we construct a strip of \( m \) quads with overlapping end faces by sampling

\[
x_i = \mathbf{f}(2i\pi(1 + \frac{1}{m-2})), i \in \{0, \ldots, m-1\},
\]

\[
y_i = 0.01x_i + 0.99p,
\]

where \( x_i \) forms the growth front and \( y_i \) is the truncated point of the cone. The Huffman nested cones model in the main text (Fig. 4C) uses \( m = 102, h = 1 \) and \( \theta = \pi/4 \). The twisted squares model in the main text (Fig. 4B) uses \( m = 6, h = .1 \) and \( \theta = \pi/4 \).

We use a simple design angle difference cost function to search for flap angles \( \alpha_1 \) that regularize the pattern design:

\[
C(\alpha_1) = \frac{1}{2(m-1)\pi^2} \sum_{i=1}^{m-2} \sum_{j=1}^{2} (\theta_{i,j} - \theta_{i+1,j})^2.
\]

In both cases, we use neighborhoods of minima of a continuation avoidance cost function (shown as dashed curves in Fig. S13)

\[
C_{con}(\alpha_1) = \frac{1}{m-1} \sum_{i=1}^{m-1} \mathbf{r}_i \cdot \mathbf{r}_i,
\]
Fig. S12. Non-uniqueness of origami approximants for any given target surface. We use our additive framework to produce two surface-fitting origami structures with the same resolution approximating the same target surface. Here, the corrugation width of the seed strip of the right example is set to be twice of that of the left example. With the two different seed strips, it can be observed that both the 3D folded structures and the 2D crease patterns are significantly different.

where \( \bar{r}_i = (x_i - y_i) / \| x_i - y_i \|_2 \). \( x_i \) is a point on the current growth front and \( y_i \) is its corresponding point on the previous growth front, to bound the flap angle optimization and thus avoid selecting trivial solutions.

In the closed loop growth pattern, boundary growth directions \( r_0 \) and \( r_m \) overlap interior growth directions \( r_{m-1} \) and \( r_1 \), respectively, and so we choose boundary design angles such that \( r_0 = r_{m-1} \) and \( r_m = r_1 \).

Consider a set of planes parameterized by

\[
X(u, v) = u < \cos(j\phi), 0, \sin(j\phi) > + v < 0, 1, 0 > .
\]

[536]

Edge lengths in the \( j \)th new strip are given by the intersection of growth directions \( r \) and the plane \( X_d(u, v) \) displaced by \( d \) in the normal direction from \( X(u, v) \) where \( d = (-1)^j h (1 + (j - 1) \Delta h) / 2 \). The Huffman nested cones model in the main text (Fig. 4C) uses \( \Delta h = 0.3 \) and \( \phi = \pi / 18 \). The twisted squares model in the main text (Fig. 4B) uses \( \Delta h = 0 \) and \( \phi = \pi / 180 \).

A gallery of curved fold results created by our additive approach are presented in Fig. S14. It can be observed that our approach is capable of creating curved fold models with different geometry and topology. More specifically, note that each of the small constitutive folds in the model is a straight fold, as is necessary when working in a discrete setting. Nevertheless, by increasing the resolution of the folds, we can achieve models with folds resembling smooth curves.

C. Alternative curved fold model (main text Fig. 4D). In the Huffman example in the main text, we grow a curved fold surface by adding strips in the transverse direction to the curved folds (i.e. each growth front is a curved fold). In this example, we grow a curved fold surface by growing in the direction of the folds.

Consider a corrugated parabola given by

\[
x_i = < -1 + idx, (-1 + idx)^2 + dy(-1)^{i-1}, 0 > ,
\]

where \( dx = 2/n \), \( dy = .05 \) and \( i = 0, \ldots, n \). We use the corollary with \( k_i = 1.04\pi \) to construct a strip of quads incident to \( x \).

We choose a flap angle of \( \alpha_1 = 7\pi/8 \) for every strip.

We choose boundary design angles such that the boundary growth directions \( r_0 \) and \( r_m \) are parallel to their adjacent interior rays \( r_1 \) and \( r_{m-1} \), respectively.
![Diagram](image_url)

**Fig. S13. Flap angle optimization for main text Figs. 4B,C.** Dashed lines show the continuation cost function $C_{\text{con}}$, neighborhoods of whose minima give bounds for the optimization of cost function $C$ (solid line), which gives the final flap angle choice (dot). (Left) Cost functions landscapes used to design the curved twist model from main text Fig. 4B. (Right) Cost functions landscapes used to design the curved twist model from main text Fig. 4C.

**Fig. S14. Additional curved fold results.** Using our additive approach, it is possible to create a wide range of curved fold models with different geometry and topology.

We choose all new new edge lengths to be $l_i = .1$.  

