HOMOTOPY THEORY OF MODEL CATEGORIES

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In this paper some questions in the homotopy theory of model categories are answered. The important results of this paper are Theorems B, C and D which state that pushouts, sequential direct limits, and realizations of simplicial objects respect weak equivalences, provided sufficient cofibrancy is present.

Section 1 presents a model category structure on the simplicial objects over a model category. This is done partly to provide a justification for the term cofibrant, as applied to certain simplicial objects, and also to show that any object can be "approximated" by a cofibrant object. The lemmas in this section are presented without proof, since the proofs are easy, and of an entirely category theoretic nature.

In section 2 it is shown that, in model categories, weak equivalences respect pushouts and sequential direct limits. That weak equivalences respect pushouts is particularly important since this shows that any closed model category is a suitable category for homology theory (see [1]).

In section 3 the result that the realization of a weak equivalence of cofibrant objects is a weak equivalence is proven. Certain stronger results in the case of simplicial topological spaces are also mentioned. Section 4 discusses special results about simplicial simplicial sets, including the result that realization is isomorphic to the diagonal, a useful result which is not widely known, and not original with the author.

The category theory in this paper is standard and may be found in Mac Lane [3]; and, the notation conforms with that in Quillen [4], except that $\amalg$ (disjoint union) is used for coproduct.

1. Simplicial Objects over a Model Category

Given any category $\mathcal{C}$, define $\mathrm{S}\,\mathcal{C}$, the simplicial objects over $\mathcal{C}$, as the covariant functor category from the category $\mathcal{O}$, the opposite of the category of finite ordered sets, to $\mathcal{C}$. Let $\mathcal{O}_n \subset \mathcal{O}$ be the full sub-category whose objects are $\{\mathbf{0}, \ldots, \boldsymbol{n}\}$, and $\mathrm{S}_n\,\mathcal{C}$ the corresponding functor category.

The nth skeleton (sk_n) and nth coskeleton (ck_n) functors, are the left and right adjoints (respectively) to the restriction functor of $\mathrm{S}\,\mathcal{C}$ to $\mathrm{S}_n\,\mathcal{C}$, if such functors exist. These functors exist if the category $\mathcal{C}$ has finite limits and colimits. If $X \in \mathrm{S}_k\,\mathcal{C}$ and $n < k$ (or $X \in \mathrm{S}\,\mathcal{C}$) then define $\mathrm{sk}_n(X)$ and $\mathrm{ck}_n(X)$ by first restricting to $\mathrm{S}_n\,\mathcal{C}$.

In principle, the kth degree of $\mathrm{sk}_n(X)$ should consist of one copy of X_n for every degeneracy map from $\boldsymbol{n}$ to $\boldsymbol{k}$, with identifications corresponding to degeneracies from $\boldsymbol{n-1}$ to $\boldsymbol{k}$. The coskeleton should be constructed by taking one copy of S_n for every face map from $\boldsymbol{k}$ to $\boldsymbol{n}$, and making identifications for the face maps from $\boldsymbol{k}$ to $\boldsymbol{n-1}$. If $\mathcal{C}$ has finite limits and colimits we can formalize this by:

Lemma 1.1. a. *The following is a pushout:*

$$\begin{array}{ccc}
\coprod_{\alpha\colon \boldsymbol{n}\to\boldsymbol{k}} \mathrm{sk}_{n-1}(X)_n & \longrightarrow & \coprod_{\alpha\colon \boldsymbol{n}\to\boldsymbol{k}} X_n \\
\downarrow & & \downarrow \\
\mathrm{sk}_{n-1}(X)_k & \longrightarrow & \mathrm{sk}_n(X)_k
\end{array}$$

where the sums are over the degeneracies from $\boldsymbol{n}$ *to* $\boldsymbol{k}$*, and the vertical maps are sums of degeneracies in* $\mathrm{sk}_n(X)$*.*

b. *The following is a pullback (fiber-product):*

$$\begin{array}{ccc}
\mathrm{ck}_n(X)_k & \longrightarrow & \mathrm{ck}_{n-1}(X)_k \\
\downarrow & & \downarrow \\
\prod_{\alpha\colon \boldsymbol{k}\to\boldsymbol{n}} X_n & \longrightarrow & \prod_{\alpha\colon \boldsymbol{k}\to\boldsymbol{n}} \mathrm{ck}_{n-1} X_n
\end{array}$$

where the products are over the faces from $\boldsymbol{k}$ *to* $\boldsymbol{n}$ *and the vertical maps are products of face maps of* $\mathrm{ck}_n(X)$*.*

