PUSHOUTS AND ADJOINT FUNCTORS

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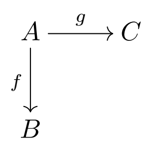
To explain the statement "a colimit is a left adjoint to the constant diagram functor", we'll discuss the special case of pushouts.

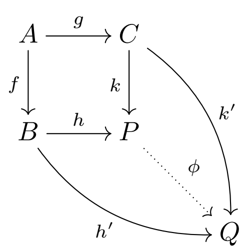
1. Pushouts

Definition 1.1. Let $\mathcal{M}$ be a category, and suppose that we have the diagram in $\mathcal{M}$

(1.2)



A *pushout* of that diagram consists of an object P together with maps $h\colon B \to P$ and $k\colon C \to P$ such that $hf = kg$ and such that if Q is another object of $\mathcal{M}$ and $h'\colon B \to Q$ and $k'\colon C \to Q$ are maps such that $h'f = k'g$, then there exists a unique map $\phi\colon P \to Q$ such that $\phi h = h'$ and $\phi k = k'$.



If we want to say that a pushout is "left adjoint" to something, then we must first describe a pushout as a functor between two categories. Since a colimit is a construction defined on diagrams of the form Diagram 1.2, we need to describe such diagrams as the objects of a category.

Definition 1.3. Let $\mathcal{I}$ be the category with three objects $\{a, b, c\}$ and with two non-identity maps $f\colon a \to b$ and $g\colon a \to c$.

Since any composition of maps in the category $\mathcal{I}$ must involve at least one identity map, it's clear what all the compositions are. We will sometimes refer to $\mathcal{I}$ as the *indexing category*.

Definition 1.4. If $\mathcal{M}$ is a category, then the category $\mathcal{M}^{\mathcal{I}}$, the *category of $\mathcal{I}$-diagrams in* $\mathcal{M}$, is the category in which the objects are the functors $\mathcal{I} \to \mathcal{M}$ and the morphisms between two such functors are the natural transformations of functors.

Date: March 9, 2021.

Note that since $\mathcal{I}$ is a *small* category (i.e., the collection of objects of $\mathcal{I}$ is a set), there is only a set of natural transformations between any two objects of $\mathcal{M}^{\mathcal{I}}$, and so this definition makes sense.

We can now describe a diagram of the form Diagram 1.2 as a functor $X : \mathcal{I} \to \mathcal{M}$:

$$X_b \xleftarrow{X_f} X_a \xrightarrow{X_g} X_c$$

How do we translate the rest of Definition 1.1 into this form? Well, Definition 1.1 mentions an object P and maps $h \colon B \to P$ and $k \colon C \to P$, but there's another map that's not mentioned explicitly, namely the composition $hf \colon A \to P$ (which is the same as the composition $kg \colon A \to P$). Thus, we really have *three* maps to P, one from each of A, B, and C. Having these maps, together with the assumption that $hf = kg$, can be described as having a commutative diagram

$$\begin{array}{ccccc} B & \xleftarrow{f} & A & \xrightarrow{g} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle hf} & & \downarrow{\scriptstyle k} \\ P & \xleftarrow[1_P]{} & P & \xrightarrow[1_P]{} & P \end{array}$$

The bottom row of that diagram is *the constant functor from $\mathcal{I}$ to $\mathcal{M}$ at the object P.*

Definition 1.5. If $\mathcal{I}$ is a category, $\mathcal{M}$ is a category, and P is an object of $\mathcal{M}$, then the *constant functor from $\mathcal{I}$ to $\mathcal{M}$ at the object P* (or the *constant diagram at P*) is the functor from $\mathcal{I}$ to $\mathcal{M}$ that takes every object of $\mathcal{I}$ to P and takes every morphism of $\mathcal{I}$ to the identity map of P.

Thus, we have the following characterization of a pushout.