**D. Crumpled sheets (main text Fig. 4E).** Finally, we deploy our strip construction to sample disordered, or “crumpled” folded sheets.

Consider three vectors

$$
\begin{align*}
a &= \langle 0, \cos \nu, -\sin \nu \rangle, \\
b &= \langle 0, -\cos \nu, -\sin \nu \rangle, \\
c &= \langle -\sin \phi, 0, -\cos \phi \rangle,
\end{align*}
$$

with $\nu = \pi/2 - \beta/2$ and $\phi = \cos^{-1}(\cos((2\pi - k)/2)/\cos(\beta/2))$. Then the points

$$
\begin{align*}
\bar{x} &= \{a, 0, b\} \\
\bar{y} &= \{a + c, c, b + c\}
\end{align*}
$$

describe a pair of quads with growth front $\bar{x}$ described by the familiar scalars $\beta$ and $k$. This pair can be repeated in space by a shift of $b - a$ to form a Miura-like strip of quads given by two rows of points $\{x'_i\}$ and $\{y_i\}$ (with overlapping points deleted). To introduce disorder to the growth front, we perturb $\bar{x}$ with Gaussian noise according to

$$
x_i = (x'_i - y_i)(1 + X),
$$

where $X \sim N(\mu = 0, \sigma = .4)$, which gives our final disordered seed with growth front $\bar{x}$. The seed from the disordered example in Fig. 4E in the main text consists of 20 quads and uses $\beta = 0.6\pi$ and $k = 1.2\pi$.  

Levi H. Dudte, Gary P. T. Choi, L. Mahadevan 13 of 19
Given the growth vertices $x_i, i \in \{0, \ldots, m\}$, we compute intervals that give local self-intersection at each potential flap angle locations $i \in \{1, \ldots, m - 1\}$ along the growth front (see Fig. S15). We can then use the inverse of $g_i$, the adjacent flap angle transfer function described in the main text, to map those local self-intersection intervals to a single growth front flap angle location. Let $g_i^{-1}(\alpha_{i+1}) = \alpha_i$ be given by

$$\begin{align*}
\alpha_i' &= \text{mod}(\alpha_{i+1} + \phi_i, 2\pi) \\
\theta_{i,2} &= \text{mod}(\tan^{-1}(\cos(k_i) - \cos(\beta_i))/\sin(\beta_i) \cos(\alpha_i') - \sin(k_i)), \pi) \\
\theta_{i,1} &= k_i - \theta_{i,2} \\
\cos \alpha_i &= (\cos(\theta_{i,2}) - \cos(\theta_{i,1}) \cos(\beta_i))/\sin(\theta_{i,1}) \sin(\beta_i) \\
\sin \alpha_i &= \sin(\theta_{i,2})/\sin(\theta_{i,1}) \sin(\alpha_i') \\
\alpha_i &= \text{atan2}(\sin \alpha_i, \cos \alpha_i),
\end{align*}$$

then we map the $m - 1$ intervals to the $m - 2$ location, the mapped $m - 1$ intervals and the $m - 2$ intervals to the $m - 3$ location and so forth until we have collected all local flap angle self-intersection intervals at the first flap angle $\alpha_1$ location along the growth front. Once the local self-intersection intervals are mapped to a common flap angle location, we sample $\alpha_1$ from the exterior of their union to give $r_i, i \in \{1, \ldots, m - 1\}$.

Boundary design angles are chosen such that the boundary growth directions $r_0$ and $r_m$ are parallel to their adjacent interior rays $r_1$ and $r_{m-1}$, respectively.

Define $L = \min(2, \min(\tilde{l}_i))$, where $\tilde{l}_i$ is the upper bound on the new edge length at growth front vertex $x_i$. The edge length at location $i$ is then sampled according to

$$\begin{align*}
l_i &= X_i (1 - k_i/(2\pi))^p, i \in \{1, \ldots, m - 1\}, \\
l_0 &= X_i (1 - k_1/(2\pi))^p, \\
l_m &= X_i (1 - k_{m-1}/(2\pi))^p,
\end{align*}$$

Fig. S15. Disordered strip self-intersection. (Top) A disordered strip with non self-intersecting (left) and self-intersecting flap angle choices (Bottom) Local (left) and mapped (right) self-intersection bounds corresponding to the above growth front. The union of all mapped intervals is shown in bold black at the bottom of the right plot, along with the flap angles values producing the above strip geometries.
Fig. S16. Additional disordered results. The top row depicts other disordered examples using $p = 2$ and the bottom row examples use $p = 1$.

where $X_i \sim \mathcal{U}(0.1L, 0.9L)$. Intuitively, the scaling prefactor provides for a kind of anti-preferential attachment. Locations with large $k_i$ (close to $2\pi$) have highly convex developments and thus, at least locally, ample space to grow. Locations with small $k_i$ (close to 0) have highly concave developments and thus, at least locally, restricted space to grow. The scaling prefactor, empirically, prevents caustics from forming and allows restricted locations to “catch up” with their relatively unrestricted counterparts by, somewhat counter-intuitively, encouraging growth in restricted locations and discouraging growth in unrestricted locations. The result in the main text uses $p = 2$. A gallery of other disordered results sampled in the same batch as that shown in the main text and a batch using $p = 1$ are presented in Fig. S16.