The important properties of sk_n and ck_n are the adjointness to restriction and the following:

Lemma 1.2. *If* $X \in \mathrm{S}_n\,\mathcal{C}$*, then an extension of* X *to* $\mathrm{S}_{n+1}\,\mathcal{C}$ *(i.e. an object of* $\mathrm{S}_{n+1}\,\mathcal{C}$ *which restricts to* X *in* $\mathrm{S}_n\,\mathcal{C}$*) is completely specified by a factorization* $\mathrm{sk}_n(X)_{n+1} \to X_{n+1} \to \mathrm{ck}_n(X)_{n+1}$*, of the standard map* $\mathrm{sk}_n \to \mathrm{ck}_n$*. If* $f\colon X \to Y$ *is a map in* $\mathrm{S}_n\,\mathcal{C}$ *then an extension of* f *to* $\mathrm{S}_{n+1}\,\mathcal{C}$ *is given by a map* $f_{n+1}\colon X_{n+1} \to Y_{n+1}$*, making the appropriate diagrams commute.*

Take $\mathcal{M}$ to be a closed model category. Define a cofibration in $\mathrm{S}\,\mathcal{M}$ as a map $f\colon X \to Y$ in which, for all n, $\mathrm{sk}_{n-1}(Y)_n \vee_{\mathrm{sk}_{n-1}(X)_n} X_n \to Y_n$ is a cofibration in $\mathcal{M}$. Define f to be a fibration if $X_n \to \mathrm{ck}_{n-1}(X)_n \times_{\mathrm{ck}_{n-1}(Y)_n} Y_n$ is for all n; and f is a weak equivalence if $f_n\colon X_n \to Y_n$ is for all n.

Theorem A. *With these definitions* $\mathrm{S}\,\mathcal{M}$ *is a closed model category.*

To prove this we need the following lemmas.

Lemma 1.3. *Given a diagram:*

$$\begin{array}{ccccc}
A_2 & \longleftarrow & A_1 & \longrightarrow & A_3 \\
{\scriptstyle f_2}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_3}\downarrow \\
B_2 & \longleftarrow & B_1 & \longrightarrow & B_3
\end{array}$$

with f_3 *and* $A_2 \vee_{A_1} B_1 \to B_2$ *trivial cofibrations, then* $A_2 \vee_{A_1} A_3 \to B_2 \vee_{B_1} B_3$ *is a trivial cofibration.*

Proof. $A_2 \vee_{A_1} A_3 \to A_2 \vee_{A_1} B_3 \cong (A_2 \vee_{A_1} B_1) \vee_{B_1} B_3 \to B_2 \vee_{B_1} B_3$ expresses the desired map as a composition of trivial cofibrations. □

Note that the dual of lemma 1.3 is also true.

Lemma 1.4. a. *A map* $f\colon X \to Y$ *is a trivial cofibration if and only if*

$$\mathrm{sk}_{n-1}(Y)_n \vee_{\mathrm{sk}_{n-1}(X)_n} X_n \to Y_n$$

is a trivial cofibration for all n*.*

b. *A map $f\colon X \to Y$ is a trivial fibration if and only if*

$$X_n \to \mathrm{ck}_{n-1}(X)_n \times_{\mathrm{ck}_{n-1}(Y)_n} Y_n$$

is a trivial fibration, for all n.

Proof. (a) Note that the two conditions are identical for $n = 0$. ($\mathrm{sk}_{-1}(X)$ is the initial object.) Lemmas 1.1 and 1.3 show that the second condition for degrees less than n implies that $\mathrm{sk}_{n-1}(X)_k \to \mathrm{sk}_{n-1}(Y)_k$ is a trivial cofibration for all k.

