THE QUILLEN MODEL CATEGORY OF TOPOLOGICAL SPACES

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Abstract. We give a complete and careful proof of Quillen's theorem on the existence of the standard model category structure on the category of topological spaces. We do not assume any familiarity with model categories.

Contents

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1. Introduction

Quillen defined model categories (see Definition 2.2) in [5, 6] to apply the techniques of homotopy theory to categories other than topological spaces or simplicial sets. Of course, one of his main examples was a model category structure on the category of topological spaces, but the proof that this example satisfied the axioms was only sketched in places, and filling in the details is actually rather hard. Hovey [4] presents a complete proof, but uses some difficult arguments involving function spaces and adjointness. In this note we assume no previous knowledge of model categories and we present a complete statement and proof of Quillen's theorem. We assume nothing more sophisticated than homotopy groups and the exact homotopy sequence of a fibration.

In early work in homotopy theory, the *homotopy category* was defined to be a category in which the morphisms were homotopy classes of maps, and so the isomorphisms were the morphisms represented by homotopy equivalences. Weak homotopy equivalences (i.e., maps that induced an isomorphism of the set of path components and of all homotopy groups at all basepoints) induced isomorphisms of all homology and cohomology groups, but they were not in general homotopy equivalences unless both the domain and target were CW-complexes. For this reason, the *homotopy category* was often defined to have as objects only the CW-complexes, or spaces homotopy equivalent to a CW-complex. Whenever a construction led to a space that was not homotopy equivalent to a CW-complex, it was replaced by a weakly equivalent CW-complex; this preserved all the relevant algebraic invariants. Thus, weakly equivalent spaces were recognized as being somehow "equivalent", even if that was never made explicit. The *homotopy category of a model category* (see [3, Def. 8.3.2]) is constructed by making the weak equivalences into isomorphisms. This identifies homotopic maps (see [3, Lemma 8.3.4]) and takes homotopy equivalences into isomorphisms, but it also explicitly takes all weak equivalences into isomorphisms.

Early work considered the fibrations that are now called *Hurewicz fibrations*, maps with the homotopy lifting property with respect to all spaces. When Serre [7] was developing his spectral sequence for the homology of fiber spaces, he realized that a weaker condition, that of having the homotopy lifting property with respect to polyhedra (or, equivalently, with respect to cubes), was sufficient for work involving homotopy and homology groups, and this larger class of *Serre fibrations* is now more commonly used. Quillen's model category of topological spaces takes as *weak equivalences* the weak homotopy equivalences, as *fibrations* the Serre fibrations, and as *cofibrations* the relative cell complexes (see Definition 4.2) and their retracts.

There is also another model category structure on the category of topological spaces. Strøm [10] has shown that there is a model category structure on the category of topological spaces in which the weak equivalences are the homotopy equivalences, the fibrations are the Hurewicz fibrations, and the cofibrations are the closed inclusions with the homotopy extension property.

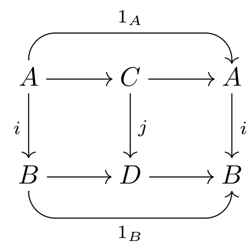
2. Definitions and the main theorem

This note can be read as applying to any of the standard complete and cocomplete categories (i.e., a category that contains both a limit and a colimit for every small diagram in the category) of topological spaces, e.g.,

- the category of all topological spaces,
- the category of compactly generated spaces (see, e.g., [1, Appendix]),
- the category of compactly generated weak Hausdorff spaces (see, e.g., [9], or [2, Appendix]), or
- the category of compactly generated Hausdorff spaces (see, e.g., [8]).

It also applies to any other category of spaces that contains all inclusions of a subcomplex of a CW-complex and in which Proposition 4.10 (that a compact subset of a relative cell complex intersects the interiors of only finitely many cells) holds. The main theorem (on the existence of the model category structure) is Theorem 2.5.

Definition 2.1. If there is a commutative diagram



then we will say that the map i is a *retract* of the map j.

We use the definition of model category from [3, Def. 7.1.3]; this differs from Quillen's original definition of a *closed model category* ([5,6]) in that we require the existence of *all* colimits and limits (not just the finite ones), and we require that the two factorizations be functorial.

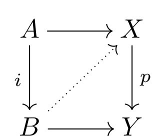
Definition 2.2. A *model category* is a category $\mathcal{M}$ together with three classes of maps (called the *weak equivalences*, the *cofibrations*, and the *fibrations*), satisfying the following five axioms:

M1: (Limit axiom) The category $\mathcal{M}$ is complete and cocomplete (i.e., contains both a limit and a colimit for every small diagram in the category).

M2: (Two out of three axiom) If f and g are maps in $\mathcal{M}$ such that gf is defined and two of f, g, and gf are weak equivalences, then so is the third.

M3: (Retract axiom) If f and g are maps in $\mathcal{M}$ such that f is a retract of g (in the category of maps of $\mathcal{M}$; see Definition 2.1) and g is a weak equivalence, a fibration, or a cofibration, then so is f.

M4: (Lifting axiom) Given the commutative solid arrow diagram in $\mathcal{M}$



the dotted arrow exists if either

(1) i is a cofibration and p is both a fibration and a weak equivalence or
(2) i is both a cofibration and a weak equivalence and p is a fibration.

M5: (Factorization axiom) Every map g in $\mathcal{M}$ has two functorial factorizations:

(1) $g = pi$, where i is a cofibration and p is both a fibration and a weak equivalence, and
(2) $g = qj$, where j is both a cofibration and a weak equivalence and q is a fibration.

Definition 2.3. Let $\mathcal{M}$ be a model category.

(1) A *trivial fibration* is a map that is both a fibration and a weak equivalence.
(2) A *trivial cofibration* is a map that is both a cofibration and a weak equivalence.
(3) An object is *cofibrant* if the map to it from the initial object is a cofibration.
(4) An object is *fibrant* if the map from it to the terminal object is a fibration.
(5) An object is *cofibrant-fibrant* if it is both cofibrant and fibrant.

Definition 2.4. We will say that a map $f\colon X \to Y$ of topological spaces is

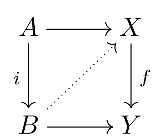
- a *weak equivalence* if it is a weak homotopy equivalence, i.e., if either X and Y are both empty, or X and Y are both nonempty and for every choice of basepoint $x \in X$ the induced map $f_*\colon \pi_i(X, x) \to \pi_i(Y, f(x))$ of homotopy groups (if $i > 0$) or sets (if $i = 0$) is an isomorphism,
- a *cofibration* if it is a relative cell complex (see Definition 4.2) or a retract (see Definition 2.1) of a relative cell complex, and
- a *fibration* if it is a Serre fibration.

Theorem 2.5. *There is a model category structure on the category of topological spaces in which the weak equivalences, cofibrations, and fibrations are as in Definition* 2.4.

The proof of Theorem 2.5 is in Section 9.

3. Lifting

Definition 3.1. If $i\colon A \to B$ and $f\colon X \to Y$ are maps such that for every solid arrow diagram



there exists a diagonal arrow making both triangles commute, then i is said to have the *left lifting property* with respect to f and f is said to have the *right lifting property* with respect to i.