E. Brownian ribbon (main text Fig. 4F). To illustrate the flexibility of the corollary to our main theorem, we construct a set of folds that develop a Brownian path in $\mathbb{R}^3$ to a circle in $\mathbb{R}^2$. In this example, the growth front is no longer a front, but instead a curve in space not associated with an existing origami surface. This requires choosing the angular material $k$ associated with the interior nodes of the target discrete curve in $\mathbb{R}^3$, rather than computing them from a the developability condition for an existing origami surface.

We sample a path $x_i$ of Brownian motion with $n$ steps in $\mathbb{R}^3$ according to

$$x_i = \sum_{j=1}^{i} X_j,$$  \[S53\]

where $X \sim \mathcal{N}(\mu = 0, \sigma = 1)$ and $x_0 = (0, 0, 0)$. The corollary allows us to inverse design the shape of the folding pattern up to the bounds $k_i \in (\beta_i, 2\pi - \beta_i)$, where $\beta_i$ is the usual angle in space formed by the points $x_{i-1}, x_i$ and $x_{i+1}$. Given a sampled path $x$, we choose $k$ according to

$$k_i = \cos^{-1}\left(\frac{\|e_{i+1}\|}{2r}\right) + \cos^{-1}\left(\frac{\|e_i\|}{2r_{\text{outer}}}\right),$$  \[S54\]

where $r$ satisfies

$$\sum_{i=1}^{n} \sin^{-1}\left(\frac{\|e_i\|}{2r_{\text{outer}}}\right) = \pi$$  \[S55\]

so that the development of $x_i$ falls on a circle of radius $r_{\text{outer}}$ and forms a closed loop. The result in the main text has $n = 400$ steps. To make visualization of the construction easier to follow, we show an example with $n = 10$ steps in Fig. S17.

We choose an arbitrary flap angle of $\alpha_1 = \pi/4$.

Boundary design angles are chosen according to

$$\theta_{0,2} = \cos^{-1}\left(\frac{\|e_1\|}{2r_{\text{outer}}}\right)$$  \[S56\]

and

$$\theta_{n,1} = \cos^{-1}\left(\frac{\|e_n\|}{2r_{\text{outer}}}\right).$$  \[S57\]
so that the developments of the growth directions \( r_0 \) and \( r_m \) are equal and fall on the line containing the development of \( x_0 \) (which is equal to the development of \( x_n \)) and the origin of the pattern’s circumscribed circle.

Edge lengths \( l_i \) are chosen such that the points on the other side of the developed ribbon fall on a circle of smaller radius than \( r_{\text{inner}} \) with the same origin:

\[
\theta_{i,r} = \cos^{-1} \left( \frac{-e_i \cdot r_i}{\|e_i\|_2 \|r_i\|_2} \right) - \cos^{-1} \left( \frac{\|e_i\|_2}{2r_{\text{outer}}} \right),
\]

\[
l_i = r_{\text{outer}} \cos \theta_{i,r} - \sqrt{r_{\text{inner}}^2 - r_{\text{outer}}^2 \sin^2 \theta_{i,r}}.
\]

The above expression is valid when \( r_{\text{inner}} / r_{\text{outer}} > \sin \theta_{i,r} \), where \( \theta_{i,r} \) is the angle the development of \( r_i \) makes with the radial direction.

**S5. Folding simulation**

As all models generated by our additive framework are developable, there is always an elastic folding process that allows them to transform from a 2D flat sheet to a 3D structure, possibly with geometrical frustration at the intermediate states (as the models are not necessarily rigid-foldable). To show this, we apply the fast, interactive origami simulator (3) available online (4) to simulate the folding process of our models. More specifically, for each model, we provide the 2D crease pattern and the target fold angle for each edge in the final 3D structure as the inputs for the simulator, which then folds every crease simultaneously using a compliant dynamic simulation method, using reasonable (default) simulation settings (numerical integration scheme = Euler (first order), axial stiffness = 20, face stiffness = 0.2, fold stiffness = 0.7, facet crease stiffness = 0.7, damping ratio = 0.45). Fig. S18 shows the snapshots of the folding simulation for several models created by our framework, and Video S10 shows an animation of one of them.
S6. Description of supplementary videos

Movie S1. An animation of the single and adjacent vertex origami in Fig. 1 in the main text.