The factorization $X_n \to \mathrm{sk}_{n-1}(Y)_n \vee_{\mathrm{sk}_{n-1}(X)_n} X_n \to Y_n$ then shows that the two conditions are equivalent in degree n, which completes part a. Part b is proved similarly. □

Proof of Theorem A. It is immediate that weak equivalences compose and cancel, and that a map which is a retract of a cofibration, fibration or weak equivalence is one also. Liftings are constructed degreewise using Lemma 1.2 and the fact that a lifting problem:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

which has been solved for degrees less than n can be extended to degree n if there is a lifting in $\mathcal{M}$ in the diagram:

$$\begin{tikzcd} \mathrm{sk}_{n-1}(B)_n \vee_{\mathrm{sk}_{n-1}(A)_n} A_n \arrow[r] \arrow[d] & X_n \arrow[d] \\ B_n \arrow[r] & \mathrm{ck}_{n-1}(X)_n \times_{\mathrm{ck}_{n-1}(Y)_n} Y_n \end{tikzcd}$$

Factorizations are also constructed degreewise. To factor $f\colon X \to Y$ as a cofibration followed by a fibration, one of which is trivial, assume we have constructed $Z \in \mathrm{S}_{n-1}\,\mathcal{M}$ which factors f, and then extend Z to $\mathrm{S}_n\,\mathcal{M}$ by factoring

$$\mathrm{sk}_{n-1}(A)_n \vee_{\mathrm{sk}_{n-1}(X)_n} X_n \to Z_n \to \mathrm{ck}_{n-1}(Z)_n \times_{\mathrm{ck}_{n-1}(Y)_n} Y_n$$

as a cofibration and a fibration, the correct one being trivial. Lemmas 1.2 and 1.4 show that the constructed maps factor f appropriately. Thus, $\mathcal{M}$ is a closed model category. □

If $\mathcal{M}$ is a closed simplicial model category, then given a simplicial set K, and $X \in \mathrm{S}\,\mathcal{M}$, we can define $(X \otimes K)_n = X_n \otimes K$, and $(X^K)_n = {X_n}^K$. It can be verified that this makes $\mathrm{S}\,\mathcal{M}$ a closed simplicial model category. This shows the weakness of this model category structure, since just the degreewise structure is measured, and not the simplicial structure; i.e. two maps which are simplicially homotopic need not be homotopic in this structure.

2. Colimits in Closed Model Categories

Let $\mathcal{M}$ be a closed model category. The following lemma provides a useful description of weak equivalences of cofibrant objects in closed model categories.

Lemma 2.1. *If A and B are cofibrant objects in $\mathcal{M}$ then $f\colon A \to B$ is a weak equivalence if and only if any lifting problem*

$$\begin{array}{ccc} A & \xrightarrow{u} & X \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle p} \\ B & \xrightarrow{v} & Y \end{array}$$

where p is a fibration, can be solved to the extent that there exists a map $\hat{v}\colon B \to X$ and a homotopy $h\colon A \times \mathrm{I} \to X$, such that $p \circ \hat{v} = v$, h is a homotopy from u to $\hat{v} \circ f$, and $p \circ h = v \circ f \circ \mathrm{pr}_A$, where $\mathrm{pr}_A\colon A \times \mathrm{I} \to A$ is the projection.

Proof. If f is a weak equivalence factor $f = k \circ j$, where j is a trivial cofibration, k a trivial fibration, $j\colon A \to Z$, $k\colon Z \to B$. Find $s\colon B \to Z$ a section, and $H\colon Z \times \mathrm{I} \to Z$ a homotopy from the identity to $s \circ k$, covering the identity of B. Find $\hat{u}\colon Z \to X$ with $p \circ \hat{u} = v \circ k$ and $\hat{u} \circ j = u$, by lifting j against p. Let $\hat{v} = \hat{u} \circ s$, and $h = \hat{u} \circ H \circ \ell$, where ℓ is the inclusion $A \times \mathrm{I} \to Z \times \mathrm{I}$. Then $\hat{v}$ and h are the desired maps.