Thus, the lifting axiom M4 (see Definition 2.2) asserts that

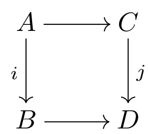
- cofibrations have the left lifting property with respect to all trivial fibrations, and
- fibrations have the right lifting property with respect to all trivial cofibrations.

In this section, we show that the class of maps with the left lifting property with respect to a map $f\colon X \to Y$ is closed under pushouts (see Lemma 3.3), coproducts (see Lemma 3.4), transfinite compositions (see Lemma 3.5) and retracts (see Lemma 3.6). These results are combined in Proposition 4.14 to show that if a map has the right lifting property with respect to the inclusions $S^{n-1} \to D^n$ for all $n \geq 0$ then it has the right lifting property with respect to all relative cell complexes (see Definition 4.2) and their retracts. This is a key step in proving that the lifting axiom M4 is satisfied (see the proof of Theorem 8.1). We also present the *retract argument* (see Proposition 3.7), which is often used to show that a map has the (left or right) lifting property with respect to other maps (see the proof of Theorem 8.2).

3.1. **Pushouts, pullbacks, and lifting.**

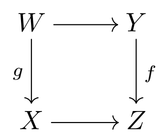
Definition 3.2.

(1) If there is a pushout diagram



then we will say that the map j is a *pushout* of the map i.

(2) If there is a pullback diagram



then we will say that the map g is a *pullback* of the map f.

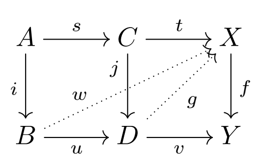
Lemma 3.3.

(1) *If the map j is a pushout of the map i (see Definition 3.2) and if i has the left lifting property with respect to a map $f\colon X \to Y$, then j has the left lifting property with respect to f.*

(2) *If the map g is a pullback of the map f and if f has the right lifting property with respect to a map i, then g has the right lifting property with respect to i.*

Proof. We will prove part 1; the proof of part 2 is similar.

Suppose that we have the solid arrow diagram

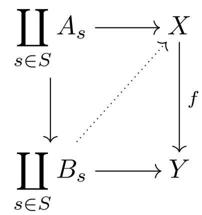


in which the left hand square is a pushout, so that j is a pushout of i. Since i has the left lifting property with respect to f, there is a map $w\colon B \to X$ such that $wi = ts$ and $fw = vu$. Since the left hand square is a pushout, this induces a map $g\colon D \to X$ such that $gu = w$ and $gj = t$. Since $(fg)u = fw = (v)u$ and $(fg)j = ft = (v)j$, the universal property of the pushout now implies that $fg = v$, and so j has the left lifting property with respect to f. $\square$

3.2. **Coproducts, transfinite composition, and lifting.**

Lemma 3.4. *Let $f\colon X \to Y$ be a map. If S is a set and for every $s \in S$ we have a map $A_s \to B_s$ that has the left lifting property with respect to f, then the coproduct $\coprod_{s\in S} A_s \to \coprod_{s\in S} B_s$ has the left lifting property with respect to f.*

Proof. Given the solid arrow diagram



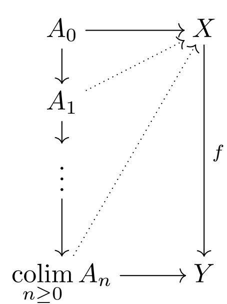
the diagonal arrow can be chosen on each summand B_s, and these together define it on the coproduct. □

Lemma 3.5. *Let $f\colon X \to Y$ be a map. If*

$$A_0 \to A_1 \to A_2 \to \cdots$$

is a sequence of maps such that $A_n \to A_{n+1}$ has the left lifting property with respect to $f\colon X \to Y$ for all $n \geq 0$, then the map $A_0 \to \mathrm{colim}_{n\geq 0} A_n$ has the left lifting property with respect to f.

Proof. Given the solid arrow diagram



the diagonal arrow can be chosen inductively on each A_n and these combine to define it on the colimit. □

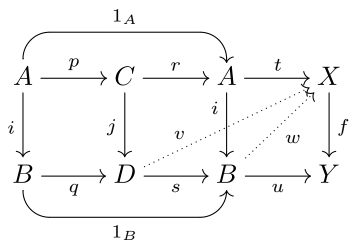
3.3. **Retracts and lifting.**

Lemma 3.6 (Retracts and lifting)**.**

(1) *If the map i is a retract of the map j (see Definition 2.1) and j has the left lifting property with respect to a map $f\colon X \to Y$, then i has the left lifting property with respect to f.*
(2) *If the map f is a retract of the map g and g has the right lifting property with respect to a map $i\colon A \to B$, then f has the right lifting property with respect to i.*

Proof. We will prove part 1; the proof of part 2 is similar.

Suppose that we have the solid arrow diagram



Since j has the left lifting property with respect to f, there exists a map $v\colon D \to X$ such that $vj = tr$ and $fv = us$. We define $w\colon B \to X$ as $w = vq$. We then have $wi = vqi = vjp = trp = t$ and $fw = fvq = usq = u$. □

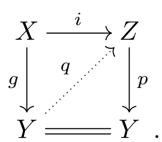
The next result, known as the *retract argument*, is often used to show that a map has the (left or right) lifting property with respect to other maps. This is because Lemma 3.6 shows that if one map is a retract of another, and if the second map has the (left or right) lifting property with respect to another map, so does the first. We will use Proposition 3.7 in Section 8 to prove that the lifting axiom M4 is satisfied (see the proof of Theorem 8.2).

Proposition 3.7 (The retract argument)**.**

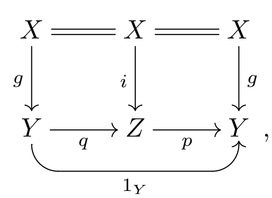
(1) *If the map g can be factored as $g = pi$ where g has the left lifting property with respect to p, then g is a retract of i.*
(2) *If the map g can be factored as $g = pi$ where g has the right lifting property with respect to i, then g is a retract of p.*

Proof. We will prove part 1; the proof of part 2 is dual.

We have the solid arrow diagram



Since g has the left lifting property with respect to p, the dotted arrow q exists. This yields the commutative diagram



and so g is a retract of i. □

4. Relative cell complexes

In this section we define and study *relative cell complexes* (see Definition 4.2) which, together with their retracts (see Definition 2.1), are the cofibrations of the model category.

Definition 4.1. If X is a subspace of Y such that there is a pushout square

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ D^n & \longrightarrow & Y \end{array}$$

for some $n \geq 0$, then we will say that Y *is obtained from* X *by attaching a cell.*

Definition 4.2. A *relative cell complex* is an inclusion of a subspace $f\colon X \to Y$ such that Y can be constructed from X by a (possibly infinite) process of repeatedly attaching cells (see Definition 4.1), and it is a *finite relative cell complex* if it can be constructed by attaching finitely many cells. The topological space X is a *cell complex* if the map $\emptyset \to X$ is a relative cell complex, and it is a *finite cell complex* if X can be constructed from $\emptyset$ by attaching finitely many cells.