Proposition 1.6. *A pushout of the diagram*

$$X_b \xleftarrow{X_f} X_a \xrightarrow{X_g} X_c$$

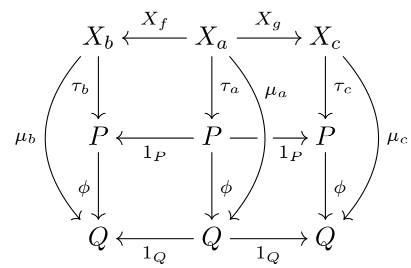
is a natural transformation τ from the functor X to a constant functor at an object P

$$\begin{array}{ccccc} X_b & \xleftarrow{X_f} & X_a & \xrightarrow{X_g} & X_c \\ {\scriptstyle \tau_b}\downarrow & & \downarrow{\scriptstyle \tau_a} & & \downarrow{\scriptstyle \tau_c} \\ P & \xleftarrow[1_P]{} & P & \xrightarrow[1_P]{} & P \end{array}$$

such that for every natural transformation μ from the functor X to a constant functor at an object Q

$$\begin{array}{ccccc} X_b & \xleftarrow{X_f} & X_a & \xrightarrow{X_g} & X_c \\ {\scriptstyle \mu_b}\downarrow & & \downarrow{\scriptstyle \mu_a} & & \downarrow{\scriptstyle \mu_c} \\ Q & \xleftarrow[1_Q]{} & Q & \xrightarrow[1_Q]{} & Q \end{array}$$

there is a unique morphism $\phi\colon P \to Q$ in $\mathcal{M}$ such that $\phi\tau_b = \mu_b$, $\phi\tau_a = \mu_a$, and $\phi\tau_c = \mu_c$ and so the following diagram commutes:



2. Adjoint functors

Suppose now that we have a functorial pushout defined for diagrams of the form Diagram 1.2 in the category $\mathcal{M}$. We will abuse language and denote the object produced by our pushout functor on the diagram X as $\operatorname{colim}_{\mathcal{J}} X$; that is, our functorial pushout applied to $X_b \leftarrow X_a \rightarrow X_c$ will produce

$$\begin{array}{ccc} X_a & \longrightarrow & X_c \\ \big\downarrow & & \big\downarrow \\ X_b & \longrightarrow & \operatorname*{colim}_{\mathcal{J}} X \end{array}$$

Thus, the colimit (i.e., the pushout) of that diagram will consist of the object $\operatorname{colim}_{\mathcal{J}} X$ together with the maps to it from X_a and X_b. This is an abuse of language, because technically the colimit consists of the object together with the maps to that object from the objects in the diagram.

We also have the constant diagram functor c, which takes the object P of $\mathcal{M}$ to the constant diagram $\mathrm{c}(P)$, i.e., $P \leftarrow P \rightarrow P$, and for every natural transformation from the diagram $X_b \leftarrow X_a \rightarrow X_c$ to a constant diagram at an object Q Proposition 1.6 defines a map in $\mathcal{M}$ from $\operatorname{colim}_{\mathcal{J}} X$ to Q. Thus, Proposition 1.6 defines an isomorphism of sets

$$\mathcal{M}^{\mathcal{J}}\bigl(X, \mathrm{c}(Q)\bigr) \to \mathcal{M}(\operatorname*{colim}_{\mathcal{J}} X, Q) \tag{2.1}$$

where $\mathcal{M}^{\mathcal{J}}\bigl(X, \mathrm{c}(Q)\bigr)$ is the set of natural transformations from the diagram X to the diagram $\mathrm{c}(Q)$. Thus, the colimit functor (i.e., the pushout functor) is left adjoint to the constant diagram functor. The counit of the adjunction takes the colimit of the constant diagram at an object Q back to Q, and is an isomorphism for every object Q. The unit of the adjunction is a natural transformation from a diagram X to the constant diagram at the object $\operatorname{colim} X$, and the maps of this natural transformation (one for each object of the indexing category) are the "structure maps" that are a part of the usual definition of a pushout.