If f has the property it is easily verified that f induces an epimorphism on left homotopy classes $\pi_\ell(B, Y) \to \pi_\ell(A, Y)$, and a monomorphism on right homotopy classes $\pi_r(B, Y) \to \pi_r(A, Y)$, where Y is a fibrant object. Since A and B are cofibrant, then f induces and isomorphism $[B, Y] \to [A, Y]$, and since $\mathcal{M}$ is closed, f is a weak equivalence. □

Note that the lemma is true if the condition for weak equivalence is weakened to a lifting exists such that both triangles homotopy commute. However, this stronger condition is needed later.

In the following material this lemma will be used as a characterization of weak equivalences of cofibrants. The proposition below gives an important independent application of Lemma 2.1.

Proposition 2.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be closed simplicial model categories, and $\mathrm{L}\colon \mathcal{M} \to \mathcal{N}$ and $\mathrm{R}\colon \mathcal{N} \to \mathcal{M}$ left and right adjoints, respectively. Further, assume that L and R are simplicial adjoints in the sense that one of the equivalent conditions*

i. $\mathrm{L}(X \otimes K) \cong \mathrm{L}(X) \otimes K$,
ii. $\mathrm{R}(X^K) \cong \mathrm{R}(X)^K$, *or*
iii. $\mathbf{Hom}_{\mathcal{M}}\big(X, \mathrm{R}(Y)\big) \cong \mathbf{Hom}_{\mathcal{N}}\big(\mathrm{L}(X), Y\big)$

holds. ($\mathbf{Hom}_{\mathcal{M}}$ and $\mathbf{Hom}_{\mathcal{N}}$ are the simplicial set valued hom *functors of $\mathcal{M}$ and $\mathcal{N}$.) Then, if R preserves fibrations and weak equivalences then L preserves cofibrations and weak equivalences of cofibrants.*

Proof. By adjointness, cofibrations are preserved by L, since R preserves trivial fibrations. We can then apply Lemma 2.1 and the fact that $\mathrm{L}(X \otimes \mathrm{I}) \cong \mathrm{L}(X) \otimes \mathrm{I}$, and conclude the result. □

This result is useful in the case of simplicial algebras when L is extension of theories. (See [6].) In this case R is the forgetful functor; thus, well behaved. The Proposition shows that L must also be well behaved.

Theorem B. *If*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} \\ C & \xrightarrow{j} & D \end{array}$$

is a pushout diagram in $\mathcal{M}$ (i.e. $D = B \vee_A C$), with i a cofibration, f a weak equivalence, and A and C are cofibrant, then g is a weak equivalence.

Proof. We wish to apply Lemma 2.1. Given a diagram:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "u"] \arrow[d, "g"] & X \arrow[d, "p"] \\ C \arrow[r, "j"] & D \arrow[r, "v"] & Y \end{tikzcd}$$

with p a fibration, we know by Lemma 2.1 that there exists a $\bar{v}\colon C \to X$ such that $p \circ \bar{v} = v \circ j$, and $H\colon A \times \mathrm{I} \to X$ such that H is a homotopy from $u \circ i$ to $\bar{v} \circ f$ and $p \circ H = v \circ j \circ f \circ \mathrm{pr}_A$.

Since A and B are cofibrant then there exists a cylinder object $B \times \mathrm{I}$ such that the map $B \vee_A A \times \mathrm{I} \to B \times \mathrm{I}$ is a cofibration and a weak equivalence (the wedge is over the zero inclusion $A \to A \times \mathrm{I}$). Lift in the diagram

$$\begin{tikzcd} B \vee_A A \times \mathrm{I} \arrow[r, "{(u,H)}"] \arrow[d] & X \arrow[d, "p"] \\ B \times \mathrm{I} \arrow[r, "v \circ g \circ \mathrm{pr}_B"] & Y \end{tikzcd}$$

to obtain h. Define $\hat{v} = (h_1, \bar{v})\colon D = B \vee_A C \to X$, where h_1 is the one end of h. The conditions of lemma 2.1 are now satisfied by $\hat{v}$ and h. Q.E.D. □

Note: By an example in [4] it is known that the cofibrancy hypothesis can not be omitted, in general.