Example 4.3. Every relative CW-complex is a relative cell complex, and every CW-complex is a cell complex. Since the attaching map of a cell in a cell complex is not required to factor through the union of lower dimensional cells, not all cell complexes are CW-complexes.

Remark 4.4. We will often construct a relative cell complex by attaching more than one cell at a time. That is, given a space X_0, a set S, and for each $s \in S$ a map $S^{(n_s-1)} \to X_0$, we may construct a pushout

$$\begin{array}{ccc} \coprod_{s\in S} S^{(n_s-1)} & \longrightarrow & X_0 \\ \big\downarrow & & \big\downarrow \\ \coprod_{s\in S} D^{n_s} & \longrightarrow & X_1 \end{array}$$

and then perform a similar construction with X_1, repeating a possibly infinite number of times.

Remark 4.5. While a CW-complex can be built by a countable process of attaching coproducts of cells, a general cell complex may require an arbitrarily long transfinite construction. This is because the attaching map of a cell in a cell complex is not required to factor through the union of lower dimensional cells.

Remark 4.6. Definition 4.2 implies that a relative cell complex is a map that can be constructed as a transfinite composition of pushouts of inclusions of the boundary of a cell into that cell, but there will generally be many different possible such constructions. When dealing with a topological space that is a cell complex or a map that is a relative cell complex, we will often assume that we have chosen some specific such construction (see Definition 4.9). Furthermore, we may choose a construction of the map as a transfinite composition of pushouts of *coproducts* of cells, i.e., we will consider constructions as transfinite compositions in which more than one cell is attached at a time.

Definition 4.7. We adopt the definition of the ordinals in which an ordinal is the well ordered set of all lesser ordinals, and every well ordered set is isomorphic to a unique ordinal (see [3, Sec. 10.1.1]). We will often view an ordinal as a small category with objects equal to the elements of the ordinal and a single map from α to β when $\alpha \leq \beta$.

Definition 4.8. If C is a class of maps and λ is an ordinal, then a *λ-sequence in C* is a functor $X\colon \lambda \to \mathrm{Top}$

$$X_0 \to X_1 \to X_2 \to \cdots X_\beta \to \cdots \qquad (\beta < \lambda)$$

such that the map $X_\beta \to X_{\beta+1}$ is in C for $\beta + 1 < \lambda$ and for every limit ordinal $\gamma < \lambda$ the induced map $\mathrm{colim}_{\beta<\gamma} X_\beta \to X_\gamma$ is an isomorphism. Equivalently, a λ-sequence in $\mathcal{C}$ is a colimit-preserving functor from λ to Top that takes every map $\beta \to \beta + 1$ to an element of $\mathcal{C}$. The *composition* of the λ-sequence is the map $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ (see [3, Sec. 10.2]).

Definition 4.9. If $f\colon X \to Y$ is a relative cell complex, then a *presentation* of f is a λ-sequence of pushouts of coproducts of elements of I

$$X_0 \to X_1 \to X_2 \to \cdots X_\beta \to \cdots \qquad (\beta < \lambda)$$

such that the composition $X_0 \to \mathrm{colim}_{\beta<\lambda} X_\beta$ is isomorphic to f. If e is a cell of the relative cell complex, then the *presentation ordinal* of e is the first ordinal β such that e is in X_β.

4.1. **Compact subsets of relative cell complexes.** The main result of this section is Corollary 4.12, which will be used in the construction of the factorizations required by axiom M5 (see the proof of Proposition 5.1).

Proposition 4.10. *If $X \to Y$ is a relative cell complex, then a compact subset of Y can intersect the interiors of only finitely many cells of $Y - X$.*

Proof. Let C be a compact subset of Y. We construct a subset P of C by choosing one point of C from the interior of each cell whose interior intersects C. We will show that this subset P of C has no accumulation point in C, which implies that P is finite, which implies that C intersects the interiors of only finitely many cells of $Y - X$.

Let $c \in C$; we will show that there is an open subset U of Y such that $c \in U$ and $U \cap P$ is either empty or contains the one point c, which will imply that c is not an accumulation point of P.

Let e_c be the unique cell of $Y - X$ that contains c in its interior. Since there is at most one point of P in the interior of any cell of $Y - X$, we can choose an open subset U_c of the interior of e_c that contains no points of P (except for c, if $c \in P$). We will use Zorn's lemma to show that we can enlarge U_c to an open subset of Y that contains no points of P (except for c, if $c \in P$).

Let α be the presentation ordinal of the cell e_c. If the presentation ordinal of the relative cell complex $X \to Y$ is γ, consider the set T of ordered pairs (β, U) where $\alpha \leq \beta \leq \gamma$ and U is an open subset of Y^β such that $U \cap Y^\alpha = U_c$ and U contains no points of P except possibly c. We define a preorder on T by defining $(\beta_1, U_1) < (\beta_2, U_2)$ if $\beta_1 < \beta_2$ and $U_2 \cap Y^{\beta_1} = U_1$.

If $\{(\beta_s, U_s)\}_{s\in S}$ is a chain in T, then $(\bigcup_{s\in S} \beta_s, \bigcup_{s\in S} U_s)$ is an upper bound in T for the chain, and so Zorn's lemma implies that T has a maximal element (β_m, U_m). We will complete the proof by showing that $\beta_m = \gamma$.

If $\beta_m < \gamma$, then consider the cells of presentation ordinal $\beta_m + 1$. Since Y has the weak topology determined by X and the cells of $Y - X$, we need only enlarge U_m so that its intersection with each cell of presentation ordinal $\beta_m + 1$ is open in that cell, and so that it still contains no points of P except possibly c. If $h\colon S^{n-1} \to Y^{\beta_m}$ is the attaching map for a cell of presentation ordinal $\beta_m + 1$, then $h^{-1}U_m$ is open in S^{n-1}, and so we can "thicken" $h^{-1}U_m$ to an open subset of D^n, avoiding the (at most one) point of P that is in the interior of the cell. If we let U' equal the union of U_m with these thickenings in the interiors of the cells of presentation ordinal $\beta_m + 1$, then the pair $(\beta_m + 1, U')$ is an element of T greater than the maximal element (β_m, U_m) of T. This contradiction implies that $\beta_m = \gamma$, and so the proof is complete. □

Proposition 4.11. *Every cell of a relative cell complex is contained in a finite subcomplex of the relative cell complex.*

Proof. Choose a presentation of the relative cell complex $X \to Y$ (see Definition 4.9); we will prove the proposition by a transfinite induction on the presentation ordinal of the cell. The induction is begun because the very first cells attached are each in a subcomplex with only one cell. Since the presentation ordinal of every cell is a successor ordinal, it is sufficient to assume that the result is true for all cells of presentation ordinal at most some ordinal β and show that it is also true for cells of presentation ordinal the successor of β.