Given two boundary design angles $\theta_{i,3}, \theta_{i,4}$ (blue) and the angle in space between the two existing quads (green), we can freely choose the flap angle $\alpha_i$ (red). These angles together uniquely determine the new design angles $\theta_{i,1}$ (yellow) and $\theta_{i,2}$ (dashed) by the local developability constraint. In other words, $\alpha_i$ parameterizes the ellipse $\gamma_i$ of spherical arcs $\theta_{i,1}, \theta_{i,2}$ which forms a closed loop around the line containing $e_i$. The choice of $\alpha_i$ does not only determine the geometry of the single vertex origami but also propagates to all other vertices at the growth front via adjacency.

- **00:00-00:09** Flap angle $\alpha_i$ (red) rotates a half-plane about growth front edge $e_i$ incident to growth front vertex $x_i$.
- **00:09-00:20** The first new design angle $\theta_{i,1}$ moves in the plane determined by $\alpha_i$ and intersects $\gamma_i$ uniquely, determining $r_i, \theta_{i,1}$ and $\theta_{i,2}$ for a given $\alpha_i$.
- **00:20-00:28** $\alpha_i$ parameterizes $\gamma_i$ uniquely: for every choice of $\alpha_i$ there is a unique growth direction $r_i$.
- **00:28-00:33** Rotating 3D view of single vertex system at growth front vertex $x_i$.
- **00:33-00:46** An adjacent growth front vertex $x_{i+1}$ is introduced. Flap angle $\alpha_{i+1}$ of this new vertex is determined by the choice of the preceding flap angle $\alpha_i$, giving the half-plane of action for the new design $\theta_{i+1}$, which intersects $\gamma_{i+1}$ uniquely.
- **00:46-00:54** This is true for every $\alpha_i$, so $\alpha_i$ parameterizes $\gamma_{i+1}$ and adjacent growth direction $r_{i+1}$ and design angles $\theta_{i+1,1}$ and $\theta_{i+1,2}$ are determined by $\alpha_i$.
- **00:54-00:58** Rotating 3D view of adjacent vertex system at growth front vertices $x_i$ and $x_{i+1}$.

Movie S2. An animation illustrating the choice of the flap angle for a pair of folded quads in Fig. 3A and B in the main text.

- **00:00-00:10** Sweeping flap angle $\alpha$ through $[0, 2\pi]$ gives the range of single vertex origami models possible at a growth front vertex (left). Mountain-valley patterns (top left) and values of new design angle $\theta_1$ are shown for each flap angle value scanned.
- **00:10-00:16** Special vertex solution given by $\alpha = 0$ is shown rotating in 3D. This flap angle gives a trivially locked configuration with coplanar $\theta_1$ and $\theta_2$ faces.
- **00:19-00:21** Flat-foldable $\theta_1 + \theta_3 = \theta_2 + \theta_4 = \pi$ special vertex solution given by $\alpha_{ff}$
- **00:22-00:24** Equal new design angles ($\theta_1 = \theta_2$) special vertex solution given by $\alpha_{eq}$
- **00:24-00:30** Special vertex continuation solution given by $\alpha_{con}$ is shown rotating in 3D. This flap angle continues the existing fold on the interior of the growth front, giving a single vertex origami with no new folds and coplanar $\theta_1$ and $\theta_4$ faces and $\theta_2$ and $\theta_3$ faces.
Movie S3. An animation illustrating the choice of the flap angle for a quad strip in Fig. 3C in the main text.

Given a folded quad strip, changing the flap angle \( \alpha_1 \) of the first new face produces a wide range of compatible growth directions for the new strip (top). The new interior design angles \( \theta_{i,j} \) in the new strip, fold angles transverse to the growth front \( \phi_{1,t} \) and parallel to the growth front \( \phi_{1,p} \) all vary as a function of the flap angle \( \alpha_1 \) (bottom right). Boundary growth directions and edge lengths are chosen arbitrarily. The corresponding mountain-valley pattern is also shown (bottom left).

- 00:03-00:11 The new strip geometry given by \( \alpha_{1,con} \) has no new folds.
- 00:14-00:22 The new strip geometry given by \( \alpha_{1,fold} \) has new, non-trivial folds along the growth front and in the new strip.

Movie S4. An animation of the surface fitting model in Fig. 4A in the main text.

Movie S5. An animation of the twisted squares model in Fig. 4B in the main text.

Movie S6. An animation of the Huffman nested cones model in Fig. 4C in the main text.

Movie S7. An animation of the curved fold model in Fig. 4D in the main text.

Movie S8. An animation of the disordered model in Fig. 4E in the main text.

Movie S9. An animation of the Brownian fold model in Fig. 4F in the main text.

Movie S10. A simulation of the folding process for a surface fitting model obtained by our additive approach.
References