Corollary. *Given a diagram*

$$\begin{tikzcd} A_2 \arrow[d, "f_2"] & A_1 \arrow[l, "i_1"'] \arrow[r, "i_2"] \arrow[d, "f_1"] & A_3 \arrow[d, "f_3"] \\ B_2 & B_1 \arrow[l, "j_1"] \arrow[r, "j_2"'] & B_3 \end{tikzcd}$$

where f_1, f_2 and f_3 are weak equivalences i_1 and j_1 are cofibrations, and A_i, B_i are cofibrant ($i = 1, 2, 3$), then $A_2 \vee_{A_1} A_3 \to B_2 \vee_{B_1} B_3$ is a weak equivalence.

Proof. By Theorem B we can assume i_2 and j_2 are also cofibrations. Then

$$\begin{aligned} A_2 \vee_{A_1} A_3 \to A_2 \vee_{A_1} (A_3 \vee_{A_1} B_1) &\cong (A_2 \vee_{A_1} B_1) \vee_{B_1} (A_3 \vee_{A_1} B_1) \\ &\to B_2 \vee_{B_1} (A_3 \vee_{A_1} B_1) \to B_2 \vee_{B_1} B_3 \end{aligned}$$

expresses the desired map as a composition of weak equivalences. □

We also need one more result for use in Section 3.

Theorem C. *If $\mathcal{M}$ has sequential direct limits (i.e. every sequence $A_0 \to A_1 \to \cdots$ has a direct limit), then given a commuting diagram of sequences*

$$\begin{tikzcd} A_0 \arrow[r, "i_0"] \arrow[d, "f_0"] & A_1 \arrow[r, "i_1"] \arrow[d, "f_1"] & \cdots \\ B_0 \arrow[r, "j_0"'] & B_1 \arrow[r, "j_1"'] & \cdots \end{tikzcd}$$

where each f_n is a weak equivalence, each i_n and j_n is a cofibration, and A_0 and B_0 are cofibrant, then $f_\infty \colon A_\infty \to B_\infty$, the limit, is a weak equivalence.

Proof. The proof of Lemma 2.1 can be extended to show that a lifting and homotopy on B_n and $A_n \times \mathrm{I}$ can be chosen which extend the ones on B_{n-1} and $A_{n-1} \times \mathrm{I}$. Since $(A \times \mathrm{I})_\infty$ will be a cylinder object for A_∞ if we chose $A_n \times \mathrm{I}$ so that $(A_n \amalg A_n) \vee A_{n-1} \times \mathrm{I} \to A_n \times \mathrm{I}$ is a cofibration, then the limit of the lifting and homotopy will show that f_∞ is a weak equivalence, by Lemma 2.1. $\square$

3. Realizations

We now assume that $\mathcal{M}$ is a closed simplicial model category having sequential direct limits. If $X \in \mathrm{S}\,\mathcal{M}$ we define the realization of X $(\lvert X \rvert)$ to be the standard identification space made up of the objects $X_n \otimes \Delta^n$ in $\mathcal{M}$. Realization is the left adjoint to the singular complex functor, given by $\mathrm{Sing}(X)_n = X^{\Delta^n}$.

Lemma 3.1. *If $X \in \mathrm{S}\,\mathcal{M}$ define $\lvert X \rvert_0 = X_0$. For $n > 0$ define $\lvert X \rvert_n$ by the following pushout diagram:*

$$(*) \qquad \begin{array}{ccc} \mathrm{sk}_{n-1}(X)_n \otimes \Delta^n \vee_{\mathrm{sk}_{n-1}(X)_n \otimes \partial\Delta^n} X_n \otimes \partial\Delta^n & \longrightarrow & X_n \otimes \Delta^n \\ \downarrow & & \downarrow \\ \lvert X \rvert_{n-1} & \longrightarrow & \lvert X \rvert_n \end{array}$$

where the maps are the standard ones. Then X is the direct limit $\lvert X \rvert_0 \to \lvert X \rvert_1 \to \cdots \to \lvert X \rvert$.