The image of the attaching map of any cell of presentation ordinal the successor of β is compact, and so Proposition 4.10 implies that it intersects the interiors of only finitely many cells, each of which (by the induction hypothesis) is contained in a finite subcomplex. The union of those finite subcomplexes and this new cell is then a finite subcomplex containing the new cell. □

Corollary 4.12. *A compact subset of a relative cell complex is contained in a finite subcomplex of the relative cell complex.*

Proof. Proposition 4.10 implies that a compact subset intersects the interiors of only finitely many cells, and Proposition 4.11 implies that each of those cells is contained in a finite subcomplex; the union of those finite subcomplexes thus contains our compact subset. □

4.2. **Relative cell complexes and lifting.**

Definition 4.13. We will let I denote the set of maps

$$I = \{S^{n-1} \to D^n \mid n \geq 0\}$$

and we will call I the *set of generating cofibrations*.

Proposition 4.14. *If a map has the right lifting property (see Definition 3.1) with respect to every element of I (see Definition 4.13), then it has the right lifting property with respect to all relative cell complexes and their retracts (see Definition 2.1).*

Proof. If a map has the right lifting property with respect to every element of I, then Lemma 3.4 implies that it has the right lifting property with respect to every coproduct of elements of I, and so Lemma 3.3 implies that it has the right lifting property with respect to every pushout of a coproduct of elements of I, and so Lemma 3.5 implies that it has the right lifting property with respect to every composition of pushouts of coproducts of elements of I, and so Lemma 3.6 implies that it has the right lifting property with respect to every retract of such a composition. □

5. The small object argument

In this section we construct two functorial factorizations of maps (see Proposition 5.1 and Proposition 5.9) that will be shown in Section 6 to be the two factorizations required by the factorization axiom M5 (see Definition 2.2). An important point for both of these factorizations is that the second map in the factorization must have the right lifting property with respect to a set of maps, each of which has a compact domain. Compact spaces are "small" in the sense that, because of Corollary 4.12, every map from a compact space into the colimit of a λ-sequence (see Definition 4.8) of relative cell complexes must factor through some intermediate stage of the λ-sequence, and this is the key fact that allows us to show that the second map in the factorization has the required lifting property. Both of these factorizations are examples of *the small object argument* (see [3, Proposition 10.5.16]).

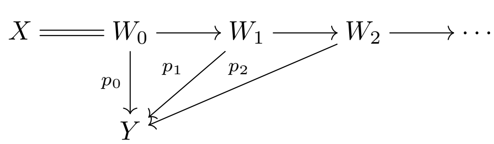
5.1. **The first factorization.**

Proposition 5.1. *There is a functorial factorization of every map $f\colon X \to Y$ as*

$$X \xrightarrow{\ i\ } W \xrightarrow{\ p\ } Y$$

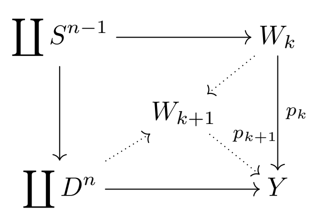
such that i is a relative cell complex (see Definition 4.2) and p has the right lifting property (see Definition 3.1) with respect to every element of I (see Definition 4.13).

Proof. We will construct the space W as the colimit of a sequence of spaces W_k



where each space W_k comes equipped with a map $p_k\colon W_k \to Y$ that makes the diagram commute. We construct the W_k inductively, and we begin the induction by letting $W_0 = X$ and letting $p_0 = f$.

For the inductive step, we assume that $k \geq 0$ and that we have created the diagram through W_k. To create W_{k+1}, we have the solid arrow diagram



where the coproducts on the left are indexed by

$$\coprod_{n\geq 0} \mathrm{Map}(S^{n-1}, W_k) \times_{\mathrm{Map}(S^{n-1},Y)} \mathrm{Map}(D^n, Y) \ ,$$

which is the set of commutative squares

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & W_k \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & Y \end{array} \ ;$$

there is one summand for each such square. We let W_{k+1} be the pushout

$$\begin{array}{ccc} \coprod S^{n-1} & \longrightarrow & W_k \\ \Big\downarrow & & \vdots \\ \coprod D^n & \cdots\cdots\rightarrow & W_{k+1} \end{array} .$$

That defines the space W_{k+1} and the map $p_{k+1}\colon W_{k+1} \to Y$. We let $W = \operatorname{colim}_{k\geq 0} W_k$, we let $p\colon W \to Y$ be the colimit of the p_k, and we let $i\colon X \to W$ be the composition $X = W_0 \to \operatorname{colim} W_k = W$.

Since W was constructed by attaching cells to X, the map $i\colon X \to W$ is a relative cell complex.

To see that $p\colon W \to Y$ has the right lifting property with respect to every element of I, suppose that $n \geq 0$ and we have the solid arrow diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & W \\ \Big\downarrow & \nearrow & \Big\downarrow{\scriptstyle p} \\ D^n & \longrightarrow & Y \end{array} . \qquad (5.2)$$

Since S^{n-1} is compact, Corollary 4.12 implies that there is a positive integer k such that the map $S^{n-1} \to W$ factors through W_k. Thus, we have the solid arrow diagram

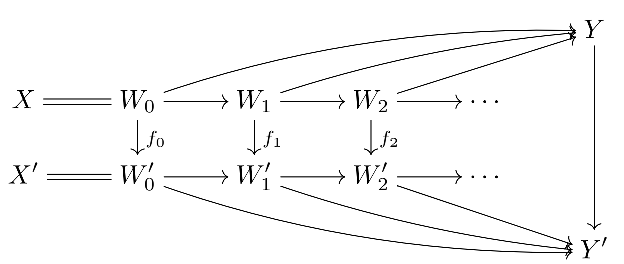
$$\begin{array}{ccccccc} S^{n-1} & \longrightarrow & W_k & \longrightarrow & W_{k+1} & \longrightarrow & W \\ \Big\downarrow & & & \nearrow & & & \Big\downarrow{\scriptstyle p} \\ D^n & & & \longrightarrow & & & Y \end{array}$$

and the map $S^{n-1} \to W_k$ is one of the attaching maps in the pushout diagram that built W_{k+1} out of W_k. Thus, there exists a diagonal arrow $D^n \to W_{k+1}$ that makes the diagram commute, and its composition with $W_{k+1} \to W$ is the diagonal arrow required in (5.2).

To see that the construction is functorial, suppose we have a commutative square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ {\scriptstyle f}\Big\downarrow & & \Big\downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array} ;$$

we will show that the construction of the factorization of $f\colon X \to Y$ maps to that of $f'\colon X' \to Y'$, i.e., that there is a commutative diagram



We let $f_0 = f$. Suppose that we've defined $f_n : W_n \to W_n'$. The space W_{n+1} is constructed by attaching an n-cell to W_n for each commutative square

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\alpha} & W_n \\ \downarrow & & \downarrow \\ D^n & \xrightarrow[\beta]{} & Y \end{array} .$$

We map the cell attached to W_n by α to the cell attached to W_n' by the map $f_n \circ \alpha$ indexed by the outer commutative rectangle

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{\alpha} & W_n & \xrightarrow{f_n} & W_n' \\ \downarrow & & \downarrow & & \downarrow \\ D^n & \xrightarrow[\beta]{} & Y & \longrightarrow & Y' \end{array} .$$

Doing that to each cell attached to W_n defines $f_{n+1} : W_{n+1} \to W_{n+1}'$. □

5.2. **The second factorization.**

Definition 5.3. We will let J denote the set of maps

$$J = \{ D^n \times \{0\} \to D^n \times I \mid n \geq 0 \}$$

and we will call J the *set of generating trivial cofibrations.*

Proposition 5.4. *A map $f : X \to Y$ is a Serre fibration if and only if it has the right lifting property (see Definition 3.1) with respect to every element of J (see Definition 5.3).*

Proof. This is just a restatement of the definition of a Serre fibration. □

Definition 5.5. If X is a subspace of Y such that there is a pushout diagram

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & X \\ \downarrow & & \downarrow \\ D^n \times I & \longrightarrow & Y \end{array}$$

for some $n \geq 0$, then we will say that Y is obtained from X by *attaching a J-cell* (see Definition 5.3).

A *relative J-cell complex* is an inclusion of a subspace $f : X \to Y$ such that Y can be constructed from X by a (possibly infinite) process of repeatedly attaching J-cells.

Lemma 5.6. *Every element of J is a relative cell complex (see Definition 4.2) with two cells. If Y is obtained from X by attaching a J-cell, then $X \to Y$ is a relative cell complex in which you attach a single n-cell and then a single $(n+1)$-cell (for some $n \geq 0$).*

Proof. There is a homeomorphism between $D^n \times I$ and D^{n+1} that takes $D^n \times \{0\}$ onto one of the two n-disks whose union is ∂D^{n+1}. Thus, D^{n+1} is homeomorphic to the result of first attaching an n-cell to $D^n \times \{0\}$ and then attaching an $(n+1)$-cell to the result, and the pushout of

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & X \\ \big\downarrow & & \\ D^n \times I & & \end{array}$$

is homeomorphic to the result of first attaching an n-cell to X and then attaching an $(n+1)$-cell to the result. □

Remark 5.7. We will often construct a relative J-cell complex by attaching more than one J-cell at a time. That is, given a space X_0, a set S, and for each $s \in S$ a map $D^{n_s} \times \{0\} \to X_0$, we may construct a pushout

$$\begin{array}{ccc} \coprod_{s\in S} D^{n_s} \times \{0\} & \longrightarrow & X_0 \\ \big\downarrow & & \big\downarrow \\ \coprod_{s\in S} D^{n_s} \times I & \longrightarrow & X_1 \end{array}$$

and then perform a similar construction with X_1, repeating a possibly infinite number of times.

Proposition 5.8. *Every fibration has the right lifting property (see Definition* 3.1*) with respect to all relative J-cell complexes and their retracts (see Definition* 2.1*).*

Proof. Proposition 5.4 implies that a fibration has the right lifting property with respect to every element of J, and so Lemma 3.4 implies that it has the right lifting property with respect to every coproduct of elements of J, and so Lemma 3.3 implies that it has the right lifting property with respect to every pushout of a coproduct of elements of J, and so Lemma 3.5 implies that it has the right lifting property with respect to every composition of pushouts of coproducts of elements of J, and so Lemma 3.6 implies that it has the right lifting property with respect to every retract of such a composition. □

Proposition 5.9. *There is a functorial factorization of every map $f\colon X \to Y$ as*

$$X \xrightarrow{\ j\ } W \xrightarrow{\ q\ } Y$$

such that j is a relative J-cell complex (see Definition 5.5*) and q has the right lifting property (see Definition* 3.1*) with respect to every element of J (see Definition* 5.3*).*

Proof. The construction of the factorization is the same as in Proposition 5.1, except that we use the set J of generating trivial cofibrations (see Definition 5.3) in place of the set I of generating cofibrations (see Definition 4.13).

Since the space W was constructed by attaching J-cells to X, the map $j\colon X \to W$ is a relative J-cell complex.

Since constructing W_{k+1} from W_k consists of attaching many copies of $D^n \times I$ along $D^n \times \{0\}$ (for all $n \ge 0$), Lemma 5.6 implies that it can be viewed as a 2-step process:

(1) Attach many n-cells (for all $n \ge 0$) to W_k to create a space we'll call W'_k.

(2) Attach many $(n+1)$-cells (for all $n \geq 0$) to W'_k to form W_{k+1}.

If, for $k \geq 0$, we let $V_{2k} = W_k$ and $V_{2k+1} = W'_k$, then W is the colimit of the sequence

$$X = V_0 \to V_1 \to V_2 \to \cdots$$

where each space V_{k+1} is built from V_k by attaching cells to V_k. Thus, the map $X \to W$ is a relative cell complex, and Corollary 4.12 implies that any map $D^n \times \{0\} \to W$ factors through W_k for some $k \geq 0$.

The proof that $q\colon W \to Y$ has the right lifting property with respect to every element of J and the proof that this construction is functorial now proceed exactly as in the proof of Proposition 5.1. □

Proposition 5.10. *The relative J-cell complex constructed in the proof of Proposition* 5.9 *is a relative cell complex.*

Proof. This follows from the proof of Proposition 5.9. □

6. The factorization axiom

In this section we show that the two factorizations of maps constructed in Section 5 are the two factorizations required by the factorization axiom M5 (see Definition 2.2).

6.1. **Cofibration and trivial fibration.**

Proposition 6.1. *The functorial factorization constructed in Proposition* 5.1 *of a map $f\colon X \to Y$ as $X \xrightarrow{i} W \xrightarrow{p} Y$ where i is a relative cell complex and p has the right lifting property with respect to every element of I (see Definition* 4.13*) is a factorization into a cofibration followed by a map that is both a fibration and a weak equivalence.*

Proof. Since the cofibrations are defined to be the relative cell complexes and their retracts (see Definition 2.4), the map i is a cofibration.

Lemma 5.6 implies that every element of J is a relative cell complex and Proposition 4.14 implies that the map p has the right lifting property with respect to all relative cell complexes. Thus, Proposition 5.4 implies that the map p is a fibration.

To see that the map p is a weak equivalence, first note that because it has the right lifting property with respect to the map $\emptyset \to D^0$, it is surjective, and so every path component of Y is in the image of a path component of W. Thus, X and Y are either both empty or both nonempty.

To see that every map of homotopy groups (or sets) $\pi_i W \to \pi_i Y$ (for $i \geq 0$) at every basepoint of W is injective, note that every element of the kernel gives rise to a commutative solid arrow diagram

$$\begin{array}{ccc} S^i & \longrightarrow & W \\ \big\downarrow & \nearrow & \big\downarrow p \\ D^{i+1} & \longrightarrow & Y \end{array} \qquad (6.2)$$

and the existence of the diagonal arrow shows that that element of the kernel is the zero element of $\pi_i W$. (If $i = 0$, then the fact that this is true for *every* choice of basepoint implies that $\pi_0 W \to \pi_0 Y$ is a monomorphism.)