The proof is easy, and is omitted. Note that Lemma 3.1 shows that $\lvert X \rvert$ in fact exists under the given conditions.

Theorem D. *If X and Y are cofibrant in $\mathrm{S}\,\mathcal{M}$, and $f \colon X \to Y$ is a weak equivalence, then $\lvert f \rvert \colon \lvert X \rvert \to \lvert Y \rvert$ is a weak equivalence.*

Proof. First $\lvert f \rvert_0 \colon \lvert X \rvert_0 \to \lvert Y \rvert_0$ is a weak equivalence, since $\lvert f \rvert_0 = f_0$. By using Lemma 1.1a and Theorem B, we know that $\mathrm{sk}_n(X) \to \mathrm{sk}_n(Y)$ is a weak equivalence. Thus we can apply Theorem B to conclude that the induced map corresponding to the upper left hand corner of the diagram $(*)$ is a weak equivalence. The upper map of $(*)$ is a cofibration, since X and Y are cofibrant. Now use Theorem B and Lemma 3.1 to conclude that $\lvert X \rvert_n \to \lvert Y \rvert_n$ is a weak equivalence. Finally, using Theorem C, noting that $\lvert X \rvert_{n-1} \to \lvert Y \rvert_{n-1}$ is a cofibration, we conclude that $\lvert X \rvert \to \lvert Y \rvert$ is a weak equivalence. $\square$

Note: The dual of this theorem must also hold. The dual statement is that a map between fibrant cosimplicial objects over a closed simplicial model category $\mathcal{M}$ (having filtered inverse limits), which is a degreewise weak equivalence, induces a weak equivalence on the corealization. When $\mathcal{M}$ is the category of simplicial sets it is easily seen that the dual model category structure agrees with the Bousfield-Kan structure [2], since the weak equivalences and fibrations are the same. The defined corealization agrees with the Bousfield-Kan codiagonal; and, in fact, the constructed filtration agrees with the filtration of the codiagonal by $\mathrm{Tot}_n(X)$.

In the special case of topological spaces, it should be noted that the following stronger result is true:

Theorem E. *If $X, Y \in \mathrm{S\,Top}$, and $f\colon X \to Y$ is a degreewise weak homotopy equivalence (respectively homotopy equivalence, or homology isomorphism), and X and Y are cofibrant, in the sense that $\mathrm{sk}_{n-1}(X)_n \subset X_n$ has homotopy extension property (same for Y), then $\left|f\right|\colon \left|X\right| \to \left|Y\right|$ is a weak homotopy equivalence (respectively homotopy equivalence, or homology isomorphism).*

Proof. Use the proof of Theorem D, noting that Theorems B and C are true for all the mentioned classes of equivalence, when cofibration mean homotopy extension property. □

Notes:

1. This shows that the standard spectral sequence for the homology of the realization of a simplicial space works as long as the skeleta sit cofibrantly into each higher degree.
2. Other forms of Lemma 3.1 and Theorem D can also be used to show that weak and strong realizations are equivalent for cofibrant simplicial sets, simplicial simplicial sets, topological spaces, etc.

4. Simplicial Sets

In the case of simplicial sets Theorem D becomes more interesting in view of the following result.

Theorem F. *Let* $\mathrm{S\,Set}$ *be the category of simplicial sets, and* $\mathrm{Bi\,S\,Set}$ *the category of simplicial simplicial sets. Then the realization functor* $\mathrm{Bi\,S\,Set} \to \mathrm{S\,Set}$ *is naturally isomorphic to the diagonal functor.*

Proof. Note the following:

Lemma 4.1. *If $\mathcal{M}$ is a closed simplicial model category and $X \in \mathrm{S}\,\mathcal{M}$, then*

$$\left|X \otimes K\right| \cong \left|X\right| \otimes K.$$

Proof. Realization is a colimit of a certain diagram in $\mathcal{M}$. Since $- \otimes K$ is a left adjoint (to $(-)^K$) then $- \otimes K$ preserves colimits. □