To see that every map of homotopy groups (or sets) $\pi_i W \to \pi_i Y$ (for $i \geq 0$) at every basepoint of W is surjective, we first note that since every path component of Y is in the image of a path component of W, we know that $\pi_0 W \to \pi_0 Y$ is always surjective. Now let $i \geq 0$, let $w \in W$, and let $a \in \pi_{i+1}\big(Y, p(w)\big)$. The element a can be represented by a map $D^{i+1} \to Y$ that takes the entire S^i that is the boundary of D^{i+1} to the point $p(w)$, and so we have a commutative solid arrow diagram as in Diagram 6.2 in which the upper horizontal map is the constant map to the point w and the left vertical map is the inclusion. There must then exist a diagonal arrow making the diagram commute, and this is a map $D^{i+1} \to W$ that takes the entire boundary S^i to the point w, and thus defines an element of $\pi_{i+1}\big(W, w\big)$ that goes under p_* to a. □

6.2. **Trivial cofibration and fibration.**

Proposition 6.3. *The functorial factorization constructed in Proposition* 5.9 *of a map $f\colon X \to Y$ as $X \xrightarrow{j} W \xrightarrow{q} Y$ where j is a relative J-cell complex (see Definition* 5.5*) and q has the right lifting property with respect to every element of J (see Definition* 5.3*) is a factorization into a map that is both a cofibration and a weak equivalence followed by a map that is fibration.*

Proof. The proof of Proposition 5.9 showed that the relative J-cell complex is a relative cell complex, and so the map j is a cofibration.

Since each inclusion $D^n \times \{0\} \to D^n \times I$ is the inclusion of a strong deformation retract, each map $W_k \to W_{k+1}$ in the construction of the factorization is also the inclusion of a strong deformation retract, and is thus a weak equivalence. Thus, for every $i \geq 0$ the sequence

$$\pi_i W_0 \to \pi_i W_1 \to \pi_i W_2 \to \cdots$$

is a sequence of isomorphisms. Since the S^i and D^{i+1} are all compact, Corollary 4.12 implies that every map from S^i or D^{i+1} to W factors through W_k for some $k \geq 0$, and so the map $\operatorname{colim}_k \pi_i W_k \to \pi_i W$ is an isomorphism. Thus, the map $\pi_i X \to \pi_i W$ is also an isomorphism, and the map j is a weak equivalence.

Proposition 5.4 implies that the map q is a fibration. □

7. Homotopy groups and maps of disks

The main result of this section is Proposition 7.10, which will be used in Section 8 to prove that the lifting axiom M4 (see Definition 2.2) holds. Given a solid arrow diagram as in Diagram 7.11, the map h defines an element of $\pi_{n-1}X$ (at some basepoint), and the existence of the map g implies that the image of that element under f_* is the zero element of $\pi_{n-1}Y$. Since the map f is a weak equivalence, this implies that the map h defines the zero element of $\pi_{n-1}X$, and so h can be extended to a map $G\colon D^n \to X$, but there's no reason for the composition fG to equal g. All that we know is that fG and g agree on the boundary of D^n.

We are thus led, in Section 7.1, to study the situation in which we have two maps $\alpha, \beta\colon D^n \to X$ that agree on the boundary of D^n. We use these maps to define (in Definition 7.2) a *difference map* $d(\alpha, \beta)\colon S^n \to X$, using α on the upper hemisphere and β on the lower hemisphere. This difference map defines an element of $\pi_n X$ (at some basepoint), and the remaining results of Section 7.1 show that these homotopy group elements behave as you would expect. The results in Section 7.2 then use the results of Section 7.1 to replace the map G with one for which the composition fG equals g.

7.1. **Difference maps.**

Definition 7.1. For $n \geq 1$ we let S^n_+ be the upper hemisphere of S^n

$$S^n_+ = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots x_{n+1}^2 = 1, x_{n+1} \geq 0\}$$

and we let S^n_- be the lower hemisphere of S^n

$$S^n_- = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots x_{n+1}^2 = 1, x_{n+1} \leq 0\} \ .$$

We let $p_+ \colon S^n_+ \to D^n$ be the homeomorphism

$$p_+(x_1, x_2, \dots, x_{n+1}) = (x_1, x_2, \dots, x_n)$$

and we let $p_- \colon S^n_- \to D^n$ be the homeomorphism

$$p_-(x_1, x_2, \dots, x_{n+1}) = (x_1, x_2, \dots, x_n) \ .$$

Definition 7.2. If X is a space and $\alpha, \beta \colon D^n \to X$ are maps that agree on ∂D^n, then we let $d(\alpha, \beta) \colon S^n \to X$ be the map that is $\alpha \circ p_+ \colon S^n_+ \to X$ on the upper hemisphere of S^n and $\beta \circ p_- \colon S^n_- \to X$ on the lower hemisphere of S^n, and we call it the *difference map* of α and β.

Lemma 7.3. *Let X be a space and let $\alpha \colon D^n \to X$ be a map. If $[g] \in \pi_n\big(X, \alpha(p_0)\big)$ is any element of $\pi_n\big(X, \alpha(p_0)\big)$ (where p_0 is the basepoint of D^n), then there is a map $\beta \colon D^n \to X$ such that $\beta|_{\partial D^n} = \alpha|_{\partial D^n}$ and $[d(\alpha, \beta)] = [g]$ in $\pi_n\big(X, \alpha(p_0)\big)$.*

Proof. The basepoint of D^n is a strong deformation retract of D^n, and so any two maps $D^n \to X$ are homotopic relative to the basepoint. Thus, the restriction of g to S^n_+ is homotopic relative to the basepoint to $\alpha \circ p_+$. Since the inclusion $S^n_+ \hookrightarrow S^n$ has the homotopy extension property, there is a homotopy of g to a map $h \colon S^n \to X$ such that $h|_{S^n_+} = \alpha \circ p_+$; we let $\beta = h \circ (p_-^{-1})$, and we have $h = d(\alpha, \beta)$. □

Lemma 7.4 (Additivity of difference maps)**.** *If X is a space, $n \geq 1$, and $\alpha, \beta, \gamma \colon D^n \to X$ are maps that agree on ∂D^n, then in $\pi_n\big(X, \alpha(p_0)\big)$ (where p_0 is the basepoint of D^n) we have*

$$[d(\alpha, \beta)] + [d(\beta, \gamma)] = [d(\alpha, \gamma)]$$

(where, if $n = 1$, addition should be replaced by multiplication).

Proof. Let $T^n = S^n \cup D^n$, where we view D^n as the subset of $\mathbb{R}^{n+1}$

$$D^n = \{(x_1, x_2, \dots, x_{n+1}) \mid x_1^2 + x_2^2 + \cdots x_n^2 \leq 1, x_{n+1} = 0\} \ ;$$

T^n is then a CW-complex that is the union of the three n-cells S^n_+, S^n_-, and D^n, which all share a common boundary. We let $t(\alpha, \beta, \gamma) \colon T^n \to X$ be the map such that

$$\begin{aligned} t(\alpha, \beta, \gamma)|_{S^n_+} &= \alpha \circ p_+ \\ t(\alpha, \beta, \gamma)|_{D^n} &= \beta \\ t(\alpha, \beta, \gamma)|_{S^n_-} &= \gamma \circ p_- \ . \end{aligned}$$

We then have that

- the composition $S^n \hookrightarrow T^n \xrightarrow{t(\alpha,\beta,\gamma)} X$ is $d(\alpha, \gamma)$,

- the composition $S^n \to S^n_+ \cup D^n \subset T^n \xrightarrow{t(\alpha,\beta,\gamma)} X$ (where that first map is the identity on S^n_+ and is p_- on S^n_-) is $d(\alpha,\beta)$, and
- the composition $S^n \to D^n \cup S^n_- \subset T^n \xrightarrow{t(\alpha,\beta,\gamma)} X$ (where that first map is p_+ on S^n_+ and is the identity on S^n_-) is $d(\beta,\gamma)$.

The basepoint of D^n is a strong deformation retract of D^n, and so the map $\beta\colon D^n \to X$ is homotopic relative to the basepoint to the constant map to $\alpha(p_0)$ (where p_0 is the common basepoint of S^n and D^n). Since the inclusion $D^n \hookrightarrow T^n$ has the homotopy extension property, there is a homotopy of $t(\alpha,\beta,\gamma)$ relative to the basepoint to a map $\hat{t}(\alpha,\beta,\gamma)$ that takes all of D^n to the basepoint $\alpha(p_0)$, and $d(\alpha,\gamma)$ is homotopic to the composition

$$S^n \hookrightarrow T^n \xrightarrow{\hat{t}(\alpha,\beta,\gamma)} X \ .$$

If $T^n \to S^n \vee S^n$ is the map that collapses D^n to a point, then $\hat{t}(\alpha,\beta,\gamma)$ factors as $T^n \to S^n \vee S^n \xrightarrow{\alpha_\beta \vee \beta_\gamma} X$, where $\alpha_\beta\colon S^n \to X$ is homotopic to $d(\alpha,\beta)$ and $\beta_\gamma\colon S^n \to X$ is homotopic to $d(\beta,\gamma)$. Thus, $d(\alpha,\gamma)$ is homotopic to the composition

$$S^n \hookrightarrow T^n \longrightarrow S^n \vee S^n \xrightarrow{\alpha_\beta \vee \beta_\gamma} X \ .$$

and so we have $[d(\alpha,\gamma)] = [d(\alpha,\beta)] + [d(\beta,\gamma)]$ if $n > 1$ and $[d(\alpha,\gamma)] = [d(\alpha,\beta)] \cdot [d(\beta,\gamma)]$ if $n = 1$. □

Lemma 7.5. *If X is a space, $\alpha,\beta\colon D^n \to X$ are maps that agree on ∂D^n, and $[d(\alpha,\beta)]$ is the identity element of $\pi_n X$, then α and β are homotopic relative to ∂D^n.*

Proof. Since $[d(\alpha,\beta)]$ is the identity element of $\pi_n X$, there is a map $h\colon D^{n+1} \to X$ whose restriction to ∂D^{n+1} is $d(\alpha,\beta)$. View $D^n \times I$ as the cone on $\partial(D^n \times I) = (D^n \times \{0\}) \cup (S^{n-1} \times I) \cup (D^n \times \{1\})$ with vertex at the center of $D^n \times I$. Let $p\colon D^n \times I \to D^{n+1}$ be the map that

- on $D^n \times \{0\}$ is the composition $D^n \times \{0\} \xrightarrow{\text{pr}} D^n \xrightarrow{(p_+)^{-1}} S^n_+ \hookrightarrow D^{n+1}$,
- on $D^n \times \{1\}$ is the composition $D^n \times \{0\} \xrightarrow{\text{pr}} D^n \xrightarrow{(p_-)^{-1}} S^n_- \hookrightarrow D^{n+1}$,
- on $S^{n-1} \times I$ is the composition $S^{n-1} \times I \xrightarrow{\text{pr}} S^{n-1} \hookrightarrow S^n_+ \cap S^n_- \subset D^{n+1}$,
- takes the center point of $D^n \times I$ to the center point of D^{n+1}, and
- is linear on each straight line connecting the center point of $D^n \times I$ to its boundary.

The composition $D^n \times I \xrightarrow{p} D^{n+1} \xrightarrow{h} X$ is then a homotopy from α to β relative to ∂D^n. □

7.2. **Lifting maps of disks.** In this section, we use the results of Section 7.1 to prove Proposition 7.10. Given a solid arrow diagram as in Diagram 7.11, Proposition 7.6 shows that, even if the weak equivalence $f\colon X \to Y$ isn't a fibration, there exists a diagonal arrow that makes the upper triangle commute exactly and makes the lower triangle commute up to a homotopy relative to the boundary of D^n. Proposition 7.9 then shows that if the weak equivalence f is also fibration, then there exists a diagonal arrow that makes both triangles commute exactly. Both Proposition 7.6 and Proposition 7.9 actually only apply to one value of n at a time, and Proposition 7.10 is the statement that if *all* homotopy groups (and sets) go by an isomorphism under f_*, then the diagonal arrow exists for all values of n.

Proposition 7.6. *Let $f\colon X \to Y$ be a map, let $n \geq 1$, and suppose that we have the solid arrow diagram*

$$\begin{array}{ccc} \partial D^n & \xrightarrow{h} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ D^n & \xrightarrow[g]{} & Y \end{array} .$$

If F is the homotopy fiber of f over some point in the image of g and if $\pi_{n-1}F = 0$, then there exists a map $G\colon D^n \to X$ such that $Gi = h$ and $fG \simeq g$ relative to ∂D^n.

Proof. The map h defines an element $[h]$ of $\pi_{n-1}X$ (at some basepoint) such that $f_*\big([h]\big) = 0$ in $\pi_{n-1}Y$. Since $\pi_{n-1}F = 0$, the long exact homotopy sequence of a fibration implies that $[h] = 0$ in $\pi_{n-1}X$, and so there is a map $j\colon D^n \to X$ such that $j \circ i = h$.

The maps $fj\colon D^n \to Y$ and $g\colon D^n \to Y$ agree on ∂D^n, and so there is a difference map $d(fj, g)\colon S^n \to X$ (see Definition 7.2) that defines an element α of $\pi_n Y$. Since $\pi_{n-1}F = 0$, the long exact homotopy sequence implies that there is an element β of $\pi_n X$ such that $f_*(\beta) = -\alpha$ (if $n > 1$) or $f_*(\beta) = \alpha^{-1}$ (if $n = 1$), and Lemma 7.3 implies that we can choose a map $G\colon D^n \to X$ that agrees with $j\colon D^n \to X$ on ∂D^n such that $[d(G, j)] = \beta$ in $\pi_n X$. Thus, $Gi = h$, and since $[d(fG, fj)] = [f \circ d(G, j)] = f_*[d(G, j)] = f_*(\beta) = -\alpha$ (if $n > 1$) or α^{-1} (if $n = 1$), Lemma 7.4 implies that $[d(fG, g)] = [d(fG, fj)] + [d(fj, g)] = -\alpha + \alpha = 0$ (with a similar statement if $n = 1$), and so Lemma 7.5 implies that fG is homotopic to g relative to ∂D^n. ☐

Lemma 7.7. *A map $f\colon X \to Y$ is a Serre fibration if and only if for every $n \geq 0$ and every solid arrow diagram*

$$\begin{array}{ccc} (D^n \times \{0\}) \cup (\partial D^n \times I) & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ D^n \times I & \longrightarrow & Y \end{array} \tag{7.8}$$

there exists a diagonal arrow making both triangles commute.