Let $\Delta^{[i]}$ represent the discrete i simplex in $\mathrm{Bi\,S\,Set}$. I.e. $\Delta^{[i]}_k$ is the discrete set Δ^i_k. Let $\Delta^{[i]} \otimes \Delta^j$ be the simplicial simplicial set whose kth degree is $\Delta^{[i]}_k \otimes \Delta^j$. By the lemma

$$\left|\Delta^{[i]} \otimes \Delta^j\right| \cong \left|\Delta^{[i]}\right| \otimes \Delta^j = \Delta^i \otimes \Delta^j,$$

which is also the diagonal of $\Delta^{[i]} \otimes \Delta^j$. The objects $\Delta^{[i]} \otimes \Delta^j$ in $\mathrm{Bi\,S\,Set}$ perform the same function as the Δ^is in $\mathrm{S\,Set}$. In particular, any object $X \in \mathrm{Bi\,S\,Set}$ is the colimit of the objects $\Delta^{[i]} \otimes \Delta^j$ over all maps of these objects into X. Realization preserves colimits, since it is a left adjoint. Diagonal trivially preserves colimits. Thus, since every $X \in \mathrm{Bi\,S\,Set}$ is naturally a colimit of objects on which diagonal and realization agree, then $\left|X\right|$ and the diagonal of X are isomorphic. □

Theorem D implies that the diagonal of a map of simplicial simplicial sets which is a degreewise weak equivalence, is also a weak equivalence.

The note at the end of section one also applies here. In particular, since a simplicial simplicial set has two different structures as a simplicial simplicial set; then, we have constructed two distinct model category structures on this category, one distinguishing the vertical structure and one distinguishing the horizontal structure.

The existence of a model category structure which combines both structures simultaneously is an interesting question. The following result provides an indication of what such a structure might be.

Proposition 4.2. *If the weak equivalences in some model category structure on* $\mathrm{Bi\,S\,Set}$ *include both the horizontal and vertical degreewise weak equivalences, then the weak equivalences include all maps whose diagonals are weak equivalences.*

Proof. Write $X_{k,*}$ for the kth vertical section and $X_{*,k}$ for the kth horizontal section of an object $X \in \mathrm{Bi\,S\,Set}$. Let X be a fibrant object in the vertical model category structure constructed in section 1. Then a lifting in the diagram

$$\begin{tikzcd} \Delta^{n-1} \arrow[r] \arrow[d, "d^i"] & X_{k,*} \arrow[d] \\ \Delta^n \arrow[r] & (\mathrm{ck}_{k-1,*}\, X)_{k,*} \end{tikzcd}$$

is the same as a lifting in the diagram

$$\begin{tikzcd} \partial\Delta^{n-1} \arrow[r] \arrow[d] & X_{*,n} \arrow[d, "d_i"] \\ \Delta^k \arrow[r] & X_{*,n-1} \end{tikzcd}$$

Thus in a vertically fibrant object all the vertical face maps are weak equivalences, so the inclusion $X_{*,0} \hookrightarrow \left|X\right|$ is a weak equivalence. Now consider any map $f\colon X \to Y$ which is an weak equivalence on the diagonal. Find $\mathrm{R}(X)$, $\mathrm{R}(Y)$, and $\mathrm{R}(f)$ such that the diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & \mathrm{R}(X) \arrow[d, "\mathrm{R}(f)"] \\ Y \arrow[r] & \mathrm{R}(Y) \end{tikzcd}$$

commutes, the horizontal inclusions are vertical weak equivalences, and $\mathrm{R}(X)$ and $\mathrm{R}(Y)$ are fibrant in the vertical structure. Then $\left|X\right|$ and $\mathrm{R}(X)_{*,0}$ are weakly equivalent to $\mathrm{R}(X)$; and, similarly for Y. Thus $\mathrm{R}(X)_{*,0} \to \mathrm{R}(Y)_{*,0}$ is a weak equivalence since $\left|f\right|$ is, so $\mathrm{R}(f)$ is a horizontal weak equivalence. Thus in the hypothetical structure f must be a weak equivalence, since $X \hookrightarrow \mathrm{R}(X)$, $Y \hookrightarrow \mathrm{R}(Y)$, and $\mathrm{R}(f)$ are. $\square$

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