Proof. For every $n \geq 0$ there is a homeomorphism of pairs

$$\big(D^n \times I, (D^n \times \{0\}) \cup (\partial D^n \times I)\big) \longrightarrow \big(D^n \times I, D^n \times \{0\}\big)$$

under which diagrams of the form (7.8) correspond to diagrams of the form

$$\begin{array}{ccc} D^n \times \{0\} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow{\scriptstyle f} \\ D^n \times I & \longrightarrow & Y \end{array}$$

and Proposition 5.4 implies that there always exists a diagonal arrow making the diagram commute if and only if f is a fibration. ☐

Proposition 7.9. *Let $f\colon X \to Y$ be a fibration, let $n \geq 1$, and suppose that we have the solid arrow diagram*

$$\begin{array}{ccc} \partial D^n & \xrightarrow{h} & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ D^n & \xrightarrow[g]{} & Y \end{array}$$

If F is the fiber of f over some point in the image of g and if $\pi_{n-1}F = 0$, then there exists a diagonal arrow making both triangles commute.

Proof. Proposition 7.6 implies that there is a map $G\colon D^n \to X$ such that $Gi = h$ and $fG \simeq g$ relative to ∂D^n. Let $H\colon D^n \times I \to Y$ be a homotopy from fG to g relative to ∂D^n. We have a lift to X of the restriction of H to $(D^n \times \{0\}) \cup (\partial D^n \times I)$ defined as $G \circ \mathrm{pr}_{D^n}$ on $D^n \times \{0\}$ and $h \circ \mathrm{pr}_{\partial D^n}$ on $\partial D^n \times I$, and Lemma 7.7 implies that we can lift the homotopy H to a homotopy $H'\colon D^n \times I \to X$ such that the restriction of H' to $D^n \times \{0\}$ is $G \circ \mathrm{pr}_{D^n}$ and the restriction of H' to $\partial D^n \times I$ is $h \circ \mathrm{pr}_{\partial D^n}$. Let $G'\colon D^n \to X$ be defined by $G'(d) = H'(d, 1)$, and we have $G' \circ i = h$ and $f \circ G' = g$. □

Proposition 7.10. *If $f\colon X \to Y$ is both a fibration and a weak equivalence, then for every $n \geq 0$ and every solid arrow diagram*

$$\begin{array}{ccc} \partial D^n & \xrightarrow{h} & X \\ {\scriptstyle i}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ D^n & \xrightarrow[g]{} & Y \end{array} \tag{7.11}$$

there exists a diagonal arrow making the diagram commute.

Proof. Since f is a weak equivalence, the set of path components of X maps onto that of Y, and so an application of Proposition 5.4 with $n = 0$ implies that X maps onto Y. That implies the case $n = 0$, and the cases $n \geq 1$ follow from Proposition 7.9. □

8. The lifting axiom

In this section we use the results of Section 7 (in particular, Proposition 7.10) to show that the lifting axiom M4 (see Definition 2.2) is satisfied (see Theorem 8.1 and Theorem 8.2).

8.1. **Cofibrations and trivial fibrations.**

Theorem 8.1. *If $f\colon X \to Y$ is both a fibration and a weak equivalence, then it has the right lifting property with respect to all cofibrations.*

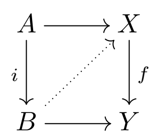
Proof. Proposition 7.10 implies that f has the right lifting property with respect to every element of I (see Definition 4.13). Since a cofibration is defined as a retract of a relative cell complex, the result follows from Proposition 4.14. □

8.2. **Trivial cofibrations and fibrations.**

Theorem 8.2. *If $i\colon A \to B$ is both a cofibration and a weak equivalence, then it has the left lifting property with respect to all fibrations.*

Proof. Proposition 5.9 implies that we can factor $i\colon A \to B$ as $A \xrightarrow{s} W \xrightarrow{t} B$ where s is a relative J-cell complex and t is a fibration (see Proposition 5.4). Proposition 6.3 implies that s is both a cofibration and a weak equivalence and t is a fibration. Since s and i are weak equivalences, the "two out of three" property of weak equivalences implies that t is also a weak equivalence, and so Theorem 8.1 implies that i has the left lifting property with respect to t. The retract argument (see Proposition 3.7) now implies that i is a retract of s, and so Proposition 5.8 implies that i has the left lifting property with respect to all fibrations. □

Corollary 8.3. *If $i\colon A \to B$ is both a relative CW-complex and a weak equivalence and $p\colon X \to Y$ is a Serre fibration, then for every solid arrow diagram*



there is a dotted arrow making the diagram commute.

Proof. Since every relative CW-complex is a relative cell complex, the map i is both a cofibration (see Definition 2.4) and a weak equivalence, and since our fibrations are Serre fibrations (see Definition 2.4), the result now follows from Theorem 8.2. □

9. The proof of Theorem 2.5

We must show that the five axioms of Definition 2.2 are satisfied by the weak equivalences, cofibrations, and fibrations of Definition 2.4.

The limit axiom M1 is satisfied because we've assumed that our category of spaces is complete and cocomplete.

For the two out of three axiom M2, we first note that if any two of f, g, and gf induce an isomorphism of the set of path components, then so does the third. If our maps are $f\colon X \to Y$ and $g\colon Y \to Z$, then if either f and g are weak equivalences or g and gf are weak equivalences, then the two out of three property applied to homomorphisms of homotopy groups implies that the third map also induces an isomorphism of homotopy groups at an arbitrary choice of basepoint. If f and gf are assumed to be weak equivalences, then we know that every choice of basepoint in Y is in the same path component as a point in the image of f, and so (using the change of basepoint isomorphism) it is sufficient to show that the homotopy groups at a basepoint in the image of f are mapped isomorphically, and that follows from the two out of three property of group homomorphisms.

For the retract axiom M3,

- a retract of a weak equivalence is a weak equivalence because a retract of a group isomorphism (or of an isomorphism of the set of path components) is an isomorphism,
- a retract of a cofibration is a cofibration because a retract of a retract of a relative cell complex is a retract of a relative cell complex, and
- a retract of a fibration is a fibration because of Proposition 5.4 and Lemma 3.6.

For the lifting axiom M4, when i is a cofibration and p is both a fibration and a weak equivalence, this is Theorem 8.1, and when i is both a cofibration and a weak equivalence and p is a fibration, this is Theorem 8.2.

For the factorization axiom M5, the factorization into a cofibration followed by a map that is both a fibration and a weak equivalence is Proposition 6.1, and the factorization into a map that is both a cofibration and a weak equivalence followed by a fibration is Proposition 6.3.

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